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On solvability of the impulsive Cauchy problem for integro-differential inclusions with non-densely defined operators

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We prove the existence of at least one integrated solution to an impulsive Cauchy problem for an integro-differential inclusion in a Banach space with a non-densely defined operator. Since we look for integrated solution we do not need to assume that A is a Hille Yosida operator. We exploit a technique based on the measure of weak non-compactness which allows us to avoid any hypotheses of compactness both on the semigroup generated by the linear part and on the nonlinear term. As the main tool in the proof of our existence result, we are using the Glicksberg–Ky Fan theorem on a fixed point for a multivalued map on a compact convex subset of a locally convex topological vector space.

This article is part of the theme issue 'Topological degree and fixed point theories in differential and difference equations'.

1. Introduction

In this paper, we study the following impulsive Cauchy problem for an integro-differential inclusion in a Banach

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space with a non-densely defined operator:

$$\begin{cases} x'(t) \in Ax(t) + F\left(t, \int_0^t x(s)\,\mathrm{d}s\right) \text{ a.e. } t \in [0,b],\ t \neq t_k,\ k = 1,\dots,N \\ x(t_k^+) = x(t_k) + c_k,\ k = 1,\dots,N \\ x(0) = x_0 \in E. \end{cases} \tag{1.1}$$

Here E is a weakly compactly generated Banach space,[1] $A : D(A) \subset E \to E$ is the generator of an integrated semigroup, $F : [0,b] \times E \multimap E$ is a multivalued map (multimap for short). We require that $0 < t_1 < t_2 < \cdots < t_N < b$ and $c_k \in E, k = 1,\dots,N$ are given elements.

The space where solutions to (1.1) naturally lie is the space $PC([0,b];E)$ of all piece-wise continuous functions $x : [0,b] \to E$ with discontinuity points at $t = t_k$, $k = 1,\dots,N$ such that all values $x(t_k^+) = \lim_{s \to t_k^+} x(s)$ and $x(t_k^-) = \lim_{s \to t_k^-} x(s)$ are finite and $x(t_k) = x(t_k^-)$ for all k.

In many situations the domain of the operator A may be non-dense in the Banach space E. It comes, for example, from restrictions on the space on which the operator A is defined (periodic continuous functions, Hölder continuous functions, etc.) or from boundary conditions (e.g. the space of smooth functions vanishing on the boundary of a domain is not dense in the space of continuous functions). Thus, several types of differential equations, such as delay differential equations, age-structure models in population dynamics, evolution equations with nonlinear boundary conditions, can be written as semilinear Cauchy differential or integro-differential problems with non-dense domain, see Da Prato & Sinestrari [1], Thieme [2] and the recent papers by Magal *et al.* [3] or by Yang [4].

In order to better describe natural phenomena it is useful to consider non-necessarily continuous propagation of the studied process, allowing that the model is subjected to short-term perturbations in time, the so-called impulses. For instance, in the periodic treatment of some diseases, impulses may correspond to administration of a drug treatment; in environmental sciences, impulses may correspond to seasonal changes or harvesting; in economics, impulses may correspond to abrupt changes of prices.

Multivalued nonlinearity is related to models where the parameters are known up to some degree of uncertainty.

It is well known that if A is a Hille–Yosida operator and it is densely defined (i.e. $\overline{D(A)} = E$), it generates a C_0-semigroup, say $\{T(t)\}_{t \geq 0}$. Thus, given an initial datum in E, the semilinear Cauchy problem

$$\begin{cases} x'(t) = Ax(t) + f(t, x(t)), \quad \text{a.e. } t \in [0,b] \\ x(0) = x_0 \in E, \end{cases} \tag{1.2}$$

where $f : [0,b] \times E \to E$, has been extensively studied, under several regularity assumptions on A and f, see, e.g., Pazy [5]. In particular, the integration of the equation in (1.2) yields the variation of constants formula

$$x(t) = T(t)x_0 + \int_0^t T(t-s) f(s, x(s))\,\mathrm{d}s.$$

When A is non-densely defined (i.e. $\overline{D(A)} \neq E$) the variation of constants formula may be not well-defined. Therefore, the well-posedness of the problem may be recovered by integrating the equation twice to introducing integrated semigroups (see [6]). Of course when f has its range in the space $\overline{D(A)}$ the usual semigroup approach is applicable, but in many real situations this condition does not hold. When A is still a Hille–Yosida operator even if non-densely defined, Da Prato & Sinestrari [1] gave a definition of integral solution obtained by integrating (1.2) in time. In particular, integrated semigroup theory was used to obtain a variation of constants formula which allows to transform the integral solutions of the evolution equation into solutions of an abstract semilinear Volterra integral equation. Indeed, if A is the generator of the integrated

[1]Recall that E is weakly compactly generated if there is a weakly compact set $M \subseteq E$ with $E = \bigcup_{\lambda \geq 0} \lambda M$. For instance, every separable space is weakly compactly generated as well as every reflexive Banach space. Also the space $C(M, \mathbb{R})$ is weakly compactly generated, if M is a weakly compact subset of a Banach space.

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semigroup $\{V(t)\}_{t\geq 0}$, then the function $t \to V(t)x$ is differentiable for each $x \in \overline{D(A)}$ and, moreover, the derivative $\{V'(t)\}_{t\geq 0}$ is a C_0-semigroup on $\overline{D(A)}$, thus the integral solution to (1.2) given by Da Prato and Sinestrari (the so-called mild solutions) is equivalent to

$$x(t) = V'(t)x_0 + \frac{\mathrm{d}}{\mathrm{d}t}\int_0^t V(t-s)f(s, x(s))\,\mathrm{d}s, \quad t \in [0, b].$$

Notice that if A is densely defined $V'(t) \equiv T(t)$ for every $t \geq 0$.

