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Overdetermined problems for p-Laplace and generalized Monge-Ampère equations

Behrouz Emamizadeh ^{*}; Yichen Liu [†]; Giovanni Porru [‡]

Abstract

We investigate overdetermined problems for p-Laplace and generalized Monge-Ampère equations. By using the theory of domain derivative we find duality results and a characterization of the overdetermined boundary conditions via minimization of suitable functionals with respect to the domain.

Key Words: Overdetermined problems, Domain derivative, Duality results, Domain functionals, p-Laplace equations, Generalized Monge-Ampère equations.

Mathematics Subject Classification: 35N25, 35A23, 35J96, 47J20, 52A40.

1 Introduction

Let D be a bounded smooth domain in \mathbb{R}^N . A point $x \in D$ will be denoted with $x = (x^1, \dots, x^N)$. We also denote $u_i = \frac{\partial u}{\partial x^i}$, $u_{ij} = \frac{\partial^2 u}{\partial x^i \partial x^j}$, etc, the partial derivatives of u .

Let us recall the following well known overdetermined problem. Let c be a constant. If there exists a solution u to the Dirichlet problem

$$(1) \quad \Delta u = 1 \text{ in } D, \quad u = 0 \text{ on } \partial D$$

such that u satisfies the additional condition

$$(2) \quad |\nabla u| = c \text{ on } \partial D$$

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then D must be a ball. This result has been proved by J. Serrin [15] on 1971 using the moving plane method. At the same time, H. Weinberger [19] yields a different proof of the same result by using a Pohozaev identity and the maximum principle applied to a suitable P-function. The method of Weinberger requires less regularity of the boundary ∂D , but the method of Serrin can be easily applied to a large class of non-linear and fully non-linear operators. These two celebrated papers have inspired a great number of mathematicians, and the corresponding literature is nowadays very prominent. We refer to [2, 3, 5, 9, 11, 17] and references therein. For recent progress on this topic, we refer to the survey [10]. Among several ideas related to this overdetermined problem, we recall the following duality result [11].

Theorem 1.1 *Let $u \in C^2(D) \cap C^1(\bar{D})$ be a solution to Problem (1). The following statements are equivalent:*

- (i) *u satisfies condition (2).*
- (ii) *The identity*

$$(3) \quad \int_D v \, dx = c \int_{\partial D} v \, d\sigma$$

holds for all functions v harmonic in D .

Motivated by this result, we shall prove duality theorems for overdetermined problems involving p-Laplace equations as well as generalized Monge-Ampère equations. In case of generalized Monge-Ampère equations, the overdetermined boundary condition is not the same as (2), but condition (27) below. In the linear case ($\kappa = 1$) condition (27) reduces to the familiar condition $|\nabla u| = c$ on ∂D . If $1 < \kappa \leq N$, this condition involves ∇u as well as the second derivatives of u throughout the Newton tensor $T_{\kappa-1}(u)$. Furthermore, we consider suitable functionals of the domain D whose minimizers must satisfy the overdetermined boundary condition (2) for the p-Laplace problem, and condition (27) for the generalized Monge-Ampère problems. A crucial tool serving us shall be the domain derivative.

The paper is organized as follows. In Section 2 we introduce the notion of domain derivative. Some of our descriptions are formal, for a precise treatment of the domain derivative we refer to [16]. In particular, we find a sort of linearized equation of the p-Laplace equation $\Delta_p u = f(u)$ (see equation (10)), as well as a linearized equation of the generalized Monge-Ampère equation $S_\kappa(u) = f(u)$ (see equation (19)). These linearized equations are crucial to get our duality results. Sections 3 and 4 contain our main results. Section 3 is made of two subsections. In Subsection 3.1 we prove a duality result for a p-Laplace boundary value problem (see Theorem 3.1). In Subsection 3.2 we prove a duality result for a boundary value problem corresponding to a generalized Monge-Ampère equation (see Theorem 3.2). Also Section 4 is made of two subsections. In Section 4.1 we introduce a special functional associated with our p-Laplace equation in a domain D . We shall prove that the minimum of such functional with respect to D under the condition $|D| = \text{constant}$ yields a condition for ∇u

on ∂D which is the same as used in Theorem 3.1 (i). In Section 4.2 we introduce a special functional associated with a generalized Monge-Ampère equation in a domain D . We shall prove that the minimum of such functional with respect to D under the condition $|D| = \text{constant}$ yields a condition for ∇u on ∂D which is the same as used in Theorem 3.2 (i).

