Bearing-Only Formation Control with Pre-Specified Convergence Time

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***Abstract*—This paper considers the bearing-only formation control problem, where the control of each agent only relies on relative bearings of their neighbors. A new control law is proposed to achieve target formations in finite time. Different from the existing results, the control law is based on a time-varying scaling gain. Hence the convergence time can be arbitrarily chosen by users, and the derivative of the control input is continuous. Furthermore, sufficient conditions are given to guarantee almost global convergence and interagent collision avoidance. Then a leader-follower control structure is proposed to achieve global convergence. By exploring the properties of the bearing Laplacian matrix, the collision avoidance and smooth control input are preserved. A multi-robot hardware platform is designed to validate the theoretical results. Both simulation and experimental results demonstrate the effectiveness of our design.**

***Index Terms*—Bearing-only formation control, finite-time formation control, prescribed-time consensus.**

I. Introduction

Formation control, as an important realm of multi-agent cooperative control, has been extensively studied in recent decades [1]. In the literature (see, [2]–[9]), numerous control laws have been designed to achieve target formations with the assumption that relative positions or distances between agents are measurable. However, this assumption is not always easy to satisfy, especially when agents have no access to an external localization system [10]. Recently, the bearing-only control laws have been proposed and attracted much attention (see, [11]–[18]). Instead of relative positions and distances, target formations of bearing-only control laws are defined by relative bearings that can be obtained by vision sensors [19] or wireless sensor arrays [20]. Due to the accessibility of relative bearings, bearing-only control laws provide potential solutions to achieve formation control merely using onboard sensing.

In two-dimensional space, some early results on bearing-constrained formation control can be found in [11], [12]. Based on the parallel rigidity theory, the authors in [11] introduce the bearing constrained rigidity matrix and propose a control law with locally asymptotic stability. Although the target formation is defined by relative bearings, the measurements of relative positions are still required. This requirement is then removed by introducing a decentralised position estimator [12]. To achieve bearing-only formation control in high-dimensional space, the authors in [13] extend the bearing rigidity theory to arbitrary dimensions and propose a control law for infinitesimally bearing rigid formations with almost global asymptotic stability. To further characterize the algebraic properties of bearing rigid formations, the bearing Laplacian matrix is proposed in [14]. This matrix can be used to examine the uniqueness of target formations in arbitrary dimensions. Based on this powerful tool, a new bearing-only control law is designed in the recent work [15], and global exponential convergence is guaranteed.

Due to the time requirement of many formation control tasks, convergence time is regarded as an important performance indicator. To achieve faster convergence rate, finite-time control has been wildly studied in multi-agent systems (see, [8], [21]–[26]). However, the intrinsic nonlinearity of bearing vectors makes the finite-time convergence analysis of bearing-only control nontrivial. A few works have been done for the finite-time bearing-only formation control (see, [16]–[18]). The authors in [16] use signum functions to suppress relative bearing errors and hence achieve finite-time convergence. However, the results can only be applied to cyclic formations. Instead of signum functions, the controllers in [17], [18] use fractional power bearing feedback and achieve alomost global convergence for infinitesimally bearing rigid formations. However, the convergence time of aforementioned results are all determined by initial conditions, and hence cannot be pre-specified by users. Moreover, the use of signum functions and fractional power feedback will lead to nonsmooth control input. In other words, bearing-only formation control in pre-specified finite time remains an open problem.

In this paper, we investigate bearing-only formation control with pre-specified convergence time. A new control law is proposed for leaderless formation control by introducing a time-varying gain to the regular feedback of relative bearings.

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Sufficient conditions are derived to achieve almost global finite-time convergence while avoiding collisions. Different from the results in [16]–[18], the convergence time can be pre-specified and arbitrarily chosen by users. Furthermore, since no fractional power feedback is used, the control input is C^1 smooth everywhere. The design of time-varying gain is partly inspired by the work on finite-time regulation of nonlinear systems [27]. However, different from relative position based formation control, for bearing-only formation control, relative bearing vectors are unit vectors, i.e., a smaller position error does not imply a smaller bearing error. This phenomenon makes it difficult to establish the boundedness of control input especially when the time-varying gain is unbounded, which implies the stability analysis method in [27] can not be directly applied to our case. Then we design a leader-follower control structure for our proposed control law. By further exploring the properties of bearing Laplacian matrix, we prove that, with the leader-follower control structure, the global convergence can be achieved in pre-specified finite time while avoiding the collisions (rather than the almost global convergence in [13]). Finally, a multi-robots hardware platform is designed, and both simulation and experimental results verify the effectiveness of the proposed control laws.

The remainder of the paper is organized as follows. Section II introduces some necessary preliminaries and problem setup. Sections III and IV presents the main results on the control law design and stability analysis for leaderless case and leader-follower case, respectively. Simulation results and experiment validation are given in Sections V and VI. Conclusions are drawn in Section VII.

II. Preliminaries and Problem Statement

A. Notations

Let $\mathbb{R}_{>0}$ denote the set of positive real numbers. $I_n \in \mathbb{R}^{n\times n}$ denotes the identity matrix, and $\mathbf{1}_n$ denotes a n-dimensional column vector with all elements equal to one. For a series of column vectors $x_1, \cdots, x_n$, $\mathrm{col}(x_1, \cdots, x_n)$ represents a column vector by stacking them together; $\mathrm{span}\{x_1, \cdots, x_n\}$ represents the linear span of the vectors. For a matrix A, $A > 0$ (or $A \geq 0$) means that A is positive definite(or positive semi-definite); $\lambda_i(A)$ is the ith eigenvalue of A; $\mathrm{null}(A)$ and $\mathrm{range}(A)$ are the null and range spaces of A, respectively. For a series of matrices $A_1, \cdots, A_n$, $\mathrm{diag}(A_i)$ denotes the block diagonal matrix with diagonal blocks $A_1, \cdots, A_n$. $\|\cdot\|$ represents the Euclidean norm of a vector or the spectral norm of a matrix. $\otimes$ denotes the Kronecker product of matrices.

B. Preliminaries

Consider a group of n mobile agents in $\mathbb{R}^d$ ($n \geq 2$ and $d \geq 2$). Let $p_i(t) \in \mathbb{R}^d$ be the position of the agent i at time t. The *configuration* of the agents is denoted as $p = \mathrm{col}(p_1, \cdots, p_n) \in \mathbb{R}^{dn}$. The interaction among agents is described by an undirected graph $\mathcal{G} = \{\mathcal{V}, \mathcal{E}\}$, where $\mathcal{V} = \{1, \cdots, n\}$ is the set of vertices and $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ is the set of edges. The edge $(i, j) \in \mathcal{E}$ if agent i can measure the relative bearing of agent j. Since the graph is undirected, we have $(i, j) \in \mathcal{E} \Leftrightarrow (j, i) \in \mathcal{E}$. The *formation*, denoted as $(\mathcal{G}, p)$, is $\mathcal{G}$ with each vertex $i \in \mathcal{V}$ mapped to the point p_i. The set of neighbours of agent i is denoted as $\mathcal{N}_i = \{j \in \mathcal{V} : (i, j) \in \mathcal{E}\}$. An *orientation* of an undirected graph is the assignment of a direction to each edge. An *oriented graph* is an undirected graph with an orientation. Let m be the number of undirected edges. Then the oriented graph has m directed edges. The *incidence matrix* of the oriented graph is denoted as $H \in \mathbb{R}^{m\times n}$, where $[H]_{ki} = -1$ if vertex i is the tail of edge k; $[H]_{ki} = 1$ if vertex i is the head of edge k; and $[H]_{ki} = 0$ otherwise. For an undirected connected graph, it holds that $H\mathbf{1}_n = \mathbf{0}$ and $\mathrm{rank}(H) = n - 1$ [28].

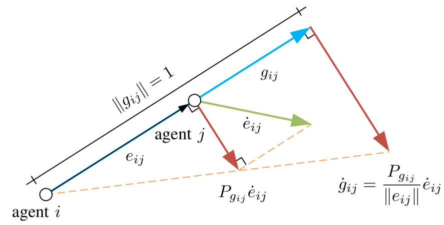


Fig. 1. Geometric relationship between g_{ij}, $\dot{g}_{ij}$, e_{ij} and $\dot{e}_{ij}$.