If A is neither a Hille–Yosida operator, the Cauchy problem (1.2) may not have an integral solution in the sense of Da Prato and Sinestrari. Thus, one can introduce the definition of integrated solution given by Thieme [2, definition 6.4] for linear equations, i.e. when $f(t, x(t)) = f(t)$ for every $t \in [0, b]$, which is obtained by integrating (1.2) twice. Thieme [2, theorem 6.5], proved that the integrated solution is equivalent to the variation of constants formula

$$y(t) = V(t)x_0 + \int_0^t V(t-s)f(s)\,\mathrm{d}s, \quad t \in [0, b].$$

After the pioneering work of Da Prato and Sinestrari, the existence of mild solutions of (1.2) with a non-densely defined operator A was extensively investigated. For instance, Kellerman & Hieber [7] and Thieme [2], assuming A a Hille–Yosida operator, studied the semilinear Cauchy problem respectively with a bounded and a Lipschitz perturbation of the closed linear operator A. Liu *et al.* [8], imposing an additional condition to assure the existence of mild solutions of the Cauchy problem, extended Thieme's results [2] to the case when the operator A is not Hille–Yosida. Semilinear equations with non-densely defined linear part and impulses were considered subject to several problems: controllability, delay, non-local initial conditions, evolution operators, fractional equations, neutral equations, integro-differential equations, see for instance [9–17] and references therein.

We considered a rather simple problem compared to the ones cited; the reason is that our focus is to weaken the assumptions as much as possible both on the solution and on the terms of the equations.

We prove an existence result for an integrated solution to (1.1), see definition 3.8. As far as we know this is the first time that integrated solutions, related to operators A which do not satisfy the Hille–Yosida condition, are introduced for nonlinear differential problems with impulses. We stress that in order to find an integrated solution it is sufficient to assume that A generates an integrated semigroup and not to pose the stronger assumption that it is a Hille–Yosida operator. Moreover, if A is a Hille–Yosida operator defined in a reflexive Banach space E, by [18, corollary 2.19] it has a dense domain. Thus, it is a contradiction to consider a non-densely defined Hille Yosida operator in a reflexive Banach space. Since we assume only that A is the generator of an integrated semigroup, unlike [9–17] we can consider also a reflexive Banach space E.

By means of the measure of weak non-compactness, we exploit a technique based on the weak topology, developed in [19] and extended to non-necessarily reflexive Banach spaces in [20]. Thus, we avoid hypotheses of compactness both on the semigroup generated by the linear part and on the nonlinear term F. In particular, in contrast to [12–15,17], this approach allows us to treat a class of nonlinear multimaps F which are not necessarily compact-valued. In fact, even for single valued nonlinearities, assuming that the nonlinear term satisfies a regularity condition in terms of measure of weak non-compactness is a much weaker assumptions than assuming the same condition with respect to strong measure of non-compactness (see remark 3.1). For example the first condition can be easily verified in a reflexive Banach space, where bounded sets are weakly relatively compact ones. Moreover, unlike [9,11,16], we can handle also linear part A generating integrated semigroups whose derivative is a non-compact semigroup.

The paper is organized as follows. In §2 we recall some known results on the integrated semigroups, as well as on the measure of non-compactness, and on Eberlein Smulian theory, we also state the Glicksberg–Ky Fan fixed point theorem we are going to use to prove our main result. Section 3 is divided in four subsections: in §3a we state the problem we are going to study, in §3b we show how the integrated solution to (1.1) is obtained, in §3c we construct a solution operator whose fixed points are integrated solutions to (1.1); finally, in §3d we prove the existence of at least one integrated solution to (1.1).

2. Preliminaries

In the whole paper $(E, \|\cdot\|)$ stands for a Banach space. We use the notation $B_r(E)$ to denote the closed ball in E centred at 0 with radius r. In the whole paper, without generating misunderstanding, we denote by $\|\cdot\|_p$ both the $L^p([0,b];E)$-norm and $L^p([0,b];\mathbb{R})$-norm and by $\|\cdot\|_0$ the $C([0,b];E)$-norm.

We recall that the space $PC([0,b];E)$ is a normed space endowed with the $\|\cdot\|_0$-norm.

Let $BV([0,b];E)$ be the space of functions of bounded variation. We recall (see [21, theorem 4.3]) that a sequence $\{x_n\}_n \subset BV([0,b];E)$ weakly converges to an element $x \in BV([0,b];E)$ if and only if

1. $\|x_n(t)\| \leq N$, for each $n \in \mathbb{N}$ and for each $t \in [0,b]$, for some constant $N > 0$;
2. $x_n(t) \rightharpoonup x(t)$ for every $t \in [0,b]$.

Thus, the above characterization of weakly convergent sequences holds also for the space $PC([0,b];E)$.

Now we recall some definitions and properties of the integrated semigroups, we refer to [2,7] for more details on this topic.

Definition 2.1 ([7, definition 1.1]). An integrated semigroup is a family $\{V(t)\}_{t\geq 0}$ of bounded linear operators on E with the following properties:

($V1$) $V(0) = 0$;
($V2$) for each $x \in E$, the function $t \to V(t)x$, $t \geq 0$ is continuous (i.e. $V(\cdot)$ is strongly continuous);
($V3$) $V(s)V(t) = \int_0^s (V(t+\tau) - V(\tau))\,\mathrm{d}\tau$ for each $t, s \geq 0$.

Definition 2.2 ([2, definition 3.1]). A linear operator $A : D(A) \subset E \to E$ not necessarily densely defined is the generator of an integrated semigroup $\{V(t)\}_{t\geq 0}$ if the following condition holds: $x \in D(A)$ and $y = Ax$ if and only if

(i) $t \to V(t)x$ is a strongly continuous differentiable function for $t \geq 0$;
(ii) $V'(t)x - x = V(t)y$ for $t \geq 0$.