2 Domain derivative

The theory of domain derivative is very useful in fields as shape optimization. From a mathematical point of view, it goes back to Hadamard [8] and Schiffer [14]. We recall shortly the definitions and refer to [16] for a careful treatment. If $\mathcal{L}(u)$ is a differential operator, we consider the Dirichlet problem:

$$(4) \quad \mathcal{L}(u) = f(u) \text{ in } D, \quad u = 0 \text{ on } \partial D,$$

where f is a smooth function such that problem (4) has a unique solution. Let I be the identity map. For a smooth (C^2 is enough) vector field $V : \mathbb{R}^N \rightarrow \mathbb{R}^N$, and $|t|$ small, define

$$D^t = (I + tV)(D).$$

Now, we consider the Dirichlet problem in D^t :

$$(5) \quad \mathcal{L}(u^t) = f(u^t) \text{ in } D^t, \quad u^t = 0 \text{ on } \partial D^t.$$

For $x \in D$ we define

$$(6) \quad v(x) = \lim_{t \rightarrow 0} \frac{u^t(x) - u(x)}{t}.$$

Clearly, since D^t depends on the vector field V , also v depends on V . By [16], v satisfies the boundary condition

$$(7) \quad v(x) = -\frac{\partial u}{\partial \nu} (V \cdot \nu) \text{ on } \partial D,$$

where $\nu = (\nu^1, \dots, \nu^N)$ is the unit exterior normal on ∂D .

To obtain the equation for v , we compute

$$(8) \quad \lim_{t \rightarrow 0} \frac{1}{t} [\mathcal{L}(u^t) - \mathcal{L}(u)] = \lim_{t \rightarrow 0} \frac{1}{t} [f(u^t) - f(u)].$$

If f is differentiable, we have

$$f(u^t) - f(u) = f'(u + \theta(u^t - u))(u^t - u). \quad 0 < \theta < 1.$$

Therefore,

$$(9) \quad \lim_{t \rightarrow 0} \frac{1}{t} [f(u^t) - f(u)] = f'(u)v.$$

The computation of the left hand side of (8) depends on the structure of the differential operator \mathcal{L} . If $\mathcal{L}(u) = \Delta u$ we find

$$\lim_{t \rightarrow 0} \frac{1}{t} [\Delta u^t - \Delta u] = \Delta v.$$

Consider now the p -Laplacian $\mathcal{L}(u) = \operatorname{div}(|\nabla u|^{p-2} \nabla u)$. We have

$$|\nabla u^t|^{p-2} \nabla u^t - |\nabla u|^{p-2} \nabla u = |\nabla u|^{p-2} (\nabla u^t - \nabla u) + (|\nabla u^t|^{p-2} - |\nabla u|^{p-2}) \nabla u^t.$$

Therefore,

$$\lim_{t \rightarrow 0} \frac{1}{t} \left(|\nabla u^t|^{p-2} \nabla u^t - |\nabla u|^{p-2} \nabla u \right) = |\nabla u|^{p-2} \nabla v + (p-2) |\nabla u|^{p-4} (\nabla u \cdot \nabla v) \nabla u.$$

Hence, in this case, the equation corresponding to (8) for v reads as

$$(10) \quad \operatorname{div} \left(|\nabla u|^{p-2} \nabla v + (p-2) |\nabla u|^{p-4} (\nabla u \cdot \nabla v) \nabla u \right) = f'(u)v.$$

Now we recall the definition of generalized Monge-Ampère operators. Let $1 \leq \kappa \leq N$, and let $S_\kappa(u)$ be the κ -th elementary symmetric function of the eigenvalues of the Hessian matrix $H = D^2 u = [u_{ij}]$ (that is, the sum of all principal minors of order κ of H). Clearly, we have $S_1(u) = \Delta u$ (Laplace operator) and $S_N(u) = \det[D^2 u]$ (Monge-Ampère operator). Given a positive smooth function $f(t)$, we consider the problem

$$(11) \quad S_\kappa(u) = f(u) \text{ in } D, \quad u = 0 \text{ on } \partial D.$$

Suppose the domain $D \subset \mathbb{R}^N$ is bounded and smooth. In addition, for κ fixed such that $2 \leq \kappa \leq N$, we assume the following property:

$$(P_\kappa) \quad \sigma_{\kappa-1} \geq \beta \text{ on } \partial D,$$

where β is a positive constant and $\sigma_{\kappa-1}$ is the $(\kappa-1)$ -th elementary symmetric function of the principal curvatures of ∂D with respect to its inner normal, see [4, 18]. If we denote by $\tau_1, \tau_2, \dots, \tau_{N-1}$ the principal curvatures of the surface ∂D we have:

$$\sigma_1 = \sum_{1 \leq i \leq N-1} \tau_i, \quad \sigma_2 = \sum_{1 \leq i_1 < i_2 \leq N-1} \tau_{i_1} \tau_{i_2}, \quad \sigma_{N-1} = \tau_1 \tau_2 \cdots \tau_{N-1}.$$

Note that condition (P_N) means that Ω is strictly convex. Moreover, if Ω enjoys property (P_κ) then also $D^t = (I+tV)(D)$, for $|t|$ small, enjoys the same property (possibly with a smaller constant β). Finally, $f(t)$ is a positive smooth function such that problem (11) has a unique admissible solution. As usual, a solution is admissible if the operator $S_\kappa(u)$ is positive definite. In this situation, the

solution u is negative in D and $\nu = \frac{\nabla u}{|\nabla u|}$ on the boundary ∂D . We refer to [4, 18] for a careful discussion of this problem.

