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On Regulated Solutions of Impulsive Differential Equations with Variable Times

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Abstract: In this paper we investigate the unified theory for solutions of differential equations without impulses and with impulses, even at variable times, allowing the presence of beating phenomena, in the space of regulated functions. One of the aims of the paper is to give sufficient conditions to ensure that a regulated solution of an impulsive problem is globally defined.

Keywords: regulated function; solution set; discontinuous function; impulsive problem with variable times

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1. Introduction

In recent years, impulse theory has been significantly developed, especially in the cases of impulsive differential equations or differential inclusions with fixed moments; see the monographs of Lakshmikantham et al. [1], Samoilenko and Perestyuk [2] and Perestyuk et al. [3] and the references therein. The study of impulsive problems with variable times presents more difficulties due to the state-dependent impulses, and in a large part of the literature, a finite number of impulses are still allowed. Some extensions to impulsive differential equations with variable times have been done by Bajo and Liz [4] and Frigon and O'Regan [5,6], and in the multivalued case, for instance, by Baier and Donchev or Gabor and Grudzka [7–9]. In the case of impulses at variable times, a "beating phenomenon" may occur, i.e., a solution of the differential equation may hit a given barrier several times (including infinitely many times). Then we will be in the presence of "pulse accumulation" whenever a solution has an infinite number of pulses which accumulate to a finite time t^*. Impulsive differential equations or inclusions have applications in physics, engineering or biology where discontinuities, which can be seen as impulses, occur [3,10]. In this paper we consider a class of initial value problems (IVPs) for differential equations with impulses at variable times on $[a,b]$, allowing pulse accumulation:

$$\begin{cases} x'(t) = f(t, x(t)), & t \notin \tau(x) \\ x(a) = x_0, & \\ x(t) - x(t^-) = I_l(x(t-)), & t \in \tau(x) \\ x(t^+) - x(t) = I_r(x(t)), & t \in \tau(x) \end{cases}$$

where $f : [a,b] \times \mathbb{R} \to \mathbb{R}$, if not otherwise stated, is a continuous function; $\tau(x) \subset [a,b]$ is at most countable; and I_r, $I_l : \mathbb{R} \to \mathbb{R}$ and $x_0 \in \mathbb{R}$. Our consideration is presented for single-valued problems, but it is still valid for multivalued problems, as can be observed in [3,11], eventually by using multivalued integration [12–14].

Note that for a given function x the set $\tau(x)$ need not be a singleton. We study the case of accumulation points for the set $\tau(x)$. For an interesting discussion in this topic; see [15], where necessary and sufficient conditions are given to assure pulse accumulation. For problems having more than one common point of a solution and a barrier sufficient conditions are described in [16] (Theorem 4) or [1–3,17].

In this paper we study impulsive IVPs in the space $G([a,b])$ of regulated functions, which seems to be the natural space of solutions for impulsive problems (see [18–20]), and we investigate properties of solutions as elements of this space. This allows us to cover and extend earlier approaches. Note that usual IVPs should be treated as impulsive problems with negligible jumps. In this case the space $C([a,b])$ or $C^1([a,b])$ are considered, and they are subspaces of $G([a,b])$. We should note that impulsive differential equations with varying times of impulses are treated in [21] (Section 5) as generalized ordinary differential equations, but accumulation points for the set of discontinuity points are not allowed and solutions are functions of bounded variation. In [22] BV solutions are expected for impulsive problems. This approach was initiated by Silva and Vinter for the study of optimality problems driven by impulsive controls, but this space is not a proper choice in our study, as we need to consider only operators preserving bounded variation of functions and the norm in $BV([a,b])$ is not directly related to the supremum norm in $C([a,b])$. One of our goals is to unify the study for impulsive and non-impulsive problems. In the literature, IVPs with impulses at finite and fixed times have been studied in the subspace $PC([a,b],t_1,t_2,...,t_k)$ of the space $PC([a,b])$ of piecewise continuous functions, so that the space of solutions depends on times of jumps. In [23,24] the case of finite number of jumps is considered and the space of solutions is independent on times of jumps. In case of impulsive problems with variable times of jumps (state dependent jumps), a new space $CJ_k([a,b])$ is considered in [8,9,11] (for multivalued problems); it is a good choice for problems having the property that every solution has exactly k jumps; still, the space of solution depends on the choice of impulsive problem. We generalize previous approaches; indeed we have (some inclusions are taken in the sense of isometric copies)

$$C^1([a,b]) \subset C([a,b]) \subset PC([a,b],t_1,...,t_k) \subset CJ_k([a,b]) \subset PC([a,b]) \subset G([a,b]).$$

One of the advantages is that we are able to cover the case of beating phenomenon, till now studied separately and in very particular cases.

The paper is organized as follows. In Section 2 we recall basic notions on the space $G([a,b])$, and introduce, as space of solutions, the subspace Z_{G^L} of regulated functions which admit only left accumulation points and have a canonical decomposition. We consider impulsive IVPs and provide conditions on the barriers which guarantee that solutions are global. In particular, condition [B4] requires that the sum of jumps (left and right) is finite and this condition implies that any solution is continuable to the point b. In Section 3 we give the equivalent representation of impulsive IVPs by means of operators acting on the space of regulated functions, and in the remaining part of the section we provide sufficient conditions for [B4]. An example is given in Section 4. Finally, in Section 5 we compare our results with earlier ones.

2. Impulsive Problems, Regulated Functions and Barriers

We denote by $G([a,b])$ the space of all real-valued regulated functions x defined on the interval $[a,b]$; that is, $G([a,b])$ is the set of all $x : [a,b] \to \mathbb{R}$ such that there exist finite the right $x(t^+)$ and left $x(s^-)$ limits for every points $t \in [a,b)$ and $s \in (a,b]$. The space $G([a,b])$ is a Banach space when equipped with the supremum norm (see [25]). The space $C([a,b])$ of continuous functions and the space $BV([a,b])$ of functions of bounded variation on $[a,b]$ are proper subspaces of $G([a,b])$, so on $BV([a,b])$ the induced norm is considered. Every regulated function is bounded, has a countable set of discontinuities and is the limit of a uniformly convergent sequence of step functions (cf. [26]). Given a regulated function $x \in G([a,b])$ we denote its set of discontinuity points by $\tau(x)$; if necessary, we distinguish the points of left-discontinuity $\tau_L(x)$ and right-discontinuity $\tau_R(x)$.

The following result, being an immediate consequence of a result by Bajo [15] (Theorem 1), implies that we need to restrict ourselves to some subspaces of regulated functions. Some necessary properties of solutions are described in the lemma below. We focus our attention on the subspace of regulated functions, denoted by $G^L([a,b])$, of all $x \in G([a,b])$, for which $\tau(x)$ has at most a finite number of left accumulation points (see [B2] for a more precise formulation).