For edge (i, j), we define the *edge vector* and the *bearing vector* respectively as

$$e_{ij} := p_j - p_i, \qquad g_{ij} := \frac{e_{ij}}{\|e_{ij}\|},$$

where g_{ij} represents the relative bearing of p_j with respect to p_i. Obviously, we have $e_{ij} = -e_{ji}$, $g_{ij} = -g_{ji}$ and $\|g_{ij}\| = 1$. For any nonzero vector $x \in \mathbb{R}^d$, define the operator $P : \mathbb{R}^d \to \mathbb{R}^{d\times d}$ as

$$P_x := I_d - \frac{xx^{\mathrm{T}}}{x^{\mathrm{T}}x},$$

where P_x is an orthogonal projection matrix that can geometrically projects any vector onto the orthogonal complement of x. Note that P_x is positive semi-definite, $P_x^2 = P_x$ and $\mathrm{null}(P_x) = \mathrm{span}(x)$. It follows that $P_x y = 0, \forall y \in \mathbb{R}^d \Leftrightarrow y$ is parallel to x. Since P_x can be used to check whether two bearing are parallel, it is widely used in bearing-based control [14], [29]. Direct evaluation gives

$$\dot{g}_{ij} = \frac{P_{g_{ij}}}{\|e_{ij}\|}\dot{e}_{ij}.$$

Together with $g_{ij}^{\mathrm{T}} P_{g_{ij}} = \mathbf{0}$, we have that $g_{ij}^{\mathrm{T}} \dot{g}_{ij} = \mathbf{0}$ and $e_{ij}^{\mathrm{T}} \dot{g}_{ij} = \mathbf{0}$. Fig. 1 shows the geometric relationship between g_{ij}, $\dot{g}_{ij}$, e_{ij} and $\dot{e}_{ij}$ when $\|e_{ij}\| < 1$.

Suppose (i, j) corresponds to the kth directed edge in the oriented graph where $k \in \{1, \cdots, m\}$. The edge and bearing vectors of kth directed edge is defined as

$$e_k := e_{ij} = p_j - p_i, \qquad g_k := \frac{e_k}{\|e_k\|}.$$

Similarly we have $g_k^{\mathrm{T}} \dot{g}_k = \mathbf{0}$ and $e_k^{\mathrm{T}} \dot{g}_k = \mathbf{0}$. It follows from the definition of H that $e = \bar{H}p$, where $e = \mathrm{col}(e_1, \cdots, e_m)$ and $\bar{H} = H \otimes I_d$.

To characterize the properties of a formation, we introduce the *bearing Laplacian* matrix $\mathcal{B}(\mathcal{G},p) \in \mathbb{R}^{dn\times dn}$ with the (i,j)th block of submatrix as [14]

$$[\mathcal{B}(\mathcal{G},p)]_{ij} = \begin{cases} \mathbf{0}_{d\times d}, & i \neq j,\ (i,j) \notin \mathcal{E}, \\ -P_{g_{ij}}, & i \neq j,\ (i,j) \in \mathcal{E}, \\ \sum_{j\in\mathcal{N}_i} P_{g_{ij}}, & i = j,\ i \in \mathcal{V}. \end{cases}$$

To simplify the notation, we use $\mathcal{B}$ instead of $\mathcal{B}(\mathcal{G},p)$. According to the definition of bearing Laplacian matrix, we have that $\mathcal{B} \geq 0$, $\mathcal{B}p = \mathbf{0}$, $\mathcal{B}\mathbf{1}_{dn} = \mathbf{0}$ and $\mathcal{B} = \bar{H}^{\mathrm{T}}\mathrm{diag}(P_{g_k})\bar{H}$. Letting $(\mathcal{G},p)$ and $(\mathcal{G},p')$ be two formations with the same bearing Laplacian matrix, we give the following definition.

Definition 1 (Infinitesimal Bearing Rigidity [13])**.** *A formation $(\mathcal{G},p)$ is infinitesimally bearing rigid if $p'-p$ corresponds to translational and scaling motions $\Leftrightarrow \mathcal{B}(p'-p) = \mathbf{0}$.*

The above definition implies that an infinitesimally bearing rigid formation is uniquely determined up to a translation and a scaling. Note that the definition of Infinitesimal Bearing Rigidity in [13] is based on bearing rigidity matrix. In this paper, Definition 1 is based on the bearing Laplacian matrix $\mathcal{B}$.

Lemma 1 ([14])**.** *For an infinitesimally bearing rigid formation, the following properties hold*

(i) $\mathrm{null}(\mathcal{B}) = \mathrm{span}\{\mathbf{1}\otimes I_d, p\}$.

(ii) $\mathrm{rank}(\mathcal{B}) = dn-d-1$, *i.e., the eigenvalues of $\mathcal{B}$ represented as $\lambda_1(\mathcal{B}) = \cdots = \lambda_{d+1}(\mathcal{B}) = 0 < \lambda_{d+2}(\mathcal{B}) \leq \cdots \leq \lambda_{dn}(\mathcal{B})$.*

(iii) Partition $\mathcal{B}$ as

$$\mathcal{B} = \begin{bmatrix} \mathcal{B}_{ll} & \mathcal{B}_{lf} \\ \mathcal{B}_{lf}^{\mathrm{T}} & \mathcal{B}_{ff} \end{bmatrix}, \tag{1}$$

where $\mathcal{B}_{ll} \in \mathbb{R}^{dn_l}$, $\mathcal{B}_{lf} \in \mathbb{R}^{dn_f}$, n_l and $n_f \in \mathbb{R}_{>0}$ satisfying $n_l + n_f = n$. Then $\mathcal{B}_{ff} > 0$ if $n_l \geq 2$.

The above lemma bridges the gap between rigidity of a formation and algebraic properties of the bearing Laplacian matrix, which plays an important role in the stability analysis. Lemma 1 (iii) implies that if more than two points of an infinitesimally bearing rigid formation are fixed then the configuration p is uniquely determined. More results on the uniqueness of infinitesimally bearing rigid formation are given in [14].

C. Problem Statement

The dynamics of mobile agents are

$$\dot{p}_i = u_i, \quad i \in \mathcal{V},$$

where $u_i \in \mathbb{R}^d$ is the velocity input of agent i. The main objective of this paper is given below.

Problem 1. *Design control input for agent $i \in \mathcal{V}$ based on the bearing vectors $\{g_{ij}(t)\}_{j\in\mathcal{N}_i}$ such that $p \to p^*$ for $t \to t_0+T$, and $p = p^*$ for $t \geq t_0+T$, where p^* is a target configuration and $T \in \mathbb{R}_{>0}$ is a pre-specified convergence time.*

The following assumption holds throughout this paper.

Assumption 1 (Target Formation)**.** *The target formation $(\mathcal{G},p^*)$ is infinitesimally bearing rigid.*

Remark 1. *Assumption 1 is commonly used to build the connection between the target configuration p^* and the target bearing vectors $\{g_{ij}^*\}_{(i,j)\in\mathcal{E}}$ (e.g., [14], [15]). Then Problem 1 can be transferred into a stabilization problem of bearing vectors in pre-specified finite time.*

III. Bearing-Only Leaderless Formation Control

In this section, we propose a bearing-only leaderless control law to solve Problem 1. The control law of each mobile agent is designed as

$$u_i = -(a + b\frac{\dot{\mu}}{\mu})\sum_{j\in\mathcal{N}_i} P_{g_{ij}} g_{ij}^*, \quad i \in \mathcal{V}, \tag{2}$$

where $a, b \in \mathbb{R}_{>0}$ are positive feedback gains, $\mu : \mathbb{R}_{>0} \to \mathbb{R}_{>0}$ is a time-varying scaling function defined as

$$\mu(t) = \begin{cases} \frac{T^h}{(t_0+T-t)^h}, & t \in [t_0, t_0+T), \\ 1, & t \in [t_0+T, \infty), \end{cases} \tag{3}$$

and $h \in \mathbb{R}_{>0}$ is a user-chosen parameter. Note that

$$\dot{\mu}(t) = \begin{cases} \frac{h}{T}\mu^{(1+\frac{1}{h})}, & t \in [t_0, t_0+T), \\ 0, & t \in [t_0+T, \infty), \end{cases}$$

where we use right-hand derivative of $\mu(t)$ at $t = t_0+T$ as $\dot{\mu}(t_0+T)$. The time-varying scaling function $\mu(t)$ plays a key role in achieving pre-specified finite-time control. For any $c \in \mathbb{R}_{>0}$, we have $\mu^{-c}(t_0) = 1$, $\lim_{t\to(t_0+T)^-}\mu^{-c}(t) = 0$, and $\mu(t)^{-c}$ is monotonically decreasing on $[t_0, t_0+T)$.