It is convenient to define the matrix

$$(12) \quad T_{\kappa-1}^{ij}(u) = \frac{\partial S_{\kappa}(u)}{\partial u_{ij}}, \quad i, j = 1, \dots, N.$$

We put $T_0(u) = I$, the identity matrix. The matrix $T_{\kappa}(u)$ is known as the κ -th Newton tensor associated with H . We have [13]

$$T_{\kappa}(u) = S_{\kappa}(u)I - T_{\kappa-1}(u)H, \quad \kappa = 1, \dots, N-1.$$

Since H is symmetric, also T_{κ} is symmetric. It has several nice properties. For example, we have

$$(13) \quad (T_{\kappa}^{ij}(u))_i = 0, \quad j = 1, \dots, N,$$

where $(T_{\kappa}^{ij}(u))_i = \frac{\partial T_{\kappa}^{ij}(u)}{\partial x^i}$, and here and in what follows, we use the summation convention over repeated indices from 1 to N . To prove (13), we recall the definition of the generalized Kronecker symbol

$$\begin{pmatrix} i_1 & i_2 & \cdots & i_q \\ j_1 & j_2 & \cdots & j_q \end{pmatrix}, \quad 2 \leq q \leq N,$$

where i_1, \dots, i_q are distinct integers between 1 and N , and also j_1, \dots, j_q are distinct integers between 1 and N . The value of the symbol is 1 (respectively -1) if (j_1, \dots, j_q) is an even (respectively an odd) permutation of (i_1, \dots, i_q) , and is 0 in all other cases. If $1 \leq \kappa \leq N-1$ we have (see [12])

$$(14) \quad T_{\kappa}^{ij}(u) = \frac{1}{\kappa!} \begin{pmatrix} i_1 & i_2 & \cdots & i_{\kappa} & i \\ j_1 & j_2 & \cdots & j_{\kappa} & j \end{pmatrix} u_{i_1 j_1} u_{i_2 j_2} \cdots u_{i_{\kappa} j_{\kappa}}.$$

We find

$$(T_{\kappa}^{ij}(u))_i = \frac{1}{\kappa!} \begin{pmatrix} i_1 & i_2 & \cdots & i_{\kappa} & i \\ j_1 & j_2 & \cdots & j_{\kappa} & j \end{pmatrix} (u_{i_1 j_1} u_{i_2 j_2} \cdots u_{i_{\kappa} j_{\kappa}})_i.$$

Simplifying we can write

$$(T_{\kappa}^{ij}(u))_i = \frac{1}{(\kappa-1)!} \begin{pmatrix} i_1 & i_2 & \cdots & i_{\kappa} & i \\ j_1 & j_2 & \cdots & j_{\kappa} & j \end{pmatrix} u_{i_1 j_1} u_{i_2 j_2} \cdots u_{i_{\kappa} j_{\kappa}}.$$

We note that $u_{i_1 j_1 i}$ is symmetric with respect to $i_1 i$, while the Kronecker symbol is skew-symmetric with respect to those indices. Thus, the sum over $i_1 i$ vanishes, and (13) follows.

The proof in above can be extended to prove that, if also v is a smooth function, we have

$$(15) \quad \begin{pmatrix} i_1 & i_2 & \cdots & i_{\kappa} & i \\ j_1 & j_2 & \cdots & j_{\kappa} & j \end{pmatrix} (v_{i_1 j_1} u_{i_2 j_2} \cdots u_{i_{\kappa} j_{\kappa}})_i = 0, \quad j = 1, \dots, N.$$

We refer to Proposition 2.1 of [12] for details.

Another very interesting property is the following (see [12, 13])

$$(16) \quad \frac{1}{\kappa} T_{\kappa-1}^{ij}(u) u_{ij} = S_{\kappa}(u), \quad \kappa = 1, \dots, N.$$

We are now ready to find the equation for v defined as in (6) with $\mathcal{L}(u) = S_{\kappa}(u)$. Let u^t be the (admissible) solution to problem (11) corresponding to D^t . Using (16) and (13) we have

$$(17) \quad \begin{aligned} S_{\kappa}(u^t) - S_{\kappa}(u) &= \frac{1}{\kappa} \left(T_{\kappa-1}^{ij}(u^t) u_{ij}^t - T_{\kappa-1}^{ij}(u) u_{ij} \right) \\ &= \frac{1}{\kappa} \left(T_{\kappa-1}^{ij}(u^t) u_i^t - T_{\kappa-1}^{ij}(u) u_i \right)_j \\ &= \frac{1}{\kappa} \left[\left(T_{\kappa-1}^{ij}(u) (u_i^t - u_i) \right)_j + \left(T_{\kappa-1}^{ij}(u^t) - T_{\kappa-1}^{ij}(u) \right) u_i^t \right]_j. \end{aligned}$$

We have

$$\lim_{t \rightarrow 0} \frac{1}{t} T_{\kappa-1}^{ij}(u) (u_i^t - u_i) = T_{\kappa-1}^{ij}(u) v_i.$$