Lemma 1. *If $t^* \in [a,b]$ is an accumulation point for the set of discontinuity points $\tau(x)$ of a regulated function $x : [a,b] \to \mathbb{R}$, then the size of the jumps is convergent to 0 when $t_n \to t^*$; i.e.,*

$$\lim_{t\in\tau(x),t\to t^*-} |x(t) - x(t^*-)| = 0 \quad \text{and} \quad \lim_{t\in\tau(x),t\to t^*+} |x(t^*+) - x(t)| = 0.$$

Now for $x \in G^L([a,b])$, we denote the left and right jump functions, respectively, by

$$J_L(x)(t) = x(t) - x(t-) \quad \text{and} \quad J_R(x)(t) = x(t+) - x(t),$$

for $t \in [a,b]$, where $x(a^-) = x(a)$ and $x(b^+) = x(b)$. Moreover for $t \in [a,b]$ we define

$$H_L(x)(t) = \sum_{t_k\in\tau_L(x),a\ \le t_k\le t} J_L(x)(t_k)$$

and

$$H_R(x)(t) = \sum_{t_k\in\tau_R(x),a\ \le t_k< t} J_R(x)(t_k)$$

with $H_R(x)(a) = 0$. In the case of a finite number of left accumulation points it is understood that we will calculate the sum of the series of jumps separately for each such a point. Thus, we allow for conditional convergence of series as well. The key point of the paper is to decompose such a class of regulated functions as a sum of continuous and steplike functions (cf. [27]). Denote by Z_{G^L} the subspace of $G^L([a,b])$ consisting of regulated functions for which the sums $H_L(x)(t)$ and $H_R(x)(t)$ are finite for each $t \in [a,b]$. Then a function $x \in Z_{G^L}$ can be uniquely written as the sum of a continuous function and a steplike function.

The functions $x_d,\ x_c :\ [a,b] \to \mathbb{R}$ defined by setting

$$x_d(t) = H_L(x)(t) + H_R(x)(t)$$

and

$$x_c(t) = x(t) - x_d(t)$$

for $t \in [a,b]$ are called discrete and continuous parts of x. We will refer to $x = x_d + x_c$ as to the canonical decomposition of x; such a decomposition is unique with $x_d(a) = x(a)$ (cf. also [28] (Theorem 3)). We observe that for $t \in \tau(x)$ we have $J_L(x)(t) = x_d(t) - x_d(t-)$ and $x_c(t) - x_c(t-) = 0$, and analogously $J_R(x)(t) = x_d(t+) - x_d(t)$ and $x_c(t+) - x_c(t) = 0$, and

$$-\infty < \sum_{t_k\in\tau_L(x),a\ \le t_k\le b} J_L(x)(t_k) + \sum_{t_k\in\tau_R(x),a\ \le t_k< b} J_R(x)(t_k) < \infty.$$

Moreover all functions $x \in Z_{G^L}$ are characterized by the condition that $J_L(x), J_R(x) \in l_1([a,b])$.

The spaces $C([a,b])$ and $BV([a,b])$ both are subspaces of Z_{G^L}. Moreover, also the space $CJ_k([a,b])$ is a subspace of Z_{G^L}. Let us stress that the function x_d is of bounded variation, but x_c need not have this property. For the sake of completeness we have to recall that a decomposition is possible for any function $x \in G([a,b])$, but without uniqueness (see [27–29]).

Let us consider the IVP for differential equations with impulses at variable times on $[a, b]$

$$\begin{cases} x'(t) = f(t, x(t)), & t \notin \tau(x) \\ x(a) = x_0, & \\ x(t) - x(t^-) = I_l(x(t-)), & t \in \tau(x) \\ x(t^+) - x(t) = I_r(x(t)), & t \in \tau(x), \end{cases} \tag{1}$$

where $f : [a, b] \times \mathbb{R} \to \mathbb{R}$, $\tau(x) \subset [a, b]$ is at most countable and I_r, $I_l : \mathbb{R} \to \mathbb{R}$, and $x_0 \in \mathbb{R}$. Here I_r and I_l describe right and left jumps when $x(t)$ "touch" the barrier τ; i.e., $t \in \tau(x)$. If we expect one-side continuous solutions (cádlàg functions, for instance), then I_l or I_r should be trivial.

As a barrier we will understand a curve of the plane $\tau = \{(t, x) : t = \alpha(s), x = \beta(s), s \in \mathbb{R}\}$ or simply the graph of an equation $x = \gamma(t)$ for $t \in [a, b]$. Therefore, $\tau(x) = \{t \in [a, b] : \; x(t-) \in \tau\}$, and the functions I_r and I_l describe, respectively, right and left jumps of a solution $x(t)$ in the point $t \in [a, b]$ for which $x(t-)$ "touches" the barrier τ.

Throughout, we will consider the following conditions:

[B1] The point $(a, x_0) \notin \tau$.

[B2] If the set $\tau(x)$, for a solution x of (1), is not finite, then $\tau(x)$ has at most a finite number of accumulation points. For any accumulation point t^* of $\tau(x)$ there is an increasing sequence $\{t_k\}_{k\in\mathbb{N}}$ in $\tau(x)$ such that $t_k \to t^*$ and $t \notin \tau(x)$ whenever $t \in (t_k, t_{k+1})$.

[B3] In case of presence of more than one barrier (or connected components of the barrier) τ_k, they should be disjoint sets on a plane ($\tau_k \cap \tau_j = \emptyset$ for $k \neq j$). These barriers will be always assumed to be piecewise continuous curves.

[B4] For any accumulation point t^* of $\tau(x)$ the jump functions I_r, I_l have locally bounded sums of jumps in t^*; i.e.,

$$-\infty < \sum_{t_k\in\tau(x), a \le t_k < t^*} I_l(x)(t_k) + \sum_{t_k\in\tau(x), a \le t_k < t^*} I_r(x)(t_k) < \infty. \tag{2}$$

Moreover, either τ is bounded or if a solution x has the property that $x(t_k) \to \infty$ for some $t_k \in \tau(x)$, $k = 1, 2...$, then (2) holds with the sums taking over k.