Definition 2.3 ([7, p. 163]). A family of bounded linear operators $\{V(t)\}_{t\geq 0}$ is called exponentially bounded if there exist constants $M \geq 0$ and $\omega \in \mathbb{R}$ such that

$$\|V(t)\| \leq M\,\mathrm{e}^{\omega t} \quad \text{for } t \geq 0.$$

Definition 2.4 ([7, p. 163]). Let $A : D(A) \subset E \to E$ be a generator of an exponentially bounded integrated semigroup $\{V(t)\}_{t\geq 0}$. Then for $\lambda > \omega$, $\lambda I_X - A$ is invertible and for the resolvent $R(\lambda, A)$ the following relation holds:

$$R(\lambda, A) := (\lambda I_E - A)^{-1} = \lambda \int_0^\infty \mathrm{e}^{-\lambda t} V(t)\,\mathrm{d}t.$$

Proposition 2.5 ([2, theorem 4.1] and [1, remark 8.5]). *If A is the generator of the integrated semigroup $\{V(t)\}_{t\geq 0}$, then the function $t \to V(t)x$ is differentiable for each $x \in \overline{D(A)}$ and, moreover, the derivative $\{V'(t)\}_{t\geq 0}$ is a C_0-semigroup on $\overline{D(A)}$ generated by the part A_0 of the operator A which is defined by*

$$D(A_0) = \{x \in D(A) : Ax \in \overline{D(A)}\},$$

$$A_0 x = Ax \quad \text{for } x \in D(A_0).$$

We recall now some properties of the generator of an integrated semigroup that we will use in the sequel.

Lemma 2.6 ([2, lemma 3.3]). *The generator of an integrated semigroup is a closed operator.*

Lemma 2.7 ([2, lemma 3.5]). *Let A be the generator of an integrated semigroup* $\{V(t)\}_{t\geq 0}$ *then*

(a) $\int_0^t V(r)x\,\mathrm{d}r \in D(A)$ *for every* $x \in E$;
(b) $A\int_0^t V(r)x\,\mathrm{d}r = V(t)x - tx$ *for every* $x \in E$.

We recall now the definition of a measure of non-compactness, describe its main properties and consider some relevant examples. Further information and all the proofs can be found, for instance, in [22].

Definition 2.8. Let $\mathcal{P}(E)$ be the family of all non-empty subsets of E. A function $\beta\colon \mathcal{P}(E) \to [0, +\infty)$ is called a measure of non-compactness (MNC) provided

$$\beta(\overline{\mathrm{conv}}\,\Omega) = \beta(\Omega), \quad \text{for every } \Omega \in \mathcal{P}(E).$$

A MNC β is said to be

(a) *monotone* if, for $\Omega_1, \Omega_2 \in \mathcal{P}(E)$, $\Omega_1 \subset \Omega_2$ we have $\beta(\Omega_1) \leq \beta(\Omega_2)$,
(b) *non-singular*, if $\beta(\{a\} \cup \Omega) = \beta(\Omega)$, for every $a \in E$ and $\Omega \in \mathcal{P}(E)$.
(c) *algebraically semiadditive*, if $\beta(\Omega_1 + \Omega_2) \leq \beta(\Omega_1) + \beta(\Omega_2)$ for every $\Omega_1, \Omega_2 \in \mathcal{P}(E)$,
(d) *regular*, if $\beta(\Omega) = 0$ is equivalent to Ω relatively compact.

As an example of a MNC obeying all the above properties we can consider the *Hausdorff MNC* χ_E, defined as

$$\chi_E(\Omega) := \inf\{\epsilon > 0 : \Omega \ \text{ has a finite } \epsilon\text{-net}\}. \tag{2.1}$$

The aim of this paper is to relax the compactness requirements as far as possible. Thus, we consider the Banach space E endowed with the weak topology and to prove our main result we use the De Blasi measure of weak non-compactness introduced in [23] which for $\Omega \subseteq E$ is defined as

$$\beta(\Omega) = \inf\{\varepsilon \in [0, \infty] : \Omega \subseteq W + B_\varepsilon(E) \text{ for some weakly compact } W \subseteq E\}.$$

Remark 2.9. The De Blasi measure of weak non-compactness is monotone, non-singular, algebraically semiadditive and regular. Moreover, for every linear and bounded operator $L : E \to E$ we have

$$\beta(L\Omega) \leq \|L\|\beta(\Omega).$$

For weakly compactly generated E, we can use the following result which is a special case of [24, theorem 2.8]:

Theorem 2.10. *Let E be a weakly compactly generated Banach space and* Ω *be a positive measure space with a finite measure. Then for every sequence of functions* $g_n\colon \Omega \to E$ *the functions* $\beta(g_n(\cdot))$ *are measurable in* Ω, *and*

$$\beta\left(\left\{\int_\Omega g_n(s)\,\mathrm{d}s \ : x \in C\right\}\right) \leq \int_\Omega \beta(\{g_n(s)\}_n)\,\mathrm{d}s,$$

where β *denotes the De Blasi measure of weak non-compactness.*

In order to define a measure of weak noncompactness in the space $PC([0, b]; E)$, we consider the following characteristic.

Theorem 2.11. *Let E be a Banach space,* β *denotes the De Blasi measure of weak non-compactness,* $L \in \mathbb{R}$, *and* $M \subseteq PC([0, b]; E)$. *Then*

$$\beta_L(M) = \sup_{\{g_n\}_n \subseteq M} \ \sup_{t\in[0,b]} \ \beta(\{g_n(t)\}_n)\,\mathrm{e}^{-Lt},$$

is a monotone measure of weak non-compactness on $PC([0, b]; E)$.

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The proof is the same as the one in [20, proposition 3.1], we present it here for the reader's convenience.

Proof. Since β is monotone, it is sufficient to prove that $\beta_L(\overline{\mathrm{conv}}\, M) \leq \beta_L(M)$ for every $M \subseteq PC([0,b];E)$, i.e., that, given $\gamma < \beta_L(\overline{\mathrm{conv}} M)$ arbitrary, we get $\beta_L(M) > \gamma$. By definition of β_L, there exist a sequence $\{g_n\}_n \subseteq \overline{\mathrm{conv}} M$ and $t_0 \in [0,b]$ such that $\beta(\{g_n(t_0)\}_n)\mathrm{e}^{-Lt_0} > \gamma$. Since $\mathrm{conv} M$ is a convex set, then by the Hahn–Banach theorem, its closure in the weak topology coincides with its closure in the norm topology. Therefore, since $\{g_n\}_n$ is a separable subset of $\mathrm{conv} M$, according to [25, proposition 3.55], there exists a sequence $\{h_n\}_n \subseteq M$ such that $\{g_n\}_n \subseteq \overline{\mathrm{conv}}\{h_n\}_n$. And the conclusion follows from

$$\begin{aligned}\gamma &< \beta(\{g_n(t_0)\}_n)\,\mathrm{e}^{-Lt_0} \leq \beta\big(\overline{\mathrm{conv}}\{h_n(t_0)\}_n\big)\,\mathrm{e}^{-Lt_0} = \beta\big(\{h_n(t_0)\}_n\big)\,\mathrm{e}^{-Lt_0}\\ &\leq \sup_{t\in[0,b]} \beta(\{h_n(t)\}_n)\,\mathrm{e}^{-Lt} \leq \sup_{\{p_n\}_n\subseteq M}\ \sup_{t\in[0,b]} \beta(\{p_n(t)\}_n)\,\mathrm{e}^{-Lt} = \beta_L(M).\end{aligned}$$