Using (13) again we find

$$(18) \quad \lim_{t \rightarrow 0} \frac{1}{t} \left(T_{\kappa-1}^{ij}(u) (u_i^t - u_i) \right)_j = T_{\kappa-1}^{ij}(u) v_{ij}.$$

Moreover, using (14), we have

$$\begin{aligned} &\left((T_{\kappa-1}^{ij}(u^t) - T_{\kappa-1}^{ij}(u)) u_i^t \right)_j \\ &= \frac{1}{\kappa!} \begin{pmatrix} i_1 & i_2 & \cdots & i_{\kappa-1} & i \\ j_1 & j_2 & \cdots & j_{\kappa-1} & j \end{pmatrix} \left((u_{i_1 j_1}^t u_{i_2 j_2}^t \cdots u_{i_{\kappa-1} j_{\kappa-1}}^t - u_{i_1 j_1} u_{i_2 j_2} \cdots u_{i_{\kappa-1} j_{\kappa-1}}) u_i^t \right)_j. \end{aligned}$$

Hence,

$$\begin{aligned} &\lim_{t \rightarrow 0} \frac{1}{t} \left((T_{\kappa-1}^{ij}(u^t) - T_{\kappa-1}^{ij}(u)) u_i^t \right)_j \\ &= \frac{1}{\kappa!} \begin{pmatrix} i_1 & i_2 & \cdots & i_{\kappa-1} & i \\ j_1 & j_2 & \cdots & j_{\kappa-1} & j \end{pmatrix} \left((v_{i_1 j_1} u_{i_2 j_2} \cdots u_{i_{\kappa-1} j_{\kappa-1}} + \cdots + u_{i_1 j_1} u_{i_2 j_2} \cdots v_{i_{\kappa-1} j_{\kappa-1}}) u_i \right)_j \\ &= \frac{\kappa-1}{\kappa!} \begin{pmatrix} i_1 & i_2 & \cdots & i_{\kappa-1} & i \\ j_1 & j_2 & \cdots & j_{\kappa-1} & j \end{pmatrix} \left(v_{i_1 j_1} u_{i_2 j_2} \cdots u_{i_{\kappa-1} j_{\kappa-1}} u_i \right)_j \\ &= \frac{\kappa-1}{\kappa!} \begin{pmatrix} i_1 & i_2 & \cdots & i_{\kappa-1} & i \\ j_1 & j_2 & \cdots & j_{\kappa-1} & j \end{pmatrix} \left((v_{i_1 j_1} u_{i_2 j_2} \cdots u_{i_{\kappa-1} j_{\kappa-1}})_j u_i + v_{i_1 j_1} u_{i_2 j_2} \cdots u_{i_{\kappa-1} j_{\kappa-1}} u_{ij} \right). \end{aligned}$$

By using (15) and changing conveniently the indices we find

$$\begin{aligned} \lim_{t \rightarrow 0} \frac{1}{t} \left((T_{\kappa-1}^{ij}(u^t) - T_{\kappa-1}^{ij}(u)) u_i^t \right)_j &= \frac{\kappa-1}{\kappa!} \begin{pmatrix} i_1 & i_2 & \cdots & i_{\kappa-1} & i \\ j_1 & j_2 & \cdots & j_{\kappa-1} & j \end{pmatrix} u_{i_1 j_1} u_{i_2 j_2} \cdots u_{i_{\kappa-1} j_{\kappa-1}} v_{ij} \\ &= (\kappa-1) T_{\kappa-1}^{ij}(u) v_{ij}, \end{aligned}$$

where (14) has been used once more. From (17), (18) and the latter result we find

$$\lim_{t \rightarrow 0} \frac{S_{\kappa}(u^t) - S_{\kappa}(u)}{t} = \frac{1}{\kappa} \left(T_{\kappa-1}^{ij}(u) v_{ij} + (\kappa-1) T_{\kappa-1}^{ij}(u) v_{ij} \right) = T_{\kappa-1}^{ij}(u) v_{ij}.$$

Hence, recalling (9), we find the equation for v :

$$(19) \quad T_{\kappa-1}^{ij}(u) v_{ij} = f'(u) v.$$

3 Duality results

In this section, we extend Theorem 1.1 to p-Laplace equations and to generalized Monge-Ampère equations.

3.1 p-Laplace equations

Let $D \subset \mathbb{R}^N$ be a bounded smooth domain, and let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a C^1 positive function such that the problem

$$(20) \quad \operatorname{div}(|\nabla u|^{p-2} \nabla u) = f(u) \quad \text{in } D, \quad u = 0 \quad \text{on } \partial D,$$

has a unique (negative) solution $u \in C^1(\bar{D}) \cap W^{1,p}(D)$. For example, one can take, for $\tau < 0$, $f(\tau) = (-\tau)^\alpha$, $0 \leq \alpha < p$, see [6]. We have

Theorem 3.1 *Let u be the solution to problem (20). Then the following statements are equivalent.*

(i) *There is a constant c such that*

$$(21) \quad |\nabla u| = c \quad \text{on } \partial D.$$

(ii) *There is a constant d such that*

$$(22) \quad \int_D \left(f(u) - \frac{1}{p-1} u f'(u) \right) v \, dx = d \int_{\partial D} v \, d\sigma$$

for all solutions v to equation (10).