Conditions [B1]–[B4] allow one to cover existing cases and to study the problem of the solvability of the impulsive differential equation in presence of the beating phenomenon. The first three assumptions are quite natural and are usually assumed in earlier papers. In particular, [B1] implies that we have always a time $t_1 > a$ such that $x(t-)$, for $t \in [a, t_1)$, does not touch the barrier. This enables us to propose a step-by-step procedure for $t_1 < t_2 < ...$ at least to the first accumulation point of $\tau(x)$. We observe that [B1] can be relaxed, if a is a point of discontinuity, then it should be isolated in $\tau(x)$ and we need to replace the initial condition $x(a) = x_0$ by $x(a+) = x_0$. In the sequel we are interested in obtaining sufficient conditions for [B4]; we point out that condition [B4] implies that any solution is continuable to the point b. In case of more than one barrier (or connected components of the barrier), it may happen than the jump functions can transfer points between them. Let us recall that we have two jump conditions and then when $x(t_1-) \in \tau_1$ we have the first left jump. Thus, if after the jump $x(t_1) \in \tau_2$, it is still not a reason to get again the new left jump (as $x(t_1-) \notin \tau_2$). Only the right jump occurs and $x(t_1+)$ is calculated as $x(t_1+) = x(t_1) + I_r(x(t_1))$. As we assume that a couple of actions for τ_1 is always required, it is the jump function associated with the first barrier τ_1, a trajectory continues with the new initial value condition $x(t_1+)$; i.e., the mapping does not jump twice or more than once at the same moment. Condition [B3] guarantees that any solution of (1) does not jump more than once at the same moment.

Definition 1. *A function $x \in G^L([a, b])$ is said to be a regulated solution of the impulsive IVP (1) if it is differentiable except at most countable set $\tau(x) = \{t_k : k \in \mathbb{N}\}$. Moreover, if $a \notin \tau(x)$, then x coincides with the interval $[a, t_1)$, where $t_1 = \min \tau(x)$, with the solution of the differential equation $z'(t) = f(t, z(t))$ with initial condition $z(0) = x_0$, and x coincides with the interval (t_k, t_{k+1}) with the solution of the differential* *equation $z'(t) = f(t, z(t))$ with initial condition $z(t_k) = x(t_k+)$, and the function x satisfies, at the points of the set $\tau(x)$, jump conditions with functions I_l and I_r, respectively.*

Remark 1. *If we expect only that $x \in AC((t_k, t_{k+1}))$ for $k \in \mathbb{N}$, i.e., differentiability a.e. on such intervals, then the above definition can be also considered (the Carathéodory case instead of continuous functions f). In the case of lack of jumps (i.e., for all x we get $\tau(x) = \emptyset$) we have C^1-solutions. In the case of the connected components of the barrier in the form of vertical lines $\tau(x) = \{t_1, t_2, ..., t_k\}$ for any x, we have piecewise continuous solutions. For the case of CJ_k-solutions we need to identify such solutions with regulated solutions with precisely k barriers, each of them describing exactly one point t_k, i.e., $\tau_k(x) = t_k$. Let us mention that even solutions being of bounded variation considered in some papers are also included in our class of regulated solutions.*

We look for regulated solutions globally defined on $[a, b]$. Let us consider the IVP of the ODE associated with (1)

$$\begin{cases} x'(t) = f(t, x(t)) \\ x(a) = x_0. \end{cases} \tag{3}$$

If for a given solution of the impulsive IVP (1) we have only a finite number of discontinuity points, then the solution is global iff the solution of the IVP (3) is so, and thus usual assumptions guaranteeing globality of solutions are sufficient for impulsive problems too. The case of countable number of discontinuity points for some solutions is more complicated. Indeed, as claimed in [3] (p. 9), it is not true that if a solution of the IVP (3) cannot be extended to some interval, then a solution of the impulsive IVP (1) cannot also be extended to the same interval. We will show that it depends rather on the barrier and jump functions than on the solution of the impulsive IVP. So it is important to have combined assumptions for the barrier and jump functions. Note that also the growth of the function f is important. Let us discuss the following example, modified from [17] (Example 3.1).

Example 1. *Consider the following (IVP) problem in $[0, 2\pi]$.*

$$\begin{cases} & x'(t) &=& 1 & t \notin \tau(x) \\ & x(0) &=& -\pi & \\ x(t^+)- & x(t) &=& I_r(x)(t) & t \in \tau(x)\,, \\ x(t)- & x(t^-) &=& I_l(x)(t) & t \in \tau(x), \end{cases}$$

where $\tau(x) = \arctan(x) + \pi$, $I_r(x)(t) \equiv 1$ and $I_l(x)(t) \equiv 0$. Clearly, the Cauchy problem $x'(t) = 1$, $x(0) = -\pi$ has unique solution $x(t) = t - \pi$ defined globally on $[0, 2\pi]$. This solution touches the barrier, for the first time for $t_1 = \pi$, so a jump occurs and we get $x(\pi+) = 1$ and the solution of the IVP is defined as $x(t) = t - \pi + 1$ up to the next point when its trajectory touch the barrier, say t_2. We can proceed with points t_k and we get $\lim_{k\to\infty} t_k = \frac{3\pi}{2}$, so we have an accumulation point for $\tau(x)$ and the solution of the IVP is not defined globally on $[0, 2\pi]$, despite the fact that Cauchy problem has a global solution.

Now consider the same problem with $I_r(x)(t) = \frac{1}{(x(t))^2+1}$. In this case we have the same solution on $[0, t_1]$ and even the first jump is the same and the next jumps are: $x(t_2+) - x(t_2) = \frac{1}{(x(t_2))^2+1}$, etc. We can also easily calculate the points t_k and we get $\sum_{k=1}^{\infty} [x(t_k+) - x(t_k)] = M < \infty$ and as $t_k \to T < \frac{3\pi}{2}$ and we can put $x(t) = x + M - T$ for $t \in [T, 2\pi]$. We still get a global solution for the impulsive problem.

3. Integral Form of Impulsive Problems

We will study impulsive problem (1), representing it by means of operators acting on the space of regulated functions. To this end, let us consider the operator F defined on the space $G^L([a, b])$ in the following way:

$$F(x)(t) = x_0 + \int_a^t f(s, x(s))\, ds + \sum_{t_k \in \tau_L(x), a\ \le t_k \le t} I_l(x)(t_k-) + \sum_{t_k \in \tau_R(x), a\ \le t_k < t} I_r(x)(t_k). \tag{4}$$

Notice that for $x \in Z_{G^L}, t \in [a,b]$, we have

$$\sum_{t_k \in \tau_L(x), a \ \leq t_k \leq t} I_l(x)(t_k-) = \sum_{a \leq s \leq t} I_l(x_d(s-)) \quad \text{and} \quad \sum_{t_k \in \tau_R(x), a \ \leq t_k < t} I_r(x)(t_k) = \sum_{a \leq s < t} I_r(x_d(s)).$$