Since control law (2) is based on an implicit assumption that g_{ij}, $\forall(i,j) \in \mathcal{E}$ are well defined, we make the following assumption.

Assumption 2 (Collision Avoidance)**.** *During the formation evolvement, no neighboring agents collide with each other.*

Assumption 2 is widely used in the existing formation control results [30], [31], since it is nontrivial to analyze the system convergence if collision avoidance is considered. In this paper, we first analyze system convergence under Assumption 2. Then we will present sufficient conditions based on initial formation such that system convergence and collision avoidance can be simultaneously guaranteed, and hence the assumption could be dropped.

To analyze the finite time convergence of the closed-loop system, we introduce the following lemma.

Lemma 2. *Consider a continuously differentiable function $y : \mathbb{R} \to \mathbb{R}_{\geq 0}$ satisfying that*

$$\dot{y}(t) \leq -\alpha y - \beta\frac{\dot{\mu}}{\mu}y, \quad t \in [t_0, \infty), \tag{4}$$

where $\alpha, \beta \in \mathbb{R}_{>0}$. Then, it follows that

$$y(t) \begin{cases} \leq \mu^{-\beta}e^{-\alpha(t-t_0)}y(t_0), & t \in [t_0, t_0+T), \\ \equiv 0, & t \in [t_0+T, \infty). \end{cases}$$

Proof: Multiplying μ^β on both sides of (4), we get

$$\mu^\beta \dot{y} \leq -\alpha\mu^\beta y - \beta\mu^{\beta-1}\dot{\mu}y.$$

Together with $\frac{\mathrm{d}(\mu^\beta y)}{\mathrm{d}t} = \beta\mu^{\beta-1}\dot{\mu}y + \mu^\beta\dot{y}$, we have that

$$\frac{\mathrm{d}(\mu^\beta y)}{\mathrm{d}t} \leq -\alpha\mu^\beta y,$$

which further implies that

$$\begin{aligned}\mu^\beta y(t) \leq & e^{-\alpha(t-t_0)}\mu(t_0)^\beta y(t_0)\\ =& e^{-\alpha(t-t_0)}y(t_0), \qquad t\in[t_0, t_0+T). \end{aligned} \tag{5}$$

From (5), we can obtain that $y(t) \leq e^{-\alpha(t-t_0)}\mu^{-\beta}y(t_0)$, $\forall t \in [t_0, t_0+T)$. By the continuity of y and $\lim_{t\to(t_0+T)^-} y(t) = 0$, we know that $y(t_0+T) = 0$, and furthermore from $\dot{y} \leq 0$, $\forall t \in [t_0+T, \infty)$, we know that $0 \leq y(t) \leq y(t_0+T)$, $\forall t \in [t_0+T, \infty)$ and that $y(t) \equiv 0$, $\forall t \in [t_0+T, \infty)$. ■

Before analysing the convergence, we first show some useful properties of (2). Define *centroid* and *scale* of a formation as $\bar{p} := \frac{1}{n}\sum_{i=1}^n p_i$ and $s := \sqrt{\frac{1}{n}\sum_{i=1}^n \|p_i - \bar{p}\|^2}$.

Lemma 3. *Under Assumption 2 and control law* (2)*, $\bar{p}$ and s are invariant. Furthermore, $\|p_i(t) - p_j(t)\| \leq 2s\sqrt{n-1}$, $\forall i, j \in \mathcal{V}$, $\forall t \geq t_0$.*

Proof: By following the analysis in [13, Theorem 9], it can be proved that $\dot{\bar{p}} \equiv \mathbf{0}$ and $\dot{s} \equiv \mathbf{0}$, which implies the invariance of $\bar{p}$ and s.

Considering that $p_i - \bar{p} = -\sum_{j\in\mathcal{V}, j\neq i}(p_j - \bar{p})$, we can obtain that

$$\|p_i - \bar{p}\|^2 = \left\|\sum_{j\in\mathcal{V},j\neq i}(p_j-\bar{p})\right\|^2 \leq (n-1)\sum_{j\in\mathcal{V},j\neq i}\|p_j - \bar{p}\|^2,$$

which further implies that $\|p_i - \bar{p}\| \leq s\sqrt{n-1}$, $\forall i \in \mathcal{V}$, and that $\|p_i - p_j\| \leq \|p_i - \bar{p}\| + \|p_j - \bar{p}\| \leq 2s\sqrt{n-1}$, $\forall i, j \in \mathcal{V}$, $\forall t \geq t_0$. ■

Let $\delta_i = p_i - p_i^*$, $\delta = \mathrm{col}(\delta_1, \cdots, \delta_n)$, $r = p - (\mathbf{1}_n \otimes \bar{p})$, $r^* = p^* - (\mathbf{1}_n \otimes \bar{p}^*)$ and $s^* = \sqrt{\frac{1}{n}\sum_{i=1}^n \|p_i^* - \bar{p}^*\|^2}$. With the control law (2), the dynamics of δ can be written in a compact form as

$$\dot{\delta} = (a + b\frac{\dot{\mu}}{\mu})\bar{H}^{\mathrm{T}}\mathrm{diag}(P_{g_k})g^*. \tag{6}$$

Lemma 3 implies that the target centroid and the target scale can be achieved by setting $\bar{p}(t_0) = \bar{p}^*$ and $s(t_0) = s^*$. To further analyze the equilibriums of the closed-loop system (6), we present the following lemma.

Lemma 4. *Under Assumptions 1 and 2 and control law* (2)*, setting $\bar{p}(t_0) = \bar{p}^*$ and $s(t_0) = s^*$, the trajectory of δ evolves on the surface of the sphere $\mathcal{S} = \{\delta \in \mathbb{R}^{dn} : \|\delta + r^*\| = \|r^*\|\}$, and the closed-loop system* (6) *has two equilibriums $\delta = \mathbf{0}$ and $\delta = -2r^*$. Moreover the equilibrium $\delta = -2r^*$ is unstable.*

Proof: By Lemma 3, we can know that $\|r\| = \sqrt{n}s(t_0) = \sqrt{n}s^* = \|r^*\|$. Since $\bar{p} = \bar{p}^*$, we have $\delta = p - (\mathbf{1}_n \otimes \bar{p}) - (p^* - (\mathbf{1}_n \otimes \bar{p}^*))$, and consequently $\|\delta + r^*\| = \|r\| = \|r^*\|$. Hence the trajectory of δ evolves on the surface $\mathcal{S}$.

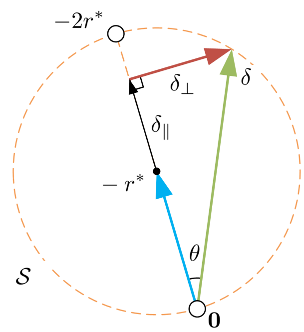


Fig. 2. Geometric relationship between δ and the surface $\mathcal{S}$.

Let $\dot{\delta}_i = (a + b\frac{\dot{\mu}}{\mu})f_i(\delta_i) = -(a + b\frac{\dot{\mu}}{\mu})\sum_{j\in\mathcal{N}_i} P_{g_{ij}}g_{ij}^*$ and $f(\delta) = \mathrm{col}(f_1(\delta_1), \cdots, f_n(\delta_n)) = \bar{H}^{\mathrm{T}}\mathrm{diag}(P_{g_k})g^*$. The equilibriums of (6) belong to $\mathcal{S}$ and satisfy $f(\delta) = \mathbf{0}$. Then it follows that

$$\begin{aligned}(p^*)^{\mathrm{T}}f(\delta) &= (p^*\bar{H})^{\mathrm{T}}\mathrm{diag}(P_{g_k})g^*\\ &= \sum_{k=1}^m \|e_k^*\|\,(g_k^*)^{\mathrm{T}}P_{g_k}g_k^* = 0.\end{aligned}$$

Due to the fact that $P_{g_k} \geq 0$, we have $g_k = \pm g_k^*$, $\forall k = 1, \cdots, m$. For the case $g_k = g_k^*$, by Assumption 1, the formation with bearing constraints $\{g_k^*\}_{k=1,\cdots,m}$ is uniquely determined up to a translation and a scaling. Together with the centroid $\bar{p} = \bar{p}^*$ and the scale $s = s^*$, the formation is uniquely determined, i.e., we have $p = p^*$ and $\delta = \mathbf{0}$. For the case $g_k = -g_k^*$, similarly, we know that the formation with bearing constraints $\{-g_k^*\}_{k=1,\cdots,m}$, $\bar{p} = \bar{p}^*$ and $s = s^*$, is uniquely determined and has the same centroid, scale and shape with $(\mathcal{G}, p^*)$. Furthermore, since $\|p - \mathbf{1}_n \otimes \bar{p}^*\| = \|p^* - \mathbf{1}_n \otimes \bar{p}^*\|$, we can conclude that $p = \mathbf{1}_n \otimes 2\bar{p}^* - p^*$ and $\delta = -2r^*$, which further implies that the formation with $\delta = -2r^*$ is geometrically a point reflection of $(\mathcal{G}, p^*)$.