Proof. Multiplying (10) by $-u$, integrating over D and recalling that $u = 0$ on ∂D we find

$$\begin{aligned}
& - \int_D f'(u)vu \, dx = - \int_D \operatorname{div} \left(|\nabla u|^{p-2} \nabla v + (p-2)|\nabla u|^{p-4} (\nabla u \cdot \nabla v) \nabla u \right) u \, dx \\
& = \int_D \left(|\nabla u|^{p-2} \nabla v + (p-2)|\nabla u|^{p-4} (\nabla u \cdot \nabla v) \nabla u \right) \cdot \nabla u \, dx \\
& = (p-1) \int_D |\nabla u|^{p-2} \nabla u \cdot \nabla v \, dx \\
& = (p-1) \int_{\partial D} |\nabla u|^{p-2} \nabla u \cdot \nu v \, d\sigma - (p-1) \int_D \operatorname{div} (|\nabla u|^{p-2} \nabla u) v \, dx.
\end{aligned}$$

Since $\nu = \frac{\nabla u}{|\nabla u|}$ on ∂D , using equation (20), from the latter result we find

$$(23) \quad \int_D \left(f(u) - \frac{1}{p-1} f'(u)u \right) v \, dx = \int_{\partial D} |\nabla u|^{p-1} v \, d\sigma.$$

If (i) holds, equation (23) yields (22) with $d = c^{p-1}$.

If (ii) holds, from equations (22) and (23) we find

$$(24) \quad \int_{\partial D} |\nabla u|^{p-1} v \, d\sigma = d \int_{\partial D} v \, d\sigma.$$

Using the boundary condition (7) we have

$$v = -\frac{\partial u}{\partial \nu} V \cdot \nu = -|\nabla u| V \cdot \nu.$$

Therefore, from (24) we find

$$(25) \quad \int_{\partial D} (|\nabla u|^p - d|\nabla u|) V \cdot \nu \, d\sigma = 0.$$

Since V is arbitrary, we must have

$$|\nabla u| (|\nabla u|^{p-1} - d) = 0 \quad \text{on } \partial D.$$

By Hopf's Lemma $|\nabla u| > 0$, hence, $|\nabla u| = d^{\frac{1}{p-1}}$ on ∂D . The theorem is proved.

3.2 Generalized Monge-Ampère equations

Let κ be an integer such that $1 \leq \kappa \leq N$. Let $D \subset \mathbb{R}^N$ be a bounded smooth domain satisfying property (P_κ) , and let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a C^1 positive function such that the problem

$$(26) \quad S_\kappa(u) = f(u) \quad \text{in } D, \quad u = 0 \quad \text{on } \partial D,$$

has a unique admissible solution $u \in C^3(D) \cap C^1(\bar{D})$. We have

Theorem 3.2 *Let u be the admissible solution to problem (26). The following statements are equivalent.*

(i) *There is a constant c such that*

$$(27) \quad T_{\kappa-1}^{ij}(u)u_iu_j = c^2 \quad \text{on } \partial D.$$

(ii) *There is a constant d such that*

$$(28) \quad \int_D (\kappa f(u) - uf'(u))v \, dx = d \int_{\partial D} \frac{v}{|\nabla u|} \, d\sigma$$

hold for all solutions v to equation (19).

Proof. Multiplying (19) by $-u$, integrating over D , using (13) and recalling that $u = 0$ on ∂D we find

$$(29) \quad \begin{aligned} - \int_D uf'(u)v \, dx &= - \int_D uT_{\kappa-1}^{ij}(u)v_{ij} \, dx \\ &= - \int_D u(T_{\kappa-1}^{ij}(u)v_i)_j \, dx = \int_D u_j T_{\kappa-1}^{ij}(u)v_i \, dx. \end{aligned}$$

Integrating by parts and using (13) again we find

$$\int_D u_j T_{\kappa-1}^{ij}(u)v_i \, dx = \int_{\partial D} u_j T_{\kappa-1}^{ij}(u)v \nu^i \, d\sigma - \int_D u_{ij} T_{\kappa-1}^{ij}(u)v \, dx.$$

Since $T_{\kappa-1}^{ij}(u)u_{ij} = \kappa f(u)$ in D and $\nu^i |\nabla u| = u_i$ on ∂D , from the latter equation we find

$$(30) \quad \int_D u_j T_{\kappa-1}^{ij}(u)v_i \, dx = \int_{\partial D} T_{\kappa-1}^{ij}(u)u_iu_j \frac{v}{|\nabla u|} \, d\sigma - \int_D \kappa f(u)v \, dx.$$

From (29) and (30) it follows that

$$(31) \quad \int_D (\kappa f(u) - uf'(u))v \, dx = \int_{\partial D} T_{\kappa-1}^{ij}(u)u_iu_j \frac{v}{|\nabla u|} \, d\sigma.$$

If (i) holds, equation (31) yields (28) with $d = c^2$.