The discrete part $F_d(x)$ of the operator F, which will depend only on x_d, has to preserve the finiteness of sums of jumps, whenever x_d has this property. This condition depends on the barrier and jump functions I_r, I_l. In case of pulse accumulation, their acting on barriers should decrease jumps and the corresponding conditions for jump functions should compensate possible divergence, so in the presence of pulse accumulation they should be rapidly decreasing in the neighborhood of such a point. We allow one to have a finite number of such points, and we will present some sufficient conditions guaranteeing that even in this case all solutions are global. In case of finite number of jumps there are no new restrictions. Let us observe that for any discontinuity point $t \in \tau(x)$ we have direct dependence of the values of both $x(t_k)$ and $x(t_k+)$ on the value $x(t_k-)$, so they also depend on the barrier τ considered in (1); indeed:

$$x(t_k+) = x(t_k) + I_r(x(t_k)) = x(t_k-) + I_l(x(t_k-)) + I_r\left[x(t_k-) + I_l(x(t_k-))\right]. \tag{5}$$

We will investigate operators on Z_{G^L} of the following form:

$$F(x)(t) = x_0 + \int_a^t f(s, x(s))\, ds + \sum_{a \leq s \leq t} I_l(x_d(s-)) + \sum_{a \leq s < t} I_r(x_d(s)). \tag{6}$$

We need to check the existence of the integral, the convergence of discrete parts and that this decomposition is canonical. Some differentiability properties of x outside of $\tau(x)$ and finite limits on $\tau(x)$ are also necessary to be solutions of (1).

Proposition 1. *Assume that the conditions [B1]–[B3] hold true and that*

(F1) $f \in C([a,b] \times \mathbb{R})$;

(J1) *for any* $x \in Z_{G^L}$ *and* $t \in [a,b]$

$$-\infty < \sum_{a \leq s \leq t} I_l(x_d(s-)) + \sum_{a \leq s < t} I_r(x_d(s)) < \infty.$$

Then F, defined in (6), maps Z_{G^L} into itself. Moreover, the operator F has the unique canonical decomposition $F(x) = F_c(x) + F_d(x)$, with

$$F_c(x)(t) = x_0 + \int_a^t f(s, x(s))\, ds$$

and

$$F_d(x)(t) = \sum_{a \leq s \leq t} I_l(x_d(s-)) + \sum_{a \leq s < t} I_r(x_d(s)),$$

so $F_c(x)$ is the continuous part of $F(x)$ and $F_d(x)$ is its discrete part.

Proof. Let us recall that if $f \in C([a,b] \times \mathbb{R})$, the superposition operator $N_f(x)(t) = f(t, x(t))$ maps $G^L([a,b])$ into itself (cf. [30] (Theorem 3.1) and [31]). Hence, the operator F_c is well-defined and $F_c(x) \in C([a,b])$. Assumption (J1) implies that $F_d : Z_{G^L} \to Z_{G^L}$; since $F_c : Z_{G^L} \to C([a,b])$, we have that F maps Z_{G^L} into itself. Let $x \in Z_{G^L}$ and decompose $F(x)$ canonically as $y_c + y_d$. We need to prove that $y_c = F_c(x)$ and $y_d = F_d(x)$. First we investigate the discrete part. As no jump occurs, due to [B1], at the point a we have $y_d(a) = 0 = F_d(x)(a)$. Clearly, both functions y_d and $F_d(x)$ should have exactly the same points of discontinuity. Thus, for $t \in [a, t_1)$ both are null functions. As $y(t_1-) = J_L(y)(t_1) = F_d(x)(t_1-)$ and $y(t_1+) = J_R(y)(t) = F_d(x)(t_1+)$ we get the same jumps at $t = t_1$, so the values $y(t_1)$ and $F_d(x)(t_1)$ are the same. Thus, the left limits at the next point of discontinuity, say t_2, are the same (both are equal to the right limits at t_1). Due to our assumption on the set of discontinuity points for x we can proceed until the endpoint of existence of both functions, so that $y_d = F_d(x)$. Then, $y_c = F(x) - y_d = F(x) - F_d(x) = F_c(x)$. □

It is important to provide a sufficient condition to check the assumption (J1) occurs (cf. also [B4]). Let us observe that we need to verify only the convergence of jumps at accumulation points t^* of sets $\tau(x)$. For an interesting discussion about the presence or absence of such points, see [15] or [32]. For a given solution function x, if the set $\tau(x)$ has no accumulation points and the barrier and jump functions are bounded, then it can be defined on a whole interval (global solutions) (cf. example in [15] (Remark)). If we allow it to have some accumulation points, the problem is much more complicated. We need to find some conditions ensuring that all solutions pass through the accumulation points of $\tau(x)$, so they are global and can be prolonged up to the point b (see [16,33], for instance). As the problem in a whole generality is very hard to be described, we restrict ourselves to one non-trivial jump function and to the barrier defined as the graph of a continuous function.

Example 2. *Let $f(t,x) = \frac{1}{\cos^2 t}$ for $0 \le t < \frac{\pi}{2}$ and $f(t,x) = 0$ for $t \ge \frac{\pi}{2}$. Consider the following problem: $x'(t) = f(t,x)$, $x(0) = 0$, $I_l(u) = -1$, $I_r(u) = 0$ and $\gamma(t) \equiv 1$. It is easy to see that this problem has a unique solution x defined on $[0,\infty)$ with $\tau(x) = \arctan(\mathbb{N})$. Clearly, $\tau(x)$ has a left dense accumulation point $t = \frac{\pi}{2}$. Despite that γ and x are bounded and defined for all $t \ge 0$, the assumption [B4] is not satisfied and $x \in G^L([0,\frac{\pi}{2})) \setminus Z_{G^L}$ and $x \notin G^L([0,\frac{\pi}{2}])$.*

Let us present some extensions for [15] (Theorem 2) and (Corollary 1).

Proposition 2. *Let $f \in C([a,b]\times\mathbb{R})$, $\gamma : [a,b] \to \mathbb{R}$ be a continuous function, the barrier τ be the graph of $x = \gamma(t)$ and $I_l \in C(\mathbb{R},\mathbb{R})$ be associated with γ. Let $t^* \in (a,b]$ and let x be a regulated solution of the problem (1) such that the point t^* is a left accumulation point for the set $\tau(x)$. Assume that the following conditions hold:*

1. *There exists a positive constant M such that $|f(t,x)| \le M$ for all $t \in [a,b]$ and $x \in Z_{G^L}$;*
2. *The barrier τ satisfies [B1]–[B3];*
3. *γ is nonincreasing on the interval (t^*-c, t^*) for some $c > 0$;*
4. *I_l is nondecreasing and $I_l(u) < 0$ for $u \in (\gamma(t_1), \gamma(t^*))$ and some $t_1 \in (t^*-c, t^*)$.*

Then $\sum_{a\le s\le t^*} -I_l(x_d(s-)) < \infty$ and x can be extended to the right of t^*, [B4] holds true, and so any solution of the problem belongs to Z_{G^L}.