■

Remark 2.12. It is easy to prove that β_L obeys all the properties of β except regularity. Indeed $\beta_L(M) = 0$ if and only if for every $\{g_n\}_n \subset M$ and for every $t \in [0,b]$ the set $\{g_n(t)\}_n$ is weakly relatively compact, but this is not sufficient to imply the weak compactness of the set M.

To prove our main result we exploit the Glicksberg–Ky Fan fixed point theorem [26,27].

Theorem 2.13. *Let X be a Hausdorff locally convex topological vector space, K a compact convex subset of X and* $G : K \multimap K$ *an upper semicontinuous multimap with closed, convex values. Then G has a fixed point, i.e. there exists* $x_* \in K$ *: such that* $x_* \in G(x_*)$.

The proof of our principal result is based upon the so-called Eberlein–Smulian theory. Let us mention the main results of this theory.

Theorem 2.14 ([28, theorem 1, p. 219]). *Let* Ω *be a subset of a Banach space X. The following statements are equivalent:*

1. *Ω is relatively weakly compact;*
2. *Ω is relatively weakly sequentially compact.*

Corollary 2.15 ([28, p. 219]). *Let* Ω *be a subset of a Banach space X. The following statements are equivalent:*

1. *Ω is weakly compact;*
2. *Ω is weakly sequentially compact.*

We recall the Krein–Smulian theorem.

Theorem 2.16 ([29, p. 434]). *The convex hull of a weakly compact set in a Banach space X is weakly compact.*

3. Impulsive differential problem

(a) Statement of the problem

As stated in the introduction we study the following problem:

$$\begin{cases} x'(t) \in Ax(t) + F\left(t, \int_0^t x(s)\,\mathrm{d}s\right) & \text{a.e. } t \in [0,b],\ t \neq t_k,\ k = 1,\dots,N\\ x(t_k^+) = x(t_k) + c_k,\ k = 1,\dots,N\\ x(0) = x_0 \in E \end{cases} \tag{3.1}$$

under the following assumptions:

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(E) E is a weakly compactly generated Banach space;

(A) $A : D(A) \subset E \to E$ is the generator of an exponentially bounded integrated semigroup $\{V(t)\}_{t\geq 0}$ and we denote by $R := M \max_{t\in[0,b]} \mathrm{e}^{\omega t}$, where M, ω are the same as in definition 2.3;

(F) $F : [0,b] \times E \multimap E$ is a multivalued map satisfying the following conditions:

- **(F0)** $F(t,x)$ is a closed bounded and convex subset of E for every $t \in [0,b]$ and $x \in E$;
- **(F1)** the multifunction $F(\cdot,c) : [0,b] \multimap E$ has a measurable selection for every $c \in E$, i.e. there exists a measurable function $f : [0,b] \to E$ such that $f(t) \in F(t,c)$ for a.e. $t \in [0,b]$;
- **(F2)** the multimap $F(t,\cdot) : E \multimap E$ is weakly sequentially closed for a.e. $t \in [0,b]$, i.e. it has a weakly sequentially closed graph;
- **(F3)** for every $r > 0$ there exists a function $\eta_r \in L^1([0,b];\mathbb{R}_+)$ such that for each $c \in E$, $\|c\| \leq r$:

$$\|F(t,c)\| = \sup\{\|x\| : x \in F(t,c)\} \leq \eta_r(t), \text{ for a.e. } t \in [0,b];$$

- **(F4)** there exists a function $k \in L^1([0,b],\mathbb{R})$ such that, for every bounded set $\Omega \subset E$,

$$\beta\big(F(t,\Omega)\big) \leq k(t)\beta(\Omega), \quad \text{for a.e. } t \in [0,b],$$

where β is the De Blasi measure of weak noncompactness in E.

Remark 3.1. In the case of a reflexive Banach space E, since a bounded set is a weakly relatively compact set, from ($F3$) it follows that $\beta(F(t,\Omega)) = \beta(\Omega) = 0$ for every bounded $\Omega \subset E$, thus assumption ($F4$) is automatically true.

(b) Definition of a solution

In order to define the concept of solution to problem (3.1) at first we consider the following linear Cauchy problem associated to a semilinear differential inclusion with non-densely defined operator:

$$\begin{cases} x'(t) \in Ax(t) + f(t) \text{ a.e. } t \in [0,b], \\ x(0) = x_0 \in E \end{cases} \tag{3.2}$$

where $f : [0,b] \to E$ is a Bochner integrable function. It is known that a classical solution of (3.2) is an absolutely continuous function $x : [0,b] \to D(A)$, satisfying (3.2) for almost every $t \in [0,b]$. To provide the existence of such a solution one must impose some smooth conditions on x_0, e.g. $x_0 \in D(A)$, and on f, either some regularity assumptions such as $f(t) \in D(A)$ for every t. Without smoothness assumptions, it is necessary to consider solutions of (3.2) in a generalized sense. The first attempt in this direction is the definition of mild solution given by Da Prato & Sinestrari [1], which is obtained by integrating (3.2) in time and assuming $x_0 \in \overline{D(A)}$.

Definition 3.2. A continuous function $x : [0,b] \to \overline{D(A)}$ is the *mild solution* to problem (3.2) if

(a) $\int_0^t x(s)\,\mathrm{d}s \in D(A)$ for every $t \in [0,b]$
(b) $x(t) = x_0 + A\int_0^t x(s)\,\mathrm{d}s + \int_0^t f(s)\,\mathrm{d}s \quad t \in [0,b]$.