If (ii) holds, from equations (28) and (31) we find

$$(32) \quad \int_{\partial D} [T_{\kappa-1}^{ij}(u)u_iu_j - d] \frac{v}{|\nabla u|} \, d\sigma = 0.$$

Finally, using the boundary condition (7) we get

$$\int_{\partial D} [T_{\kappa-1}^{ij}(u)u_iu_j - d] V \cdot \nu \, d\sigma = 0.$$

Since V is arbitrary, (27) follows with $c^2 = d$. The theorem is proved.

Let us recall a result from [1].

Theorem 3.3 *Let D be a bounded convex domain in the plane and let c be a constant. If there exists a convex solution u to the Dirichlet problem*

$$(33) \quad S_2(u) = u_{11}u_{22} - u_{12}^2 = 1 \quad \text{in } D, \quad u = 0 \quad \text{on } \partial D$$

such that u satisfies the additional condition

$$(34) \quad u_{22}u_1^2 + u_{11}u_2^2 - 2u_{12}u_1u_2 = c^2 \quad \text{on } \partial D,$$

then D must be an ellipse.

Proof. See Theorem 2.4 of [1].

Corollary 3.4 *Let D be a bounded convex domain in the plane and let c be a constant. If there exists a convex solution u to problem (33) such that the integral equations*

$$(35) \quad 2 \int_D v \, dx = c^2 \int_{\partial D} \frac{v}{|\nabla u|} \, ds,$$

hold for all solutions v to the equation

$$(36) \quad u_{22}v_{11} + u_{11}v_{22} - 2u_{12}v_{12} = 0 \quad \text{in } D,$$

then, D is an ellipse.

Proof. Since $T_1^{11}(u) = u_{22}$, $T_1^{12}(u) = T_1^{21}(u) = -u_{12}$ and $T_1^{22}(u) = u_{11}$, equation (36) can be written as $T_1^{ij}(u)v_{ij} = 0$, and condition (34) can be written as $T_1^{ij}u_iu_j = c^2$. Hence, the corollary follows from Theorem 3.2 and Theorem 3.3.

4 Minimization of functionals

In this section we present a motivation of the overdetermined conditions (2) and (27).

4.1 p-Laplace equations

Let $D \subset \mathbb{R}^N$ be a bounded smooth domain, and recall the problem (20) below

$$(37) \quad \operatorname{div}(|\nabla u|^{p-2}\nabla u) = f(u) \quad \text{in } D, \quad u = 0 \quad \text{on } \partial D,$$

where f is a positive function such that problem (37) has a unique (negative) solution. Given D and the corresponding solution u to problem (37), we consider the functional

$$(38) \quad J(D) = \int_D \left(|\nabla u|^p + p \int_0^u f(\tau) \, d\tau \right) dx.$$

Theorem 4.1 *Let $J(D)$ be defined as in (38). If \hat{D} is a minimum of $J(D)$ among all domains D having the same measure as \hat{D} , then $|\nabla u|$ is constant on $\partial\hat{D}$.*

Proof. To prove the theorem, we use the notion of domain derivative. Recall that I is the identity map. Let $V : \mathbb{R}^N \rightarrow \mathbb{R}^N$ be a smooth vector field, and let $D^t = (I + tV)\hat{D}$ be a deformation of \hat{D} . By the well known Lagrange principle, \hat{D} is a stationary point of the functional

$$I(D) \equiv J(D) + \lambda(K(D) - \mu), \quad K(D) = |D|, \quad \mu = |\hat{D}|,$$

where λ is a real parameter. Since \hat{D} is a stationary point of $I(D)$, we must have $dI(\hat{D}, V) = 0$ for every vector field V . Clearly,

$$(39) \quad dI(\hat{D}, V) \equiv dJ(\hat{D}, V) + \lambda dK(\hat{D}, V).$$

We compute first $dJ(\hat{D}, V)$. Let u be the solution of problem (37) with $D = \hat{D}$, and let u^t be the solution of problem (37) corresponding to D^t . We have

$$\begin{aligned} dJ(\hat{D}, V) &= \lim_{t \rightarrow 0} \frac{J(D^t) - J(\hat{D})}{t} \\ &= \lim_{t \rightarrow 0} \frac{1}{t} \left[\int_{D^t} (|\nabla u^t|^p + p \int_0^{u^t} f(\tau) d\tau) dx - \int_{\hat{D}} (|\nabla u|^p + p \int_0^u f(\tau) d\tau) dx \right] \\ &= \lim_{t \rightarrow 0} \frac{1}{t} \left[\int_{D^t} (|\nabla u^t|^p + p \int_0^{u^t} f(\tau) d\tau) dx - \int_{\hat{D}} (|\nabla u^t|^p + p \int_0^{u^t} f(\tau) d\tau) dx \right] \\ &\quad + \lim_{t \rightarrow 0} \frac{1}{t} \left[\int_{\hat{D}} (|\nabla u^t|^p + p \int_0^{u^t} f(\tau) d\tau) dx - \int_{\hat{D}} (|\nabla u|^p + p \int_0^u f(\tau) d\tau) dx \right]. \end{aligned}$$