Proof. Let x be a regulated solution of the impulsive problem (1) for which t^* is a left accumulation point of $\tau(x)$. Set $u^* = \gamma(t^*)$; then, due to the continuity of γ, the point $(t^*,u^*) \in \tau$. Let (t_k) be a sequence in $[a,t^*)$ convergent to t^*. Without loss of generality, we may assume that $t_1 > t^* - c$, so γ is nonicreasing on (t_1,t^*). Fix an arbitrary regulated solution x of the impulsive problem (1). Fix $k \in \mathbb{N}$. Denote $u_k = x(t_k-) = \gamma(t_k)$. Then $(t_k,u_k) \in \tau$. We can estimate the position of the next point. Consider the system of equations: $x = \gamma(t)$ and $x = M\cdot t + u_k + I_l(u_k) - M\cdot t_k$ and denote by t^*_{k+1} the first solution to right of t_k. Moreover, as $|x'(t)| = |f(t,x(t)| \le M$ for $t \notin \tau(x)$, we also have $t^*_{k+1} \le t_{k+1}$. Since t^*_{k+1} is a solution of the equation $M\cdot t + u_k + I_l(u_k) - M\cdot t_k = \gamma(t)$, using the fact that γ is nonincreasing, we have

$$M\cdot t_{k+1} + u_k - (-I_l(u_k)) - M\cdot t_k \ge M\cdot t^*_{k+1} + u_k - (-I_l(u_k)) - M\cdot t_k = \gamma(t^*_{k+1}) \ge \gamma(t_{k+1}) = u_{k+1}.$$

From the latter we deduce

$$M\cdot(t_{k+1}-t_k) - (u_{k+1}-u_k) \ge -I_l(u_k) > 0.$$

Thus, for any $N \geq 1$, we have

$$\sum_{k=1}^{N}(-I_l(u_k)) \leq M \cdot \sum_{k=1}^{N}(t_{k+1}-t_k) - \sum_{k=1}^{N}(u_{k+1}-u_k),$$

and passing to the limit, we obtain

$$\begin{aligned}\sum_{k=1}^{\infty}-I_l(u_k) &= \sum_{k=1}^{\infty}-I_l(x(t_k-)) = \sum_{a\leq s\leq t}-I_l(x_d(s-)) \\ &\leq \lim_{N\to\infty}\left(M\cdot\sum_{k=1}^{N}(t_{k+1}-t_k) - \sum_{k=1}^{N}(u_{k+1}-u_k)\right) \\ &= \lim_{N\to\infty} M\cdot(t_{N+1}-t_1) - \lim_{N\to\infty}(u_{N+1}-u_1) = M\cdot(t^*-t_1)+(u_1-u^*) < \infty.\end{aligned}$$

□

The analogy of Proposition 2 holds when γ is nondecreasing.

Proposition 3. *Let $f \in C([a,b]\times\mathbb{R})$, $\gamma : [a,b] \to \mathbb{R}$ be a continuous function, the barrier τ be a the graph of $x = \gamma(t)$ and $I_l \in C(\mathbb{R},\mathbb{R})$ be associated with γ. Let $t^* \in (a,b]$ and let x be a regulated solution of the problem (1) such that the point t^* is a left accumulation point for the set $\tau(x)$. Assume that conditions 1 and 2 of Proposition 2 hold true and also:*

3′. *γ is nondecreasing on the interval (t^*-c,t^*) for some $c>0$;*
4′. *I_l is nonincreasing and $I_l(u) > 0$ for $u \in (\gamma(t_1),\gamma(t^*))$ and some $t_1 \in (t^*-c,t^*)$.*

Then $\sum_{a\leq s\leq t^*} I_l(x_d(s-)) < \infty$ and x can be extended to the right of t^*, [B4] holds true and so any solution of the problem belongs to Z_{G^L}.

Proof. In this case we have an equation $x = -M\cdot t + u_k + I_l(u_k) + M\cdot t_k$, and if t^*_{k+1} denotes a solution of the equation $-M\cdot t + u_k + I_l(u_k) + M\cdot t_k = \gamma(t)$, then

$$M\cdot(t_{k+1}-t_k) + (u_{k+1}-u_k) \geq I_l(u_k).$$

Arguing as above we obtain

$$\sum_{k=1}^{\infty} I_l(x(t_k-)) = \sum_{a\leq s\leq t} I_l(x_d(s-)) = \sum_{k=1}^{\infty} I_l(u_k) < \infty.$$

□

In view of (5) we can formulate similar sufficient conditions considering both left and right jump functions.

Theorem 3.1. *Let $f \in C([a,b]\times\mathbb{R})$, $\gamma : [a,b] \to \mathbb{R}$ be a continuous function, the barrier τ be the graph of $x = \gamma(t)$ and $I_l,\ I_r \in C(\mathbb{R},\mathbb{R})$. Let $t^* \in (a,b]$ and let x be a regulated solution of the problem (1) such that the point t^* is a left accumulation point for the set $\tau(x)$. Assume that the following conditions hold:*

1. *There exists a positive constant M such that $|f(t,x)| \leq M$ for all $t \in [a,b]$ and $x \in Z_{G^L}$;*
2. *The barrier τ satisfies [B1]–[B3];*
3. *γ is nonincreasing on the interval (t^*-c,t^*) for some $c>0$;*
4. *I_l and I_r are nondecreasing and $I_l(u) < 0$, $I_r(u) < 0$ for $u \in (\gamma(t_1),\gamma(t^*))$ and some $t_1 \in (t^*-c,t^*)$.*

Then (J1) holds true; i.e., $-\infty < \sum_{a\leq s\leq t} I_l(x_d(s-)) + \sum_{a\leq s<t} I_r(x_d(s)) < \infty$, and x can be extended to the right of t^*.