Proposition 3.3 ([30, theorem 3.1 page 67]). *The following assertions are equivalent*

(*i*) *x is the mild solution of* (3.2)

(*ii*)

$$x(t) = V'(t)x_0 + \frac{\mathrm{d}}{\mathrm{d}t}\int_0^t V(t-s)f(s)\,\mathrm{d}s, \quad t \in [0,b] \tag{3.3}$$

(*iii*)

$$x(t) = V'(t)x_0 + \lim_{\lambda\to\infty}\int_0^t V'(t-s)\lambda R(\lambda,A)f(s)\,\mathrm{d}s \quad t \in [0,b]. \tag{3.4}$$

If we want to relax smoothness conditions even more, we can introduce the definition of integrated solution given by Thieme [2, definition 6.4], which is obtained by integrating (3.2) twice.

Definition 3.4. A continuous function $y:[0,b]\to E$ is the *integrated solution* to problem (3.2) if

(a) $\int_0^t y(s)\,\mathrm{d}s \in D(A)$ for every $t\in[0,b]$
(b) $y(t) = tx_0 + A\int_0^t y(s)\,\mathrm{d}s + \int_0^t (t-s)f(s)\,\mathrm{d}s$ $t\in[0,b]$.

Notice that the integrated solution $y:[0,b]\to E$, is the integral function of the mild solution $x:[0,b]\to E$ given by Da Prato and Sinestrari, namely $y(t)=\int_0^t x(s)\,\mathrm{d}s$. Thus, integrating (3.3) we get the following result.

Proposition 3.5 ([2, theorem 6.5]). *y is an integrated solution of* (3.2) *if and only if*

$$y(t) = V(t)x_0 + \int_0^t V(t-s)f(s)\,\mathrm{d}s, \quad t\in[0,b]. \tag{3.5}$$

Now consider the following linear Cauchy problem with impulses:

$$\begin{cases} x'(t)\in Ax(t)+f(t) \text{ a.e. } t\in[0,b],\ t\neq t_k,\ k=1,\dots,N \\ x(t_k^+) = x(t_k)+c_k,\ k=1,\dots,N \\ x(0)=x_0\in E \end{cases} \tag{3.6}$$

where $f:[0,b]\to E$ is Bochner integrable.

In order to define the integrated solution of the impulsive linear problem (3.6), we first show a formula for the mild solution of a Cauchy problem with starting point $a>0$.

Proposition 3.6. *Given $a>0$, x is the mild solution of $x'(t)\in Ax(t)+f(t)$ in $[a,b]$ if and only if*

$$x(t) = V'(t-a)x(a) + \lim_{\lambda\to\infty}\int_a^t V'(t-s)\lambda R(\lambda,A)f(s)\,\mathrm{d}s, \quad t\in[a,b]. \tag{3.7}$$

Proof. By (3.4)

$$x(a) = V'(a)x(0) + \lim_{\lambda\to\infty}\int_0^a V'(a-s)\lambda R(\lambda,A)f(s)\,\mathrm{d}s.$$

Substituting this value in (3.7) and exploiting the semigroup properties of V' we get, for every $t\geq a$,

$$\begin{aligned} &V'(t-a)x(a) + \lim_{\lambda\to\infty}\int_a^t V'(t-s)\lambda R(\lambda,A)f(s)\,\mathrm{d}s \\ &\quad = V'(t-a)\left(V'(a)x(0) + \lim_{\lambda\to\infty}\int_0^a V'(a-s)\lambda R(\lambda,A)f(s)\,\mathrm{d}s\right) + \lim_{\lambda\to\infty}\int_a^t V'(t-s)\lambda R(\lambda,A)f(s)\,\mathrm{d}s \\ &\quad = V'(t)x(0) + \lim_{\lambda\to\infty}\int_0^a V'(t-s)\lambda R(\lambda,A)f(s)\,\mathrm{d}s + \lim_{\lambda\to\infty}\int_a^t V'(t-s)\lambda R(\lambda,A)f(s)\,\mathrm{d}s \\ &\quad = V'(t)x(0) + \lim_{\lambda\to\infty}\int_0^t V'(t-s)\lambda R(\lambda,A)f(s)\,\mathrm{d}s, \end{aligned}$$

and the proof is completed. ∎

The following proposition establishes a formula for the mild solution of (3.6). We give here a direct proof. For an alternative proof see [11].

Proposition 3.7. *If x is a mild solution of problem* (3.6) *then*

$$x(t) = V'(t)x_0 + \frac{\mathrm{d}}{\mathrm{d}t}\int_0^t V(t-s)f(s)\,\mathrm{d}s + \sum_{0<t_k<t} V'(t-t_k)c_k, \quad t\in[0,b]. \tag{3.8}$$

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Proof. Let x be a mild solution of (3.6). Then, according to proposition 3.3, for every $t \in [0, t_1]$,

$$x(t) = V'(t)x_0 + \lim_{\lambda\to\infty} \int_0^t V'(t-s)\lambda R(\lambda, A)f(s)\,\mathrm{d}s.$$

By (3.7), recalling the semigroup properties of V', for $t \in (t_1, t_2]$ we get

$$\begin{aligned}
x(t) &= V'(t-t_1)x(t_1^+) + \lim_{\lambda\to\infty} \int_{t_1}^t V'(t-s)\lambda R(\lambda, A)f(s)\,\mathrm{d}s \\
&= V'(t-t_1)\Big(x(t_1) + c_1\Big) + \lim_{\lambda\to\infty} \int_{t_1}^t V'(t-s)\lambda R(\lambda, A)f(s)\,\mathrm{d}s \\
&= V'(t-t_1)\left(V'(t_1)x_0 + \lim_{\lambda\to\infty} \int_0^{t_1} V'(t_1-s)\lambda R(\lambda, A)f(s)\,\mathrm{d}s + c_1\right) \\
&\quad + \lim_{\lambda\to\infty} \int_{t_1}^t V'(t-s)\lambda R(\lambda, A)f(s)\,\mathrm{d}s \\
&= V'(t)x_0 + \lim_{\lambda\to\infty} \int_0^{t_1} V'(t-s)\lambda R(\lambda, A)f(s)\,\mathrm{d}s + V'(t-t_1)c_1 \\
&\quad + \lim_{\lambda\to\infty} \int_{t_1}^t V'(t-s)\lambda R(\lambda, A)f(s)\,\mathrm{d}s \\
&= V'(t)x_0 + \lim_{\lambda\to\infty} \int_0^t V'(t-s)\lambda R(\lambda, A)f(s)\,\mathrm{d}s + V'(t-t_1)c_1.
\end{aligned}$$