For the reason that $(a + b\frac{\dot{\mu}}{\mu}) > 0$, the stability of the equilibrium $\delta = -2r^*$ is determined by the Jacobian matrix of $f(\delta)$. Let $F = \frac{\partial f(\delta)}{\partial \delta}$ be the Jacobian matrix of $f(\delta)$ with (i, j)th block of submatrix defined as

$$[F]_{ij} = \begin{cases}\mathbf{0}_{d\times d}, & i\neq j,\ (i,j)\notin\mathcal{E},\\ \frac{\partial f_i(\delta)}{\partial \delta_j}, & i\neq j,\ (i,j)\in\mathcal{E},\\ \frac{\partial f_i(\delta)}{\partial \delta_i}, & i=j,\ i\in\mathcal{V}.\end{cases}$$

Following the similar analysis in [13, Theorem 9], it can be proved that $F|_{\delta=-2r^*} \geq 0$ and F has at least one positive eigenvalue. Hence the equilibrium $\delta = -2r^*$ is unstable. ■

Remark 2. *The geometric relationship between δ and the surface $\mathcal{S}$ is shown in Fig. 2. The angle between δ and $-r^*$ is denoted as θ. Note that δ can always be decoupled as $\delta = \delta_\parallel + \delta_\perp$, where $\delta_\parallel$ is parallel to $-r^*$ and $\delta_\perp$ is perpendicular to $-r^*$.*

Note that Lemmas 3 and 4 all based on the assumption that g_{ij}, $\forall(i, j) \in \mathcal{E}$ is well defined. The following result will show that the inter-agent distances are lower bounded by γ if some initial conditions are satisfied.

Theorem 5. *Under Assumption 1 and control law* (2)*, for a constant* $0 < \gamma < \min_{i,j\in\mathcal{V},i\neq j}\|p_i^* - p_j^*\|$*, the inter-agent distances are lower bounded by* γ*, i.e.,* $\|p_i(t) - p_j(t)\| > \gamma$*,* $\forall i,j \in \mathcal{V}$*,* $\forall t > t_0$ *if*

$$\|\delta(t_0)\| < \frac{\min_{i,j\in\mathcal{V},i\neq j}\|p_i^* - p_j^*\| - \gamma}{\sqrt{n}}. \tag{7}$$

Proof: Before analysing the inter-agent distances, we first show that $\|\delta(t)\|$ is upper bounded by $\|\delta(t_0)\|$ for $t \geq t_0$.

Consider the following Lyapunov function candidate

$$V = \frac{1}{2}\delta^{\mathrm{T}}\delta.$$

By (6), the time derivative of V is obtained as

$$\begin{aligned}\dot{V} &= (a + b\frac{\dot{\mu}}{\mu})(p - p^*)^{\mathrm{T}}\bar{H}^{\mathrm{T}}\mathrm{diag}(P_{g_k})g^* \\ &= -(a + b\frac{\dot{\mu}}{\mu})e^{*\mathrm{T}}\mathrm{diag}(P_{g_k})g^* \\ &= -(a + b\frac{\dot{\mu}}{\mu})\sum_{k=1}^{m}\|e_k^*\|(g_k^*)^{\mathrm{T}}P_{g_k}g_k^* \leq 0. \end{aligned} \tag{8}$$

It follows that $\|\delta(t)\| \leq \|\delta(t_0)\|$, $\forall t \geq t_0$.

Since $p_i - p_j = (p_i - p_i^*) - (p_j - p_j^*) + (p_i^* - p_j^*)$, we obtain that

$$\begin{aligned}\|p_i - p_j\| &\geq \|p_i^* - p_j^*\| - \|p_i - p_i^*\| - \|p_j - p_j^*\| \\ &\geq \|p_i^* - p_j^*\| - \sum_{i=1}^{n}\|p_i - p_i^*\| \\ &\geq \|p_i^* - p_j^*\| - \sqrt{n}\|p - p^*\| \\ &\geq \|p_i^* - p_j^*\| - \sqrt{n}\delta(t),\end{aligned}$$

where we have used the fact that $n\|p - p^*\|^2 \geq \sum_{i=1}^{n}\|p_i - p_i^*\|^2$. Together with (7) and $\|\delta(t)\| \leq \|\delta(t_0)\|$, we can conclude that $\|p_i(t) - p_j(t)\| > \gamma$, $\forall i,j \in \mathcal{V}$, $\forall t > t_0$. ■

Remark 3. *From* (7)*, we can observe that the upper bound of* $\|\delta(t_0)\|$ *is proportional to* $\min_{i,j\in\mathcal{V}}\|p_i^* - p_j^*\|$ *and* $\frac{1}{\sqrt{n}}$*. The intuitive explanation of condition* (7) *is that, for a large group of agents with a small target configuration, to avoid the collisions, the initial error* $\delta(t_0)$ *has to be small.*

Now we are in the position to give the first main result of this paper.

Theorem 6. *Under Assumption 1, Problem 1 is solved by control law* (2) *if the condition* (7) *is satisfied,* $\delta(t_0) \neq -2r^*$*,* $\bar{p}(t_0) = \bar{p}^*$*,* $s(t_0) = s^*$*, and*

$$\begin{aligned}bh\lambda_{d+2}(\mathcal{B}^*)(\sin^2\theta(t_0))\min_{i,j\in\mathcal{V},i\neq j}\|p_i^* - p_j^*\| \\ > 8(n-1)(s^*)^2.\end{aligned} \tag{9}$$

Furthermore, $\|p_i(t) - p_j(t)\| > \gamma$*,* $\forall i,j \in \mathcal{V}$*, and the control input* $u = \mathrm{col}(u_1, \cdots, u_n)$ *remains* C^1 *smooth and uniformly bounded over the time interval* $[t_0, \infty)$*.*

Proof: It follows from (8) that

$$\begin{aligned}\dot{V} &= -(a + b\frac{\dot{\mu}}{\mu})\sum_{k=1}^{m}\|e_k^*\|(g_k)^{\mathrm{T}}P_{g_k^*}g_k \\ &= -(a + b\frac{\dot{\mu}}{\mu})\sum_{k=1}^{m}\frac{\|e_k^*\|}{\|e_k\|^2}e_k^{\mathrm{T}}P_{g_k^*}e_k \\ &\leq -(a + b\frac{\dot{\mu}}{\mu})\frac{\min_{i,j\in\mathcal{V},i\neq j}\|p_i^* - p_j^*\|}{\max_{i,j\in\mathcal{V},i\neq j}\|p_i - p_j\|^2}p^{\mathrm{T}}\bar{H}^{\mathrm{T}}\mathrm{diag}(P_{g_k^*})\bar{H}p \\ &= -(a + b\frac{\dot{\mu}}{\mu})\frac{\min_{i,j\in\mathcal{V},i\neq j}\|p_i^* - p_j^*\|}{\max_{i,j\in\mathcal{V},i\neq j}\|p_i - p_j\|^2}\delta^{\mathrm{T}}\bar{H}^{\mathrm{T}}\mathrm{diag}(P_{g_k^*})\bar{H}\delta,\end{aligned} \tag{10}$$

where we have used the facts that $(g_k)^{\mathrm{T}}P_{g_k^*}g_k = (g_k^*)^{\mathrm{T}}P_{g_k}g_k^*$ and $P_{g_k^*}e^* = \mathbf{0}$ to get the first and last equalities, respectively. Since $s(t) = s(t_0) = s^*$, in light of Lemma 3, we obtain that $\max_{i,j\in\mathcal{V}}\|p_i - p_j\|^2 \leq 4(n-1)(s^*)^2$, and furthermore that