Proof. We consider the affine function:

$$x = M \cdot t + u_k + I_l(u_k) + I_r(u_k + I_l(u_k)) - M \cdot t_k$$

and we get similar estimation as in Proposition 2,

$$u_{k+1} = \gamma(t_{k+1}) \leq \gamma(t^*_{k+1}) = M \cdot t^*_{k+1} + u_k + I_l(u_k) + I_r(u_k + I_l(u_k)) - M \cdot t_k.$$

As $I_l(u_k) < 0$, then $u_k + I_l(u_k) < u_k$. Thus

$$-I_l(x(t_k-)) + I_r(x(t_k)) \leq M(t_{k+1} - t_k) + (u_k - u_{k+1}).$$

The convergence of the series can be deduced as previously. □

Remark 2. *An analogous result of the previous Theorem can be obtained considering hypotheses* (3′) *and* (4′) *of Proposition 3.*

Corollary 1. *Under the assumptions of Proposition 3.1 there exists constant A such that all solutions x of the IVP (1) have equi-bounded sums of jumps:*

$$\sum_{a \leq s \leq t} |I_l(x_d(s-))| + \sum_{a \leq s < t} |I_r(x_d(s))| \leq A.$$

Proof. We restrict ourselves to proving the result in the case of left jumps. Put $a_k = \gamma(t^*_k)$, where (t^*_k) is the sequence constructed in Proposition 2, and let $A = \sum_{k=1}^{\infty} a_k$. Observe that for any solution x points of jumps $t_k \geq t^*_k$, so by the property of γ we get $\gamma(t^*_k) \geq \gamma(t_k)$ and then $I_l(\gamma(t^*_k)) \geq I_l(\gamma(t_k))$. For any x we get $\sum_{a \leq s \leq t} I_l(x_d(s-)) = \sum_{k=1}^{\infty} I_l(u_k) \leq \sum_{k=1}^{\infty} a_k = A < \infty$. □

Finally, we show that existence of solutions of IVP (1) is equivalent to existence of fixed points of operator F defined in (6) that are solutions of the following integral equation:

$$x(t) = x_0 + \int_a^t f(s, x(s))\, ds + \sum_{a \leq s \leq t} I_l(x(s-)) + \sum_{a \leq s < t} I_r(x(s)). \tag{7}$$

Theorem 3.2. *Assume that the conditions* [B1]–[B3] *hold true and conditions (F1) and (J1) are satisfied. Then a function $x : [a, b] \to \mathbb{R}$ is a regulated solution of problem (1) on $[a, b]$ if and only if it is a fixed point of the operator F given by (6), i.e., a regulated solution of the integral Equation (7).*

Proof. (⇐) Let x be a solution of (7). Due to Proposition 1 we know that it belongs to $Z_{G^L} \subset Z_G \subset G([a, b])$ and has a decomposition into a continuous part $x_0 + \int_a^t f(s, x(s))\, ds$ and a discrete part $\sum_{a \leq s \leq t} I_l(x(s-)) + \sum_{a \leq s < t} I_r(x(s))$.

Immediately, we get that x satisfies the initial condition. Let $t \in [a, b]$ be a point of continuity, i.e., $t \notin \tau(x)$. Then $x'(t) = (\int_a^t f(s, x(s))\, ds)' = f(t, x(t))$ so the differential equation is satisfied at such a point t. Now, let $t \in \tau(x)$. Let us calculate the jumps at this point. We have

$$\begin{aligned} x(t) - x(t-) &= x_0 + \int_a^t f(s, x(s))\, ds + \sum_{a \leq s \leq t} I_l(x_d(s-)) + \sum_{a \leq s < t} I_r(x_d(s)) \\ &- \left[x_0 + \int_a^t f(s, x(s))\, ds + \sum_{a \leq s < t} I_l(x_d(s-)) + \sum_{a \leq s < t} I_r(x_d(s)) \right] \\ &= I_l(x(t-)), \end{aligned}$$

so the jump is precisely described by the function I_l. For the right jump we have similar calculation, so that $x(t+) - x(t) = I_r(x(t))$.

($\Rightarrow$) Let x be a regulated solution of the problem (1). As the superposition $f(\cdot, x(\cdot))$ is again regulated (cf. Proposition 1), it is an integrable function. Then if $t \in [a, b]$ is a point of continuity, we get $(\int_a^t f(s, x(s))\, ds)' = x'(t)$.

Since its left and right jumps at the points $t \in \tau(x)$ are described by jump functions $I_l(x(t))$ and $I_r(x(t))$, respectively, then by the definition of the discrete part, x_d is a sum of jumps, so $x_d(t) = \sum_{a \le s \le t} I_l(x_d(s-)) + \sum_{a \le s < t} I_r(x_d(s))$ and finally $x(t) = x_c(t) + x_d(t) = F_c(x)(t) + F_d(x)(t)$. □

Now, let us present some consequences of our approach to the theory of differential inclusions. We will restrict our attention to the case of impulsive differential inclusions considered, for example, in [34] or [10] (cf. also [8,9]):

$$\begin{cases} x'(t) \in F(t, x(t)), & t \notin \tau(x) \\ x(0) = x_0, & \\ x(t) - x(t^-) = I_l(x(t)), & t \in \tau(x) \\ x(t^+) = x(t), & t \in \tau(x) \end{cases} \tag{8}$$

where $F : [0, 1] \times \mathbb{R}^d \to \mathcal{P}_{ck}(\mathbb{R}^d)$ is a multifunction with compact non-necessarily convex values in a real Euclidean space. In order to draw the readers' attention especially to new aspects of the paper, and not to focus their attention on the concepts of multi-valued analysis, let us refer them to [34] for definitions from multivalued analysis which will be used here after. In our evidence, we will only focus on the application of the previously obtained results, and the remaining details can be found in the literature.

We need to recall that in [34] the jump condition is of the form

$$\Delta x|_{t=\tau_i(x)} = S_i(x), \quad i = 1, \ldots, p,\ x(t) \in \mathbb{R}^d. \tag{9}$$

By an R-solution we mean an absolutely continuous function on each (τ_i, τ_{i+1}) for $i = 0, 1, \ldots, p, p+1$ ($\tau_0 = 0$ and $\tau_{p+1} = 1$) with impulses $\Delta x|_{t=\tau_i(x)} = S_i(x(\tau_i(x)^-))$; i.e., $x(\tau_i(x)^+) = x(\tau_i(x)^-) + S_i(x(\tau_i(x)^-))$, which satisfy $x'(t) \in F(t, x(t))$, $x(0) = x_0$ with $t \neq \tau_i(x)$ and (9).

The definition of R-solutions is more general than continuous or piecewise continuous solutions, but still it is more restrictive than ours. Consequently, we are ready to prove some results under less restrictive assumptions. Indeed, from our point of view, the most restrictive assumptions are those relating to barriers (cf. [34] (Assumptions (A1) and (A2))), which implies existence of at most p points of discontinuity for any solution x. Clearly, any R-solution is a regulated one, but not conversely.

Let us present two immediate generalizations of Proposition 2.