By induction we obtain that, for every $t \in [0, b]$,

$$x(t) = V'(t)x_0 + \lim_{\lambda\to\infty} \int_0^t V'(t-s)\lambda R(\lambda, A)f(s)\,\mathrm{d}s + \sum_{0<t_k<t} V'(t-t_k)c_k$$

and the assertion follows from proposition 3.3. ■

Now, recalling that the integrated solution y is the integral function of the mild solution x, according to definition 2.1 and proposition 2.5, we have that, for every $t \in [0, b]$,

$$\begin{aligned}
y(t) &= \int_0^t x(s)\,\mathrm{d}s = \int_0^t \left[V'(s)x_0 + \frac{\mathrm{d}}{\mathrm{d}s}\int_0^s V(s-r)f(r)\,\mathrm{d}r + \sum_{0<t_k<s} V'(s-t_k)c_k \right] \mathrm{d}s \\
&= V(t)x_0 + \int_0^t V(t-s)f(s)\,\mathrm{d}s + \sum_{0<t_k<t} \int_{t_k}^t V'(s-t_k)c_k\,\mathrm{d}s \\
&= V(t)x_0 + \int_0^t V(t-s)f(s)\,\mathrm{d}s + \sum_{0<t_k<t} V(t-t_k)c_k.
\end{aligned}$$

Let $x_0 \in D(A)$ and $x : [0, b] \to D(A)$ be an absolutely continuous function satisfying (3.1). Integrating the equation in (3.1) we get the following integral equation

$$x(t) = x_0 + A\int_0^t x(s)\,\mathrm{d}s + \int_0^t f(s)\,\mathrm{d}s + \sum_{0<t_k<t} c_k, \quad t \in [0, b] \tag{3.9}$$

with $f(s) \in F(s, \int_0^s x(r)\,\mathrm{d}r)$ for a.e. $s \in [0, b]$.

As stated above, this integral equation (3.9) is equivalent to

$$x(t) = V'(t)x_0 + \frac{\mathrm{d}}{\mathrm{d}t}\int_0^t V(t-s)f(s)\,\mathrm{d}s + \sum_{0<t_k<t} V'(t-t_k)c_k, \quad t \in [0, b]$$

with $f(s) \in F(s, \int_0^s x(r)\,\mathrm{d}r)$ for a.e. $s \in [0,b]$. Integrating again and denoting by $y:[0,b] \to E$ the integral function of x, i.e. $y(t) = \int_0^t x(s)\,\mathrm{d}s$, $t \in [0,b]$ we get

$$y(t) = \int_0^t x(s)\,\mathrm{d}s = V(t)x_0 + \int_0^t V(t-s)f(s)\,\mathrm{d}s + \sum_{0<t_k<t} V(t-t_k)c_k, \quad t \in [0,b] \tag{3.10}$$

with $f(s) \in F(s, \int_0^s x(r)\,\mathrm{d}r) = F(s, y(s))$ for a.e. $s \in [0,b]$.

This approach motivates the following definition.

Definition 3.8. A function $y \in PC([0,b];E)$ is an *integrated solution* to problem (3.1) if satisfies (3.10) with $f \in L^1([0,b];E), f(s) \in F(s,y(s))$ for a.e. $s \in [0,b]$.

(c) Solution operator

In this section we shall prove the existence of an integrated solution of (1.1). We first justify the well posedness of the superposition operator $\mathcal{P}_F : PC([0,b];E) \multimap L^1([0,b];E)$, defined by

$$\mathcal{P}_F(q) = \{f \in L^1([0,b];E) \,:\, f(t) \in F(t,q(t)) \text{ a.a. } t \in [0,b]\}.$$

For this purpose we need the following proposition which is a special case of [20, theorem 4.4].

Proposition 3.9. *Assume conditions* (E), (F0), (F2), (F3) *and* (F4). *Consider a bounded sequence* $\{q_n\}_n \subset PC([0,b];E)$ *and a sequence of measurable functions* $\{f_n\}_n$ *such that* $f_n(t) \in F(t,q_n(t))$ *for a.a.* $t \in [0,b]$. *If* $q_n(t) \rightharpoonup q(t)$ *for a.a.* $t \in [0,b]$ *then there exists a subsequence* $\{f_{n_k}\}_k$ *weakly converging to a function* $f \in L^1([0,b];E)$ *with* $f(t) \in F(t,q(t))$ *for a.e.* $t \in [0,b]$.

Proposition 3.10. *Under assumptions* (E), (F0), (F1), (F2), (F3) *and* (F4), $\mathcal{P}_F(q) \neq \emptyset$ *for every* $q \in PC([0,b];E)$.

Proof. Since $q \in PC([0,b];E)$ there exists a sequence of step functions $\{q_n\}_n$, $q_n : [0,b] \to E$ such that

$$\sup_{t\in[0,b]} \|q_n(t) - q(t)\| \to 0, \quad \text{for } n \to \infty. \tag{3.11}$$

Hence $\{q_n\}_n$ is a bounded sequence and $q_n(t) \rightharpoonup q(t)$ for every $t \in [0,b]$. By (F1), there exists a sequence of functions $\{f_n\}_n$ such that $f_n(t) \in F(t,q_n(t))$ for a.a. $t \in [0,b]$ and $f_n : [0,b] \to E$ is measurable for any $n \in \mathbb{N}$. Then by proposition 3.9 we get the existence of a subsequence, still denoted as the sequence, weakly converging to $f \in \mathcal{P}_F(q)$ in $L^1([0,b];E)$ and the proposition is proved. ∎

Consider now the operator $\Gamma : PC([0,b];E) \multimap PC([0,b];E)$ defined by

$$\Gamma(q) = \left\{ \begin{aligned} & y \in PC([0,b);E): \\ & y(t) = V(t)x_0 + \int_0^t V(t-s)f(s)\,\mathrm{d}s + \sum_{0<t_k<t} V(t-t_k)c_k \quad t \in [0,b], \\ & f \in \mathcal{P}_F(q) \end{aligned} \right\}.$$

Under the above assumptions, the operator Γ is well defined, i.e. for every $q \in PC([0,b];E)$, $\Gamma(q) \neq \emptyset$. Clearly if y is a fixed point of Γ then there exists $f \in \mathcal{P}_F(y)$ such that

$$y(t) = V(t)x_0 + \int_0^t V(t-s)f(s)\,\mathrm{d}s + \sum_{0<t_k<t} V(t-t_k)c_k,$$

i.e. y is an integrated solution of (1.1).