$$\dot{V} \leq -(a + b\frac{\dot{\mu}}{\mu})\frac{\min_{i,j\in\mathcal{V},i\neq j}\|p_i^* - p_j^*\|}{4(n-1)(s^*)^2}\delta^{\mathrm{T}}\mathcal{B}^*\delta,$$

where $\mathcal{B}^* = \mathcal{B}(\mathcal{G}, p^*)$. By Lemma 1, we know that $\mathrm{null}(\mathcal{B}^*) = \mathrm{span}\{\mathbf{1}\otimes I_d, p^*\} = \mathrm{span}\{\mathbf{1}\otimes I_d, r^*\}$. Since $\bar{p} = \bar{p}(t_0) = \bar{p}^*$ according to Lemma 3, we have $(\mathbf{1}\otimes I_d)^{\mathrm{T}}\delta = \mathbf{0}$. Together with the facts that $\delta = \delta_\parallel + \delta_\perp$, $(\mathbf{1}\otimes I_d)^{\mathrm{T}}\delta_\parallel = (\mathbf{1}\otimes I_d)^{\mathrm{T}}r^* = \mathbf{0}$ and $\delta_\perp^{\mathrm{T}}r^* = 0$, we can conclude that $\delta_\perp \perp \mathrm{null}(\mathcal{B}^*)$, which further implies that $\delta^{\mathrm{T}}\mathcal{B}^*\delta = \delta_\perp^{\mathrm{T}}\mathcal{B}^*\delta_\perp \geq \lambda_{d+2}(\mathcal{B}^*)\delta_\perp^{\mathrm{T}}\delta_\perp$. From Lemma 4, we have δ evolves on $\mathcal{S}$ and $\delta_\perp^{\mathrm{T}}\delta_\perp = \sin^2\theta\delta^{\mathrm{T}}\delta$ (see Fig. 2). It can be observed from Fig. 2 that $\theta \in [0, \frac{\pi}{2})$. Since $\|\delta(t)\| \leq \|\delta(t_0)\|$, $\forall t > t_0$, we know that $\theta(t) \geq \theta(t_0)$.

Based on the above analysis, we obtain that

$$\begin{aligned}\dot{V} \leq &-\underbrace{\frac{a\lambda_{d+2}(\mathcal{B}^*)(\sin^2\theta(t_0))\min_{i,j\in\mathcal{V},i\neq j}\|p_i^* - p_j^*\|}{4(n-1)(s^*)^2}}_{\bar{\alpha}_1}\|\delta(t)\|^2 \\ &-\underbrace{\frac{b\lambda_{d+2}(\mathcal{B}^*)(\sin^2\theta(t_0))\min_{i,j\in\mathcal{V},i\neq j}\|p_i^* - p_j^*\|}{4(n-1)(s^*)^2}}_{\bar{\beta}_1}\frac{\dot{\mu}}{\mu}\|\delta(t)\|^2 \\ = &-2\bar{\alpha}_1 V - 2\bar{\beta}_1\frac{\dot{\mu}}{\mu}V.\end{aligned}$$

Then we can conclude from Lemma 2 that

$$\|\delta(t)\|\begin{cases}\leq \mu^{-\bar{\beta}_1}e^{-\bar{\alpha}_1(t-t_0)}\|\delta(t_0)\|, & t \in [t_0, t_0+T), \\ \equiv 0, & t \in [t_0+T, \infty),\end{cases} \tag{11}$$

which implies that p converges to p^* in user pre-specified finite time T. In the following, we will show that u remains C^1 smooth and uniformly bounded.

By (2), we obtain that

$$\|u\| \leq (a + b\frac{\dot{\mu}}{\mu})\|\bar{H}\|\|\mathrm{diag}(P_{g_k})g^*\|.$$

Since

$$\|\mathrm{diag}(P_{g_k})g^*\|^2 = g^{\mathrm{T}}\mathrm{diag}(P_{g_k^*})g = \sum_{k=1}^{m}\frac{1}{\|e_k\|^2}e_k^{\mathrm{T}}P_{g_k^*}e_k, \tag{12}$$

and $\|e_k\| \geq \gamma$ (from Theorem 5), and we have

$$\|\mathrm{diag}(P_{g_k})g^*\| \leq \frac{1}{\gamma}\sqrt{\delta^{\mathrm{T}}\bar{H}^{\mathrm{T}}\mathrm{diag}(P_{g_k^*})\bar{H}\delta}. \quad (13)$$

Incorporating this with (11), we have

$$\|\mathrm{diag}(P_{g_k})g^*\| \leq \frac{1}{\gamma}\|\mathcal{B}^*\|^{\frac{1}{2}}\mu^{-\bar{\beta}_1}e^{-\bar{\alpha}_1(t-t_0)}\|\delta(t_0)\|, \quad (14)$$

$\forall t \in [t_0, t_0+T)$ and $\|\mathrm{diag}(P_{g_k})g^*\| \equiv 0$, $\forall t \in [t_0+T, \infty)$. Hence we have

$$\|\frac{\dot{\mu}}{\mu}\mathrm{diag}(P_{g_k})g^*\| \leq \frac{1}{\gamma}\|\mathcal{B}^*\|^{\frac{1}{2}}\frac{h}{T}\mu^{-(\bar{\beta}_1-\frac{1}{h})}e^{-\bar{\alpha}_1(t-t_0)}\|\delta(t_0)\|, \quad (15)$$

$\forall t \in [t_0, t_0+T)$, and further by $\bar{\beta}_1 - \frac{1}{h} > 0$ (from (9)), we have that $\lim_{t\to(t_0+T)^-}\|\frac{\dot{\mu}}{\mu}\mathrm{diag}(P_{g_k})g^*\| = 0$ and that $\|\frac{\dot{\mu}}{\mu}\mathrm{diag}(P_{g_k})g^*\| \equiv 0$, $\forall t \in [t_0+T, \infty)$. Noting that $u = (a+b\frac{\dot{\mu}}{\mu})\bar{H}^{\mathrm{T}}\mathrm{diag}(P_{g_k})g^*$ and g_k is continuous respect to t, we can conclude that u is continuous and uniformly bounded on $[t_0, \infty)$.

Next we will show $\frac{\mathrm{d}u}{\mathrm{d}t}$ is continuous on $[t_0, \infty)$. Since

$$\begin{aligned}\frac{\mathrm{d}u}{\mathrm{d}t} =& -(a+b\frac{\dot{\mu}}{\mu})\bar{H}^{\mathrm{T}}\frac{\mathrm{d}(\mathrm{diag}(P_{g_k}))}{\mathrm{d}t}g^* \\ &- \frac{bh}{T^2}\mu^{\frac{2}{h}}\bar{H}^{\mathrm{T}}\mathrm{diag}(P_{g_k})g^*, \end{aligned} \quad (16)$$

it is clear that $\frac{\mathrm{d}u}{\mathrm{d}t}$ is continuous on $[t_0, t_0+T)$ and (t_0+T, ∞). From (14), it can be obtained that

$$\|\mu^{\frac{2}{h}}\bar{H}^{\mathrm{T}}\mathrm{diag}(P_{g_k})g^*\| \leq \frac{1}{\gamma}\|\mathcal{B}^*\|^{\frac{1}{2}}\mu^{-(\bar{\beta}_1-\frac{2}{h})}e^{-\bar{\alpha}_1(t-t_0)}\|\delta(t_0)\|, \quad (17)$$

$\forall t \in [t_0, t_0+T)$, and further by $\bar{\beta}_1 - \frac{2}{h} > 0$ (from (9)), we have that $\lim_{t\to(t_0+T)^-}\|\mu^{\frac{2}{h}}\mathrm{diag}(P_{g_k})g^*\| = 0$ and that $\|\mu^{\frac{2}{h}}\mathrm{diag}(P_{g_k})g^*\| \equiv 0$, $\forall t \in [t_0+T, \infty)$.

For the first term in (16), we have

$$\begin{aligned}&(a+b\frac{\dot{\mu}}{\mu})\bar{H}^{\mathrm{T}}\frac{\mathrm{d}(\mathrm{diag}(P_{g_k}))}{\mathrm{d}t}g^* \\ =&(a+b\frac{\dot{\mu}}{\mu})\frac{\partial f(\delta)}{\partial \delta}\dot{\delta} = (a+\frac{bh}{T}\mu^{\frac{1}{h}})^2F\bar{H}^{\mathrm{T}}\mathrm{diag}(P_{g_k})g^*,\end{aligned}$$

where F is the Jacobian matrix defined in the proof of Lemma 4. From Theorem 5, we have $\|e_{ij}\| \geq \gamma$, and further due to the definition of G_{ij} and $P_{g_{ij}}$, we know that $\|[F]_{ij}|_\delta\|$, $\forall i,j \in \mathcal{V}$ is bounded for $\delta \in [0, \delta(t_0)]$. Thus we can always define a positive constant κ such that $\kappa = \max_{\delta\in[0,\delta(t_0)]}\|F|_\delta\|$.