Proposition 4. (cf. [34] (Theorem 2.3)) *Let $F : [0, 1] \times \mathbb{R}^d \to \mathbb{R}^d$ be almost usc multifunction with convex (and compact) values. Assume that the following conditions hold:*

1. *There exists a constant C such that $|F(t, x)| \le C$ for every x and a.e. $t \in [0, 1]$.;*
2. *The barrier τ satisfies [B1]–[B3];*
3. *γ is nonincreasing on the interval $(t^* - c, t^*)$ for some $c > 0$, provided that the point t^* is a left accumulation point for the set $\tau(x)$ and for any continuous function x satisfying $x'(t) \in F(t, x(t))$ and $x(0) = x_0$;*
4. *I_l is nondecreasing and $I_l(u) < 0$ for $u \in (\gamma(t_1), \gamma(t^*))$ and some $t_1 \in (t^* - c, t^*)$.*

Then there exists at least one regulated solution x for (8) and all solutions for this problem are global, i.e., they can be extended up to the right endpoint of the interval.

Proof. The proof is quite classical, so we want to draw attention to the differences resulting from our approach and related to the new definition of regulated solutions. The boundedness of F (hypothesis (A3) of [34] (Theorem 2.3)) allows us to conclude that if $G_\varepsilon(t, x) = \overline{\text{co}}\, F(([t - \varepsilon, t + \varepsilon] \cap$

$[0,1]) \setminus A, x + \varepsilon\mathbb{B})$ then $|G_\varepsilon(t,x)| \leq C$, where A is a null set and $\mathbb{B} \subset \mathbb{R}^d$ is the open unit ball. Then the set of functions being solutions of the initial value problem $x' \in F(t,x)$, $x(0) = x_0$ is nonempty.

Let 0 be a point of impulse. Then we consider (8) with an initial condition $x_0 + I_r x(0)$. Consequently, one can suppose without loss of generality that 0 is not a point of discontinuity. Thus, the differential inclusion without impulses

$$\begin{cases} x'(t) \in F(t,x(t)) \;\; t \in [0,1] \;\text{ a.e.}, \\ x(0) = x_0 \end{cases}$$

has continuous solutions (and the set of such solutions is compact in in $C([0,1],\mathbb{R}^d)$. For any such function x, either its graph touches the barrier γ on a set $\tau(x)$ consisting of finite numer of points, so by classical procedure (cf. [2,3]) it can be prolonged up to the point 1, or there exists some left accumulation point t^* for the set of $\tau(x)$.

Now, we take a solution of the above problem on $[0,t_1]$, where $t_1 = \min \tau(x)$ (see Definition 1), and step by step we construct our regulated solution on the whole interval $[0,t^*]$. We can repeat our procedure presented in Section 3; i.e., by Proposition 2 we get a function from Z_{G^L} defined to the right of the point t^*. Recall, that this procedure is one of the main goals of this paper.

This procedure replaces the original one from [34] without any additional assumptions guarantying solutions with a number of discontinuity points prescribed by additional assumptions. Moreover, Proposition 2 implies that any solution exists on the interval $[0,1]$. □

Let us consider also the lower semicontinuous case. The main idea of how to change the proof is essentially the same as in previous proposition.

Proposition 5 (cf. [34] (Theorem 2.8))**.** *Let $F : [0,1] \times \mathbb{R}^d \to \mathbb{R}^d$ be an almost lower semi-continuous on $\mathcal{A}$, with some negligible set $\mathcal{A}$; $F(\cdot,x)$ is measurable for every x; $F(t,\cdot)$ is upper semi-continuous with convex values on $([0,1] \times \mathbb{R}^d) \setminus \mathcal{A}$.*

Assume that the following conditions hold:

1. *There exists a constant C such that $|F(t,x)| \leq C$ for every x and a.e. $t \in [0,1]$;*
2. *The barrier τ satisfies [B1]–[B3];*
3. *γ is nonincreasing on the interval $(t^* - c, t^*)$ for some $c > 0$, provided that the point t^* is a left accumulation point for the set $\tau(x)$ and for any continuous function x satisfying $x'(t) \in F(t,x(t))$ and $x(0) = x_0$;*
4. *I_l is nondecreasing and $I_l(u) < 0$ for $u \in (\gamma(t_1), \gamma(t^*))$ and some $t_1 \in (t^* - c, t^*)$.*

Then there exists at least one regulated solution x for (8) x and all solutions for this problem are global, i.e., they can be extended up to the right endpoint of the interval.

4. Example

We present an explanatory example. We consider a classical Cauchy problem without uniqueness with the impulsive "stopping condition" on the interval $[0,a]$. To show the idea, it is sufficient to consider only one surface $\tau(x)$ with the property, that any solution with its graph reaching this surface has a jump. Put $H(x)(t) = x(t) - J(x)$, where $J(x) = 0$ for $x \leq 1$ and $J(x) = 1$ for $x > 1$, so $\tau(x)$ is the set of points t with $x(t) - 1 = 0$. Clearly $H_c(x) = x$ and $H_d(x) = -J(x)$.

$$\left\{ \begin{array}{rcll} x'(t) & = & 2\sqrt{x(t)} & t \notin \tau(x) \\ x(0) & = & x(0+) = 0 & \\ x(t+) - \;\; x(t-) & = & H_d(x(t)) & t \in \tau(x). \end{array} \right. \tag{10}$$

As claimed above, let us find all the positions and the number of the points of discontinuity, i.e., the set $\tau(x)$. This set is depending on a solution x and then earlier results are not applicable in such a case.

Let us consider the integral form of this problem with $F(x)(t) = \int_0^t 2\sqrt{x(s)}\ ds + H_d(x(t))$, with $x_0 = 0$. The operator F takes the set of regulated functions Z_G into itself. For any $x \in Z_G$ we know that $H_d(x_d)$ has uniformly bounded sums $\sum_{k=0}^{N} \sqrt{k}$, where N is the number of jumps for a solution x, i.e., provided this sum is still less than a.

I. First let us present a general form for an arbitrary solution of (10). Since we know the formulae for all the solutions for the Cauchy problem (without the impulse condition), i.e., a trivial one $x_0(t) \equiv 0$ and $x_C(t) = 0$ for $t \in [0, C] \subset [0, a]$ and $x_C(t) = (t - C)^2$ for $a \geq t > C$, we can easily describe the set S_0 of all solutions for (10). All the intervals are considered here as intersections with $[0, a]$; i.e., $t \leq a$. Clearly, if $x_0(t) \equiv 0$, then $x_0 \in S_0$. Consider now an arbitrary function x_C. For $t_1 = C + 1$ we have $x_C(t_1) = 1$, so, using our condition, the function is "stopped" and $x_C(t_1+) = 0$. In such a way, we are again in the axis $y = 0$ and we are able to continue our procedure. The solution could be zero till the next point C_{k+1} in which we take $x_C(t) = (t - C_{k+1})^2$ or up to a. That means, the solution need not be determined by selecting only one point C. Then, for any set $Q = \{C_k \in [0, a] : k \in K \subset \mathbb{N}\}$, satisfying $C_{k+1} \geq C_k + \sqrt{k}$ $(k \in K)$, we associate a function x_Q having the form $x_Q(t) = (t - C_k)^2$ with some intervals $(C_k, C_k + \sqrt{k}]$ for all $C_k \in Q$ and vanishing elsewhere. Since x_Q is a bounded and regulated function, $S_0 \subset Z_G \subset G([0, a], \mathbb{R})$.