Therefore, we study the solvability of (1.1) by means of theorem 2.13. With this aim, we describe the properties of $\Gamma : PC([0,b];E) \multimap PC([0,b];E)$.

Proposition 3.11. *The operator $G: L^1([0,b];E) \to C([0,b];E)$ defined as*

$$G(f)(t) = \int_0^t V(t-s)f(s)\,\mathrm{d}s \quad t \in [0,b],$$

is a linear and bounded operator.

Proof. The linearity follows by the linearity of the operators $V(t), t \in [0,b]$ and of the integral operator. Moreover from (A), being $\{V(t)\}_{t\geq 0}$ exponentially bounded, we have

$$\|G(f)(t)\| \leq R\|f\|_1$$

for every $t \in [0,b]$. ■

Proposition 3.12. *The multioperator $\Gamma : PC([0,b];E) \multimap PC([0,b];E)$ has a weakly sequentially closed graph.*

Proof. Let $\{q_n\}_n \subset PC([0,b];E)$ and $\{y_n\}_n \subset PC([0,b];E)$ satisfying $y_n \in \Gamma(q_n)$ for all n and $q_n \rightharpoonup q$, $y_n \rightharpoonup y$ in $PC([0,b];E)$; we will prove that $y \in \Gamma(q)$.

The weak convergence of $\{q_n\}_n$ to q in $PC([0,b];E)$ implies its boundedness and the weak convergence of $\{q_n(t)\}_n$ to $q(t)$ for a.a. $t \in [0,b]$. Moreover the fact that $y_n \in \Gamma(q_n)$ means that there exists a sequence $\{f_n\}_n, f_n \in \mathcal{P}_F(q_n)$ for every n, such that for every $t \in [0,b]$,

$$y_n(t) = V(t)x_0 + \int_0^t V(t-s)f_n(s)\,\mathrm{d}s + \sum_{0<t_k<t} V(t-t_k)c_k.$$

Hence, by proposition 3.9 we have the existence of a subsequence, denoted as the sequence, and a function $f \in \mathcal{P}_F(q)$ such that $f_n \rightharpoonup f$ in $L^1([0,b];E)$.

By proposition 3.11 we have that $G(f_n) \rightharpoonup G(f)$. Thus, we have

$$y_n(t) \rightharpoonup V(t)x_0 + \int_0^t V(t-s)f(s)\,\mathrm{d}s + \sum_{0<t_k<t} V(t-t_k)c_k := y_0(t), \quad \forall\, t \in [0,b],$$

implying, for the uniqueness of the weak limit in E, that $y_0(t) = y(t)$ for all $t \in [0,b]$, i.e. that $y \in \Gamma(q)$. ■

Proposition 3.13. *The multioperator Γ has convex and closed values.*

Proof. Taking a fixed $q \in PC([0,b];E)$, since F is convex valued, from the linearity of the operator G it follows that the set $\Gamma(q)$ is convex. The closedness of $\Gamma(q)$ follows from proposition 3.12. ■

Proposition 3.14. *Under assumptions* $(E), (A), (F0), (F1), (F2), (F4)$ *and*

$(F3')$ $\sup_{\|x\|\leq n} \|F(t,x)\| \leq \varphi_n(t)$, *for a.a.* $t \in [0,b]$, *with* $\varphi_n \in L^1([0,b];\mathbb{R})$ *and such that*

$$\liminf_{n\to\infty} \frac{1}{n}\int_0^b |\varphi_n(s)|\,\mathrm{d}s = 0, \tag{3.12}$$

there exists $\overline{n} \in \mathbb{N}$ *such that* $\Gamma(Q_{\overline{n}}) \subset Q_{\overline{n}}$, *where for* $n \in \mathbb{N}$,

$$Q_n = \{y \in PC([0,b];E),\ \|y(t)\| \leq n\ \forall\, t \in [0,b]\}.$$

Proof. Assume to the contrary, that there exist two sequences $\{q_n\}_n$ and $\{y_n\}_n$ such that $q_n \in Q_n$, $y_n \in \Gamma(q_n)$ and $y_n \notin Q_n$ for all $n \in \mathbb{N}$. By the definition of Γ, there exists a sequence $\{f_n\}_n, f_n \in \mathcal{P}_F(q_n)$,

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such that

$$y_n(t) = V(t)x_0 + \int_0^t V(t-s)f_n(s)\,ds + \sum_{0<t_k<t} V(t-t_k)c_k \quad t \in [0,b].$$

Moreover, $q_n \in Q_n$ implies, by (F3′), that $\|f_n(t)\| \le \varphi_n(t)$ for a.a. $t \in [0,b]$, hence $\|f_n\|_1 \le \|\varphi_n\|_1$. Consequently,

$$\|y_n\|_0 \le R\|x_0\| + R\|f_n\|_1 + R\sum_{k=0}^{N}\|c_k\| \le R\left(\|x_0\| + \|\varphi_n\|_1 + \sum_{k=0}^{N}\|c_k\|\right). \tag{3.13}$$

According to (3.12), there exists a subsequence, still denoted as the sequence, such that

$$\lim_{n\to\infty}\frac{1}{n}\int_0^b |\varphi_n(s)|\,ds = 0, \tag{3.14}$$

thus it is possible to choose $\overline{n}$ such that, for every $n \ge \overline{n}$,

$$\frac{\|y_n\|_0}{n} \le R\left(\frac{\|x_0\|}{n} + \frac{\|\varphi_n\|_1}{n} + \frac{\sum_{k=0}^{N}\|c_k\|}{n}\right) < 1,$$

getting a contradiction with $y_n \notin Q_n$ for every n. ■

(d) Existence result

We are now ready to state our main result.