Since $u = 0$ on $\partial\hat{D}$ we find

$$\begin{aligned} &\lim_{t \rightarrow 0} \frac{1}{t} \left[\int_{D^t} (|\nabla u^t|^p + p \int_0^{u^t} f(\tau) d\tau) dx - \int_{\hat{D}} (|\nabla u^t|^p + p \int_0^{u^t} f(\tau) d\tau) dx \right] \\ &= \int_{\partial\hat{D}} (|\nabla u|^p + p \int_0^u f(\tau) d\tau) V \cdot \nu d\sigma = \int_{\partial\hat{D}} |\nabla u|^p V \cdot \nu d\sigma. \end{aligned}$$

Therefore, we find

$$\begin{aligned} (40) \quad dJ(\hat{D}, V) &= \int_{\partial\hat{D}} |\nabla u|^p V \cdot \nu d\sigma \\ &\quad + \lim_{t \rightarrow 0} \int_{\hat{D}} \frac{|\nabla u^t|^p - |\nabla u|^p}{t} dx + p \lim_{t \rightarrow 0} \int_{\hat{D}} \frac{\int_u^{u^t} f(\tau) d\tau}{t} dx \\ &= \int_{\partial\hat{D}} |\nabla u|^p V \cdot \nu d\sigma + p \left(\int_{\hat{D}} |\nabla u|^{p-2} \nabla u \cdot \nabla v dx + \int_{\hat{D}} f(u) v dx \right), \end{aligned}$$

where v is defined as

$$v(x) = \lim_{t \rightarrow 0} \frac{u^t(x) - u(x)}{t}.$$

Integrating the equation

$$-v \operatorname{div}(|\nabla u|^{p-2} \nabla u) = -f(u)v$$

over \hat{D} we find

$$- \int_{\partial \hat{D}} v |\nabla u|^{p-2} \nabla u \cdot \nu \, d\sigma + \int_{\hat{D}} |\nabla u|^{p-2} \nabla u \cdot \nabla v \, dx = - \int_{\hat{D}} f(u)v \, dx.$$

Recalling that $\nabla u = |\nabla u| \nu$ on ∂D and using the boundary condition (7), from the latter equation we find

$$\int_{\hat{D}} |\nabla u|^{p-2} \nabla u \cdot \nabla v \, dx + \int_{\hat{D}} f(u)v \, dx = - \int_{\partial \hat{D}} |\nabla u|^p V \cdot \nu \, d\sigma.$$

By (40) and the latter equation we find

$$(41) \quad dJ(\hat{D}, V) = \int_{\partial \hat{D}} |\nabla u|^p V \cdot \nu \, d\sigma - p \int_{\partial \hat{D}} |\nabla u|^p V \cdot \nu \, d\sigma = -(p-1) \int_{\partial \hat{D}} |\nabla u|^p V \cdot \nu \, d\sigma.$$

On the other hand (see [16] page 652 formula (12) with $C(u) = 1$), we have

$$(42) \quad dK(\hat{D}, V) = \int_{\partial \hat{D}} V \cdot \nu \, d\sigma.$$

Insertion of (41) and (42) into (39) yields

$$\begin{aligned} dI(\hat{D}, V) &= -(p-1) \int_{\partial \hat{D}} |\nabla u|^p V \cdot \nu \, d\sigma + \lambda \int_{\partial \hat{D}} V \cdot \nu \, d\sigma \\ &= \int_{\partial \hat{D}} \left(-(p-1) |\nabla u|^p + \lambda \right) V \cdot \nu \, d\sigma. \end{aligned}$$

Since $dI(\hat{D}, V) = 0$ for every vector field V , it follows that $|\nabla u|^p = \frac{\lambda}{p-1}$. Therefore, $|\nabla u|$ is a constant on $\partial \hat{D}$, and the theorem is proved.

4.2 Generalized Monge-Ampère equations

Now we prove a similar result for generalized Monge-Ampère equations. Assume the domain D bounded, smooth and having the property (P_κ) . Let u be an admissible solution to the problem

$$(43) \quad S_\kappa(u) = f(u) \quad \text{in } D, \quad u = 0 \quad \text{on } \partial D.$$

Here $1 \leq \kappa \leq N$ and $f(t) > 0$. Consider the functional

$$(44) \quad E(D) = \int_D \left(T_{\kappa-1}^{ij}(u) u_i u_j + (\kappa + \kappa^2) \int_0^u f(\tau) \, d\tau \right) dx,$$

where u is an admissible solution to problem (43).