II. Note that different solutions of the considered problem can have different number of discontinuity points. Clearly, we have also infinitely many continuous solutions of our problem ($x \equiv 0$ and all functions having values zero up to a point C_k for which $(t - C_k)^2 < 1$ for $t \in [C_k, a]$).

The strength of our approach is more visible when we consider multivalued problems. Such a case is of special interest for unifying continuous and discontinuous approaches. Consider a modified problem from the previous example with the differential inclusion

$$x'(t) \in \left\{0, 2\sqrt{x(t)}\right\}\ ,\ t \notin \tau(x),$$

with the same set of conditions for impulses. Now, for arbitrary solution of previously considered problem at any point of its trajectory we can either prolong it as a constant function or continue as in Example 4. However, all solutions, both continuous and discontinuous, are still in our space Z_G. The case of convexified values of the above multifunction can be studied in the same manner.

5. Remarks about an Earlier Approach

In [9] (cf. also [8]) the following multivalued impulsive problem was studied:

$$\begin{aligned} y'(t) \in F(t, y(t))\ , &\quad \text{for } t \in [0, a], t \neq \tau_j(y(t)), j = 1, \ldots, m, \\ y(0) &= y_0, \\ y(t^+) = y(t) + I_j(y(t))\ , &\quad \text{for } t = \tau_j(y(t)), j = 1, \ldots, m, \end{aligned} \tag{11}$$

where $F : [0, a] \times \mathbb{R}^N \to 2^{\mathbb{R}^N}$, $I_j : \mathbb{R}^N \to \mathbb{R}^N$, $j = 1, \ldots, m$, are given impulse functions, $\tau_j \in C^1(\mathbb{R}^N, \mathbb{R})$ with $0 < \tau_j(y) < a$, and $t_y = \{t | t = \tau_k(y(t))\}$. The hypersurface $t - \tau_j(y) = 0$ is called the j-th pulse hypersurface and will be denoted by σ_j. If for each $j = 1, \ldots, m$, τ_j is a different constant function, then impulses are in the fixed times.

The authors are looking for (discontinuous) solutions in a special space. Let $CJ_m([0,a]) := C([0,a]) \times (\mathbb{R} \times \mathbb{R}^N)^m$ with following interpretation: the element $(\varphi, (l_j, v_j)_{j=1}^m)$, where $l_j \in [0,a]$ we will interpret as the function with m jumps in the times j_k defined as follows:

$$\hat{\varphi}(t) := \begin{cases} \varphi(t), & 0 \le t \le l_{\sigma(1)}, \\ \varphi(t) + \sum_{i=1}^{j} v_{\sigma(i)}, & l_{\sigma(j)} < t \le l_{\sigma(j+1)}, \\ \varphi(t) + \sum_{i=1}^{m} v_{\sigma(i)}, & l_{\sigma(m)} < t \le a, \end{cases}$$

where σ is a permutation of $\{1,2,\ldots,m\}$ such that $l_{\sigma(i)} \le l_{\sigma(i+1)}$.

The authors announced a mutual correspondence between the functions on interval $[0,a]$ with m jumps and the sets $\{(\varphi, (l_j, v_j)_{j=1}^m) \in CJ_m([0,a]) : l_j < l_{j+1}\}$, with $\zeta \mapsto (\check{\zeta}, (l_j, I_j(\check{\zeta}(l_j)))_{j=1}^m)$, where the function $\check{\zeta}$ is ζ with reduced jumps, l_j is j-th time of jump and the function I_j is an impulse function.

The space $CJ_m([0,a])$ with the norm

$$\|(\varphi, (l_j, v_j)_{j=1}^m)\| := \sup_{t\in[0,a]} \|\varphi(t)\| + \sum_{j=1}^{m} (|l_j| + \|v_j\|)$$

is a Banach space. In our approach it means that the considered functions are sums of continuous parts and discrete parts having finite number of discontinuity points. As the nature of mutual correspondences is not investigated in [9], solutions of the considered problem are included in this space $CJ_m([0,a])$. Thus, the problem is defined on a subset of continuous functions and the solution set is in a different space. Our approach allows one to eliminate such a problem. In contrast to our approach, the number of discontinuity points for solutions is then prescribed.

It is worthwhile to stress that our approach is based on analytical rather than topological methods and can be easily used for differential problems of various types having discontinuous solutions.

Let us mention that the main result in [9] is devoted to investigate the structure of the set of solutions for (11), and it was proved that under some assumptions this set is an R_δ set in $CJ_m([0,a])$. Despite that it exceeds the scope of this paper, it is an interesting problem and will be studied. Let us mention one big difference: our approach allows one to study problems with numbers of jumps depending on the solutions, including possibly infinite numbers of jumps.

The key difference in both cases is that we do not expect that all solutions of the considered should have prescribed (finite) number of discontinuity points. In [8,9] the authors have a finite number of "barriers" such that any solution meets each barrier (exactly one time). This means that several technical assumptions on that curves are required (conditions (H1)–(H3) in [11], for instance). As claimed above (and in our Example 4), the solutions studied by us have neither finite numbers of discontinuity points, nor the same number and placements of these points. An added value is that the space of solutions is universal for all problems having discontinuous solutions.

As claimed in Section 3, the same idea of solutions for differential inclusions having limited number of (possible) discontinuity points indicated by barriers met at once can be found in [34] or [10]. The space of solutions considered there consists of all functions x which are L-Lipschitz on $[\tau_i(x)^+, \tau_{i+1}(x)]$ and have no more than p jump points $\tau_1(x) < \tau_2(x) < \cdots < \tau_p(x)$. Note that in general τ_i depends on x; i.e., the impulses are not fixed times. Clearly, all such solutions are regulated.

Remark 3. *We propose to treat all such problems in an unified manner. First, we need to choose a proper subspace of $G([a,b],Y)$ and to define an operator on this space. Then either we have already a decomposition of this operator in its continuous and discrete parts (defined as in the formulation of a problem), or we need to decompose it like in our main theorem.*

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