Theorem 3.15. *Under assumptions* (E), (A), (F0)(F1), (F2), (F3)′ *and* (F4) *the problem* (3.1) *has at least one integrated solution.*

Proof. We have to prove that Γ has a fixed point. Let

$$\mathcal{A} = \{K \subset PC([0,b];E) \text{ closed, convex} : 0 \in K, \Gamma(K) \subset K\}.$$

By proposition 3.14 $Q_{\overline{n}} \in \mathcal{A}$, hence $\mathcal{A} \neq \emptyset$. Moreover $0 \in K$ for every $K \in \mathcal{A}$. Therefore $0 \in \bigcap_{K\in\mathcal{A}} K := Q$, so Q is non-empty. The set Q is closed and convex and Q is bounded, because $Q \subset Q_{\overline{n}}$. Moreover for every $K \in \mathcal{A}$ we have

$$\Gamma(Q) \subset \Gamma(K) \subset K$$

hence $\Gamma(Q) \subset Q$. Thus,

$$\Gamma(\overline{\mathrm{conv}}(\Gamma(Q)\cup\{0\})) \subset \Gamma(\overline{\mathrm{conv}}(Q\cup\{0\})) = \Gamma(\overline{\mathrm{conv}}(Q)) = \Gamma(Q) \subset \overline{\mathrm{conv}}(\Gamma(Q)\cup\{0\}).$$

Therefore $\overline{\mathrm{conv}}(\Gamma(Q)\cup\{0\}) \in \mathcal{A}$ and

$$Q = \bigcap_{K\in\mathcal{A}} K \subset \overline{\mathrm{conv}}(\Gamma(Q)\cup\{0\}).$$

On the other side

$$\overline{\mathrm{conv}}(\Gamma(Q)\cup\{0\}) \subset \overline{\mathrm{conv}}(Q\cup\{0\}) \subset \overline{\mathrm{conv}}Q = Q.$$

Thus we get

$$Q = \overline{\mathrm{conv}}(\Gamma(Q)\cup\{0\}). \tag{3.15}$$

Now we prove that Q is weakly compact. By the definition of the measure of noncompactness and recalling remark 2.12, we have that

$$\beta_L(Q) = \beta_L(\overline{\mathrm{conv}}(\Gamma(Q)\cup\{0\})) = \beta_L(\Gamma(Q)\cup\{0\}) = \beta_L(\Gamma(Q)).$$

Consider $\{q_n\}_n \subset Q$ and $\bar{t} \in [0, b]$. Thus there exists a sequence $\{f_n\}_n$ with $f_n \in \mathcal{P}_F(q_n)$ for every n such that, according to remark 2.9 and ($F4$),

$$\begin{aligned}\beta(\{\Gamma(q_n)(\bar{t})\}) &= \beta\left(\left\{V(\bar{t})x_0 + \int_0^{\bar{t}} V(\bar{t}-s)f_n(s)\,\mathrm{d}s + \sum_{0<t_k<\bar{t}} V(\bar{t}-t_k)c_k\right\}\right)\\ &\leq \beta\left(\left\{\int_0^{\bar{t}} V(\bar{t}-s)f_n(s)\,\mathrm{d}s\right\}\right) \leq \int_0^{\bar{t}} \beta(\{V(\bar{t}-s)f_n(s)\})\,\mathrm{d}s\\ &\leq R\int_0^{\bar{t}} \beta\left(\{f_n(s)\}\right)\,\mathrm{d}s \leq R\int_0^{\bar{t}} k(s)\beta\left(\{q_n(s)\}\right)\,\mathrm{d}s\\ &= R\int_0^{\bar{t}} k(s)\,\mathrm{e}^{Ls}\,\mathrm{e}^{-Ls}\beta\left(\{q_n(s)\}\right)\,\mathrm{d}s \leq R\sup_{t\in[0,b]}\,\mathrm{e}^{-Lt}\beta\left(\{q_n(t)\}\right)\int_0^{\bar{t}}\mathrm{e}^{Ls}k(s)\,\mathrm{d}s\\ &= \beta_L(\{q_n\}_n)R\int_0^{\bar{t}}\mathrm{e}^{Ls}k(s)\,\mathrm{d}s.\end{aligned}$$

Therefore, we obtain that

$$\begin{aligned}\beta_L(Q) = \beta_L(\Gamma(Q)) &= \sup_{q_n\in Q}\sup_{t\in[0,b]}\beta(\{\Gamma(q_n)(t)\})\,\mathrm{e}^{-Lt}\\ &\leq \sup_{q_n\in Q}\beta_L(\{q_n\}_n)\sup_{t\in[0,b]} R\,\mathrm{e}^{-Lt}\int_0^t \mathrm{e}^{Ls}k(s)\,\mathrm{d}s\\ &= \beta_L(Q)\sup_{t\in[0,b]} R\int_0^t \mathrm{e}^{-L(t-s)}k(s)\,\mathrm{d}s.\end{aligned}$$

Now, we can choose $L > 0$ such that

$$\sup_{t\in[0,b]} R\int_0^t \mathrm{e}^{-L(t-s)}k(s)\,\mathrm{d}s < 1,$$

obtaining $\beta_L(Q) = 0$. By remark 2.12 and corollary 2.15, since Q is closed, $Q(t)$ is weakly compact for every $t \in [0, b]$ and so also $\mathcal{P}_F(Q)(t)$ is weakly compact by ($F4$). Moreover, by $(F3)'$, $\mathcal{P}_F(Q)$ is also bounded and uniformly integrable and so again by the Dunford Pettis theorem we get that $\mathcal{P}_F(Q)$ is weakly compact. Since G is linear and bounded we obtain that $\Gamma(Q)$ is weakly compact. Therefore, the restriction of the operator Γ to the set Q is a weakly upper semicontinuous map, mapping Q onto itself. Finally, from (3.15) and theorem 2.16 we get that Q is weakly compact and the existence of a fixed point of Γ follows from theorem 2.13. ■

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