Theorem 4.2 Let $E(D)$ be defined as in (44). If \hat{D} is a minimum of $E(D)$ among all domains D having the property (P_κ) and having the same measure as \hat{D} , then we have

$$T_{\kappa-1}^{ij}(u)u_iu_j = \text{constant} \quad \text{on } \partial\hat{D}.$$

Proof. Let us find a different formulation for $E(D)$. If we multiply (43) by u and use (16) we have

$$T_{\kappa-1}^{ij}(u)u_{ij}u = \kappa f(u)u.$$

Integration over D yields

$$-\int_D T_{\kappa-1}^{ij}(u)u_iu_j dx = \kappa \int_D f(u)u dx.$$

Hence, the functional defined by (44) can be rewritten as

$$E(D) = \int_D \left(-\kappa f(u)u + (\kappa + \kappa^2) \int_0^u f(\tau) d\tau \right) dx.$$

From now on, we shall use this formula for $E(D)$.

By the well known Lagrange principle, \hat{D} is a stationary point of the functional

$$I(D) \equiv E(D) + \lambda(K(D) - \mu), \quad K(D) = |D|, \quad \mu = |\hat{D}|,$$

where λ is a real parameter. For a smooth vector field V , let $D^t = (I + tV)\hat{D}$ be a deformation of \hat{D} . We must have $dI(\hat{D}, V) = 0$ for every vector field V . Clearly,

$$(45) \quad dI(\hat{D}, V) \equiv dE(\hat{D}, V) + \lambda dK(\hat{D}, V).$$

If u^t is the solution to problem (43) corresponding to D^t , we compute

$$\begin{aligned} dE(\hat{D}, V) &= \int_{\partial\hat{D}} \left(-\kappa f(u)u + (\kappa + \kappa^2) \int_0^u f(\tau) d\tau \right) V \cdot \nu d\sigma \\ &+ \lim_{t \rightarrow 0} \frac{1}{t} \int_{\hat{D}} \left(-\kappa (f(u^t)u^t - f(u)u) + (\kappa + \kappa^2) \int_u^{u^t} f(\tau) d\tau \right) dx. \end{aligned}$$

Since $u = 0$ on $\partial\hat{D}$, the first integral vanishes. Hence,

$$(46) \quad \begin{aligned} dE(\hat{D}, V) &= \int_{\hat{D}} \left(-\kappa (f'(u)vu + f(u)v) + (\kappa + \kappa^2)f(u)v \right) dx \\ &= \int_{\hat{D}} (-\kappa f'(u)vu + \kappa^2 f(u)v) dx. \end{aligned}$$

As usual, the function v is defined as

$$v(x) = \lim_{t \rightarrow 0} \frac{u^t(x) - u(x)}{t}.$$

Now we multiply equation (19) by $-u$ and integrate over \hat{D} . We find

$$\begin{aligned} - \int_{\hat{D}} f'(u)vu \, dx &= - \int_{\hat{D}} T_{\kappa-1}^{ij}(u)v_{ij}u \, dx = \int_{\hat{D}} T_{\kappa-1}^{ij}(u)u_i v_j \, dx \\ &= \int_{\partial\hat{D}} T_{\kappa-1}^{ij}(u)u_i \nu^j v \, d\sigma - \int_{\hat{D}} T_{\kappa-1}^{ij}(u)u_{ij}v \, dx. \end{aligned}$$

Since $\nu^j|\nabla u| = u_j$ and $v = -|\nabla u|V \cdot \nu$ on $\partial\hat{D}$, and $T_{\kappa-1}^{ij}(u)u_{ij} = \kappa f(u)$ in \hat{D} , from the latter equation we find

$$- \int_{\hat{D}} \kappa f'(u)vu \, dx = -\kappa \int_{\partial\hat{D}} T_{\kappa-1}^{ij}(u)u_i u_j V \cdot \nu \, d\sigma - \int_{\hat{D}} \kappa^2 f(u)v \, dx.$$

In view of the latter result, from (46) we get

$$(47) \quad dE(\hat{D}, V) = -\kappa \int_{\partial\hat{D}} T_{\kappa-1}^{ij}(u)u_i u_j V \cdot \nu \, d\sigma.$$

Insertion of (47) and (42) into (45) yields

$$dI(\hat{D}, V) = \int_{\partial\hat{D}} \left(-\kappa T_{\kappa-1}^{ij}(u)u_i u_j + \lambda \right) V \cdot \nu \, d\sigma.$$

Since $dI(\hat{D}, V) = 0$ for every vector field V , it follows that $T_{\kappa-1}^{ij}(u)u_i u_j = \frac{\lambda}{\kappa}$. Therefore, $T_{\kappa-1}^{ij}(u)u_i u_j$ is a constant on $\partial\hat{D}$, and the theorem is proved.

Corollary 4.3 *Let D be a convex planar domain, let u be a convex solution to problem (43) with $N = \kappa = 2$ and $f = 1$. If $E(D)$ is the corresponding functional defined as in (44), and if \hat{D} is a minimum of $E(D)$ among all convex domains D having the same measure as \hat{D} , then D is an ellipse.*

Proof. It follows from Theorems 4.2 and 3.3.

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