It then follows that

$$\begin{aligned}&\|(a+\frac{bh}{T}\mu^{\frac{1}{h}})^2F\bar{H}^{\mathrm{T}}\mathrm{diag}(P_{g_k})g^*\| \\ \leq&\kappa\|\bar{H}\|(a^2+\frac{2abh}{T}\mu^{\frac{1}{h}}+\frac{b^2h^2}{T^2}\mu^{\frac{2}{h}})\|\mathrm{diag}(P_{g_k})g^*\|.\end{aligned}$$

By using the fact $\bar{\beta}_1 - \frac{2}{h} > 0$, following similar analysis for (15) and (17), it is clear that $\lim_{t\to(t_0+T)^-}\|\frac{\mathrm{d}u}{\mathrm{d}t}\| = 0$, $\forall t \in [t_0, t_0+T)$ and $\|\frac{\mathrm{d}u}{\mathrm{d}t}\| \equiv 0$, $\forall t \in [t_0+T, \infty)$. Hence we can conclude that $\frac{\mathrm{d}u}{\mathrm{d}t}$ is continuous on $[t_0, \infty)$, and furthermore the control input u is C^1 smooth and uniformly bounded over the time interval $[t_0, \infty)$. ■

Remark 4. *Noting that $\delta = -2r^*$ is an unstable equilibrium of closed-loop system* (6)*, Theorem 6 guarantees almost global formation stabilization excepting the case $\delta(t_0) = -2r^*$. Furthermore, the invariance of scale and centroid is used such that the formation control problem can be transferred into a bearing stabilization problem.*

Remark 5. *It is worth noting that the time-varying gain $\frac{\dot{\mu}}{\mu}$ plays an important role in achieving the formation control in finite time. From* (15)*, we can see that the control gain $\frac{\dot{\mu}}{\mu}$ goes to infinite when $t \to t_0+T$. However, the control input u remains bounded and C^1 smooth. The equations (12) and (13) build the connection between $\|u\|$ and $\|\delta\|$. Intuitively, the condition* (9) *guarantees a sufficiently large b such that the decrease of $\|\delta\|$ is faster than the increase of $\frac{\dot{\mu}}{\mu}$. Different from the fractional power based finite time control law in [8], [21], [32], the converge time of control law* (2) *does not depend on the initial condition and can be any value specified by users.*

IV. Bearing-Only Leader-Follower Formation Control

To guarantee the convergence, Theorem 6 requires $\bar{p}(t_0) = \bar{p}^*$ and $s(t_0) = s^*$, which may not be easily satisfied when system has a large number of agents. In this section, we will show that these requirements can be relaxed and the global stabilization can be achieved by using a leader-follower control structure.

Without loss of generality, suppose the first $n_l \geq 2$ agents are leaders and the rest $n_f = n - n_l$ agents are followers. Let $\mathcal{V}_l = \{1, \cdots, n_l\}$ and $V_f = \{n_l+1, \cdots, n\}$ be the set of leaders and followers, respectively. The positions of agents are denoted as $p = \mathrm{col}(p_l, p_f)$, where $p_l = \mathrm{col}(p_1, \cdots, p_{n_l})$ and $p_f = \mathrm{col}(p_{n_l+1}, \cdots, p_n)$ are the positions of leaders and followers respectively. The leader-follower formation control problem is given below.

Problem 2. *With leader positions $\{p_i^*\}_{i\in\mathcal{V}_l}$, design control input for agent $i \in \mathcal{V}_f$ based on the bearing vectors $\{g_{ij}(t)\}_{j\in\mathcal{N}_i}$ such that $p \to p^*$ for $t \to t_0+T$, and $p = p^*$ for $t \geq t_0+T$, where p^* is a target configuration and $T \in \mathbb{R}_{>0}$ is the convergence time pre-specified by users.*

Since the leaders are stationary, we have $\dot{p}_i = 0$, $i \in \mathcal{V}_l$. The control law of each following mobile agent is designed as

$$u_i = -(a+b\frac{\dot{\mu}}{\mu})\sum_{j\in\mathcal{N}_i}P_{g_{ij}}g_{ij}^*, \quad i \in \mathcal{V}_f, \quad (18)$$

Note that control law (18) is same as (2). In the following, we will show that control law (18) can achieve global formation stabilization.

Since $\delta_i = p_i - p_i^*$, we have $\delta = \mathrm{col}(\delta_l, \delta_f) = \mathrm{col}(\mathbf{0}_{dn_l}, \delta_f)$, where $\delta_l = p_l - p_l^*$ and $\delta_f = p_f - p_f^*$. By (18), the dynamics of δ can be written in a compact form as

$$\dot{\delta} = (a+b\frac{\dot{\mu}}{\mu})\begin{bmatrix}\mathbf{0}_{dn_l\times dn_l} & \mathbf{0}_{dn_l\times dn_f}\\ \mathbf{0}_{dn_f\times dn_l} & I_{dn_f}\end{bmatrix}\bar{H}^{\mathrm{T}}\mathrm{diag}(P_{g_k})g^*.$$

Theorem 7. *Under Assumption 1 and control law* (18)*, the inter-agent distances are also lower bounded by γ, if condition* (7) *is satisfied.*

Proof: Here we just need to proof that $\|\delta\|$ is upper bounded by $\|\delta(t_0)\|$, for $t > t_0$ and the rest of proof follows similarly as in Theorem 5. Noting that $V = \frac{1}{2}\delta^{\mathrm{T}}\delta$, the time derivative of V is given as

$$\begin{aligned}\dot{V} &= (a + b\frac{\dot{\mu}}{\mu})\delta^{\mathrm{T}} \begin{bmatrix} \mathbf{0}_{dn_l \times dn_l} & \mathbf{0}_{dn_l \times dn_f} \\ \mathbf{0}_{dn_f \times dn_l} & I_{dn_f} \end{bmatrix} \bar{H}^{\mathrm{T}} \mathrm{diag}(P_{g_k}) g^* \\ &= -(a + b\frac{\dot{\mu}}{\mu})\delta^{\mathrm{T}} \bar{H}^{\mathrm{T}} \mathrm{diag}(P_{g_k}) g^* \\ &= -(a + b\frac{\dot{\mu}}{\mu}) \sum_{k=1}^{m} \|e_k^*\| (g_k^*)^{\mathrm{T}} P_{g_k} g_k^* \leq 0, \end{aligned} \tag{19}$$

where we have used the fact that $\delta_l = \mathbf{0}$. It follows that $\|\delta(t)\| \leq \|\delta(t_0)\|$, $\forall t \geq t_0$. ■

Remark 6. *Theorem 7 implies that although the state trajectory in leader-follower case is different with leaderless case, the conditions for collision avoidance are same. Fixing arbitrary number of agents on the target position will not change $\dot{V}$, hence the condition for the collision avoidance is not related to the number of leaders.*

Theorem 8. *Under Assumption 1, Problem 2 is solved by control law* (18) *if the condition* (7) *is satisfied and*

$$\begin{aligned} bh\lambda_{\min}(\mathcal{B}_{ff}^*) &\min\nolimits_{i,j\in\mathcal{V},i\neq j}\|p_i^* - p_j^*\| \\ &> 2\|\bar{H}\|^2(\|\delta(t_0)\| + \sqrt{n}s^*)^2, \end{aligned} \tag{20}$$

where $\lambda_{\min}(\mathcal{B}_{ff}^*)$ is the smallest eigenvalue of $\mathcal{B}_{ff}^*$. Furthermore, $\|p_i(t) - p_j(t)\| > \gamma$, $\forall i, j \in \mathcal{V}$, and the control input $u_f = \mathrm{col}(u_{n_l+1}, \cdots, u_n)$ remains C^1 smooth and uniformly bounded over the time interval $[t_0, \infty)$.

Proof: By (18) and following the analysis for (10), we have

$$\begin{aligned}\dot{V} \leq &-(a + b\frac{\dot{\mu}}{\mu}) \frac{\min_{i,j\in\mathcal{V},i\neq j}\|p_i^* - p_j^*\|}{\max_{i,j\in\mathcal{V},i\neq j}\|p_i - p_j\|^2} \delta^{\mathrm{T}} \bar{H}^{\mathrm{T}} \mathrm{diag}(P_{g_k^*}) \bar{H}\delta \\ = &-(a + b\frac{\dot{\mu}}{\mu}) \frac{\min_{i,j\in\mathcal{V},i\neq j}\|p_i^* - p_j^*\|}{\max_{i,j\in\mathcal{V},i\neq j}\|p_i - p_j\|^2} \delta^{\mathrm{T}} \mathcal{B}^* \delta. \end{aligned} \tag{21}$$

Due to the fact that

$$\delta^{\mathrm{T}} \mathcal{B}^* \delta = [\mathbf{0}^{\mathrm{T}} \ \delta_f^{\mathrm{T}}] \begin{bmatrix} \mathcal{B}_{ll}^* & \mathcal{B}_{lf}^* \\ (B_{lf}^*)^{\mathrm{T}} & \mathcal{B}_{ff}^* \end{bmatrix} [\mathbf{0}^{\mathrm{T}} \ \delta_f^{\mathrm{T}}]^{\mathrm{T}},$$

and in light of Lemma 1 (iii), we know that $\mathcal{B}_{ff}^* > 0$ and further that $\delta^{\mathrm{T}} \mathcal{B}^* \delta \geq \lambda_{\min}(\mathcal{B}_{ff}^*) \delta^{\mathrm{T}}\delta$.

Different from the leaderless case, for the leader-follower case, the invariance of the centroid $\bar{p}$ and the scale s are no longer hold. Alternatively, the following inequalities are used to characterize the upper bound of $\max_{i,j\in\mathcal{V},i\neq j}\|p_i - p_j\|^2$.

$$\begin{aligned} \max\nolimits_{i,j\in\mathcal{V},i\neq j}\|p_i - p_j\|^2 &\leq \|e\|^2 = \|\bar{H}(p - p^* + p^*)\|^2 \\ &= \|\bar{H}(\delta + r^*)\|^2 \\ &\leq \|\bar{H}\|^2(\|\delta(t_0)\| + \sqrt{n}s^*)^2, \end{aligned} \tag{22}$$

where we have used the facts $\|\delta(t)\| \leq \|\delta(t_0)\|$, $\bar{H}(p^* - (\mathbf{1}_n \otimes \bar{p}^*)) = \bar{H}p^*$ and $\|r^*\| = \sqrt{n}s^*$ to get the last inequality. It then follows from (21) and (22) that

$$\begin{aligned}\dot{V} \leq &-\underbrace{\frac{a\lambda_{\min}(\mathcal{B}_{ff}^*)\min_{i,j\in\mathcal{V},i\neq j}\|p_i^* - p_j^*\|}{\|\bar{H}\|^2(\|\delta(t_0)\| + \sqrt{n}s^*)^2}}_{\bar{\alpha}_2} \|\delta(t)\|^2 \\ &-\underbrace{\frac{b\lambda_{\min}(\mathcal{B}_{ff}^*)\min_{i,j\in\mathcal{V},i\neq j}\|p_i^* - p_j^*\|}{\|\bar{H}\|^2(\|\delta(t_0)\| + \sqrt{n}s^*)^2}}_{\bar{\beta}_2} \frac{\dot{\mu}}{\mu}\|\delta(t)\|^2 \\ = &-2\bar{\alpha}_2 V - 2\bar{\beta}_2\frac{\dot{\mu}}{\mu}V. \end{aligned}$$

In light of Lemma 2, we have

$$\|\delta(t)\| \begin{cases} \leq \mu^{-\bar{\beta}_2} e^{-\bar{\alpha}_2(t-t_0)}\|\delta(t_0)\|, & t \in [t_0, t_0 + T), \\ \equiv 0, & t \in [t_0 + T, \infty), \end{cases}$$

which implies that p converges to p^* in a user pre-specified finite time T. Note that

$$\begin{aligned}\|u_f\| = &\left\| \begin{bmatrix} \mathbf{0}_{dn_l \times dn_l} & \mathbf{0}_{dn_l \times dn_f} \\ \mathbf{0}_{dn_f \times dn_l} & I_{dn_f} \end{bmatrix} \bar{H}^{\mathrm{T}} \mathrm{diag}(P_{g_k}) g^* \right\| \\ \leq &\|\bar{H}\| \|\mathrm{diag}(P_{g_k}) g^*\|. \end{aligned}$$

Following the similar analysis in Theorem 6, it can be proved that u_f is C^1 smooth and uniformly bounded over the time interval $[t_0, \infty)$. ■

Remark 7. *Any positive a and b can guarantee $\dot{V} \leq 0$. The conditions* (9) *and* (20) *are required to guarantee the boundness and the smoothness of the control input.*

Remark 8. *In leader-follower case, the initial requirements $\bar{p}(t_0) = \bar{p}^*$ and $s(t_0) = s^*$ are removed. Intuitively, due to $n_l \geq 2$, at least two points and the edge between these two points are fixed. Since these two points and the edge can determine the translation and scaling of the target formation, together with the fact that the target formation is infinitesimally bearing rigid, the target formation is uniquely determined. Furthermore, since we have $\delta_l = \mathbf{0}$, the system will not start from the initial condition $\delta(t_0) = -2r^*$. Hence the global stability can be achieved.*

V. Simulation Example

To validate the effectiveness of control law (2), we show an example of eight agents with a cubic target formation. The initial positions are chosen to satisfy the conditions in Theorem 6 and the parameters are set as follows: $a = 0.2$, $b = 5$, $h = 5$ and $T = 4$ s.

The initial positions and the positions at 4 s are shown in Figs. 3(a) and (b). The vertexes in different colour are the agents and the solid lines are trajectories of the agents. The dashed lines in grey and the plus sign in black represent the relative bearing and the centroid $\bar{p}$, respectively. We can observe that the centroid is invariant. Fig. 3(c) shows that the minimum distance between agents is larger than 0.5. Hence there is no collision between agents. Together with Fig. 4(a), we can see that the target formation is achieved at 4 s. Furthermore, Fig. 4(b) shows that the control inputs u_i, $\forall i = 1, \cdots, 8$ are bounded and smooth.

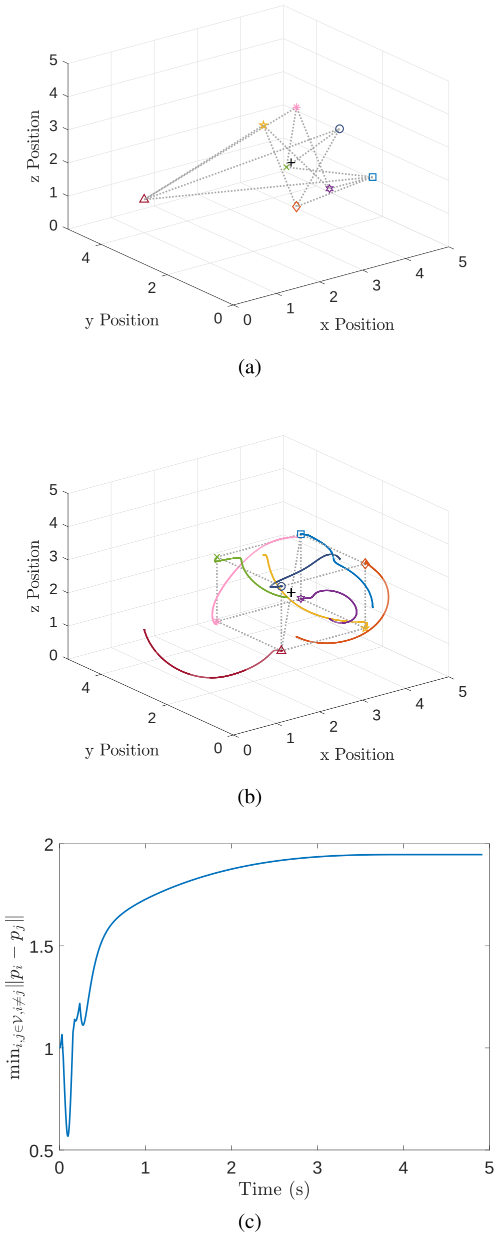


Fig. 3. Simulation results of control law (2). (a) Initial positions $p_i(0)$, $\forall i = 1, \cdots, 8$; (b) Trajectories and positions p_i at 4 s, $\forall i = 1, \cdots, 8$; (c) The minimum distance between agents $\min_{i,j\in\mathcal{V},i\neq j}\|p_i - p_j\|$.

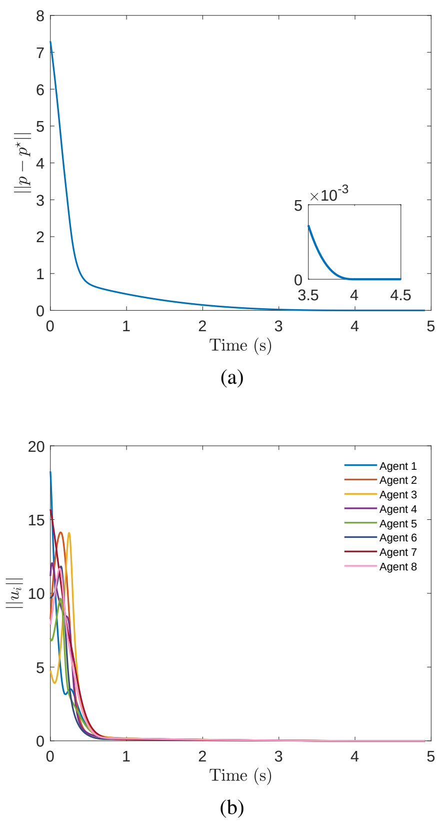


Fig. 4. Simulation results of control law (2). (a) Position error $\|p - p^*\|$; (b) The norm of control input $\|u_i\|$, $\forall i = 1, \cdots, 8$.

VI. Experiment Validation

To demonstrate the performance of control law (18), we design an experimental platform with self-fabricated mobile robots shown in Fig. 5. In this platform, a VICON motion capture system with 6 Vero X cameras are used to get the position of mobile robots. A Linux-based host computer (CPU 2.7-GHz, 4-GB RAM) is used to transfer the position data into the relative bearings, package the relative bearings into ROS (Robot Operating System) topics, and broadcast the topics through Wi-Fi. To simulate a distributed sensor network, each robot only subscribes the topics of neighbouring robots. The mobile robot is mainly composed of three levels: (1) Mona robot [33] based on arduino mini pro (designed by the University of Manchester); (2) LiPo SHIM + Raspberry Pi Zero (running control law (18) and subscribing ROS topics at 80 Hz); (3) 14 mm Pearl Markers (forming unique patterns for motion capture).

In this experiment, the target formation of six robots is given in Fig. 6 and the parameters are set as $a = 0.12, b = 0.3, h = 2$ and $T = 35$ s. It is worth noting that the noise introduced by motion capture system is inevitable and may result in an unbounded $\frac{\dot{\mu}}{\mu}\sum_{j\in\mathcal{N}_i} P_{g_{ij}} g_{ij}^*$ (with $\frac{\dot{\mu}}{\mu}$ growing unbounded, while $\sum_{j\in\mathcal{N}_i} P_{g_{ij}} g_{ij}^*$ not decaying to zero). To address this issue, inspired by [27], we set T in μ on $[t_0, t_0 + T)$ to a value $\bar{T}$ slightly larger than the user pre-specified settling time, that is,

$$\mu(t) = \frac{\bar{T}^h}{(t_0 + \bar{T} - t)^h}, \qquad t \in [t_0, t_0 + T), \tag{23}$$

where $\bar{T} = 35.4$ s $> T$. The dynamics of robots are described as unicycle [8]. To implement control law (18), we linearise the dynamics of robots by following the steps in [8] (Please refer to [8] for details).

The experiment results are shown in Fig. 7. Fig. 7(a) is plotted against an image taken by the downward-looking camera on the ceiling. Together with Fig. 7(b), we can see that the target configuration is achieved at 35 s and there is no collision between robots. Furthermore, the control inputs u_i of followers are bounded as shown in Fig. 7(c).

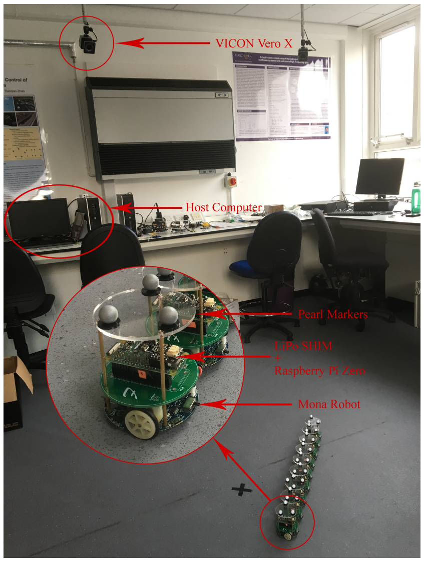


Fig. 5. The experimental platform.

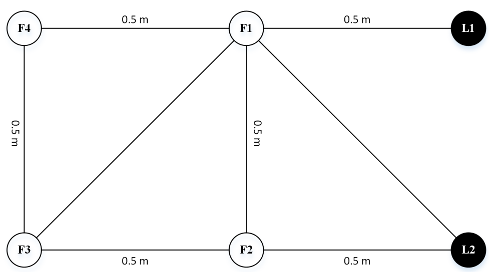
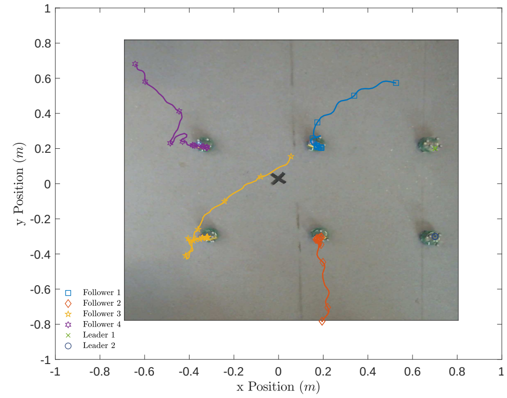


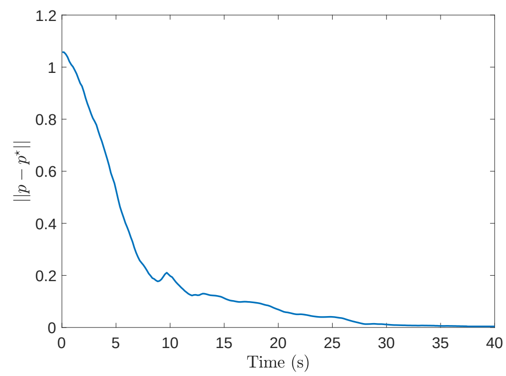
Fig. 6. The target formation with two leaders L1 and L2.

VII. Conclusion

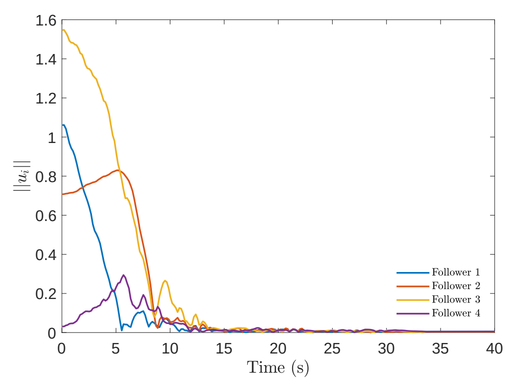
This paper proposes new bearing-only control laws to achieve target formations in finite time. The almost global convergence is guaranteed. Furthermore, the convergence time is not related to initial conditions and can be arbitrarily chosen by users. Sufficient conditions for collision avoidance are also given. Then the almost global convergence is extended to global convergence by using a leader-follower control structure. Since no signum function or fractional power feedback is used, the control action of the proposed control laws are C^1 smooth. Simulation and experimental results both demonstrate the effectiveness of our design.



(a)



(b)



(c)

Fig. 7. Experiment results of control law (18). (a) Trajectories and final configuration; (b) Position error $\|p - p^*\|$; (c) The norm of control input $\|u_i\|, \forall i \in \mathcal{V}_f$.

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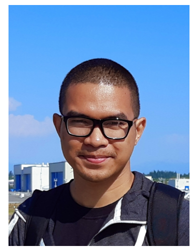

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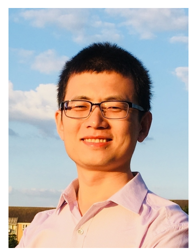

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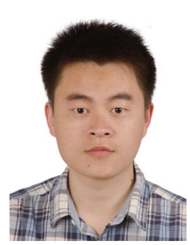

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