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Grim Under a Compensation Variant

Aaron Davis

Aaron Davis

Murray State University Honors College

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Dr. Elizabeth Donovan, Associate Professor
Mathematics & Statistics

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Dr. Warren Edminster, Executive Director
Honors College

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Author: Aaron Davis

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(Dr. Elizabeth Donovan, Advisor) (Date)

(Dr. Robert Donnelly, Committee Member) (Date)

(Dr. Leslie Wiglesworth, Committee Member) (Date)

Grim Under a Compensation Variant

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Aaron Davis

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Abstract

Games on graphs are a well studied subset of combinatorial games. Balance and strategies for winning are often looked at in these games. One such combinatorial graph game is Grim. Many of the winning strategies of Grim are already known. We note that many of these winning strategies are only available to the first player. Hoping to develop a fairer Grim, we look at Grim played under a slighlty different rule set. We develop winning strategies and known outcomes for this altered Grim. Throughout, we discuss whether our altered Grim is a fairer game then the original.

Chapter 1

Introduction

The study of games has become known in pop culture recently with the input of game shows, where millions of dollars are on the line for winning a relatively simple game. Results have been found on Jeopardy, and the Price is Right, among others [6]. But, game theory has been around much longer, dating back to 1713 [7]. Since then, games were mostly studied independently until Sprague-Grundy [5] developed a theorem in 1939 that allowed games to be studied more generally. We reap the benefits of their work in this paper. We will be looking specifically at games on graphs.

Graph theory is slightly younger than game theory, being first introduced by Euler in 1735 via the Königsberg bridges problem [4]. Graph theory today is vital within the fields of computer science and social networking. Put simply, graph theory is the study of objects and their connections. Before this year, the most prominent pop culture example of graph theory would have been the 6 degrees of Kevin Bacon. But, as of 2020, you can see its effects whenever anyone tests positive for COVID-19. People that were in direct contact must quarantine, while those two degrees away may quarantine, and so on.

Graphs are useful for games as they are naturally pictorial. In this paper, we use graphs as the base of our game Grim. Grim is an subtraction game similar to Nim, except Grim is played on graphs. This is how it got the name.

Like the Price is Right or Jeopardy, we will study the winning strategies of Grim. Unlike in those game shows, the games we study will not depend on probability or "luck." We will be able to prove who should win before anyone makes a move, assuming each player follows a winning strategy throughout the game.

We begin in Chapter 2 by introducing some graph theory definitions as well as some game theory concepts necessary for our study. In Chapter 3, we look at the known results on Grim along with some results on the compensation turn order we use in this paper. Finally, we look at some new results on Grim using our compensation turn order in Chapter 4. We begin by looking at Grim on specific families of graphs like paths and cycles and then conclude by examining Grim played on various families of complete graphs.

Chapter 2

Graph Theory Definitions

Graph theory is the study of objects and their interactions: who is connected to whom on social media; which animal eats which in nature; what highway crosses what road; just to name a few. Unlike the bulk of mathematical areas, the formal idea of graph theory is a relatively new concept, with most discoveries coming in the last 50 years. In this chapter we will introduce some basic graph theory terminology as well as introduce those ideas common to playing games on graphs. The following definitions are consistent with *Introduction to Graph Theory* by West[9]. This book is also useful for those hoping to gain a deeper understanding of graph theory than is found in this paper.

2.1 Graph Theory Definitions

Definition 2.1.1. A **graph** $G = (V(G), E(G))$ is a pair composed of a set of vertices $V(G)$ and a set of edges $E(G)$, where each element e in $E(G)$ is a pair of vertices; that is, $e = (x, y)$ for $x, y \in V(G)$. If a pair of vertices has an edge, we say those vertices are **adjacent**. The size of the vertex set is denoted as $|V(G)|$, or $|V|$ if the graph is understood.

It should be noted that edge pairs do not have to be unique. This leads us to the following definitions.

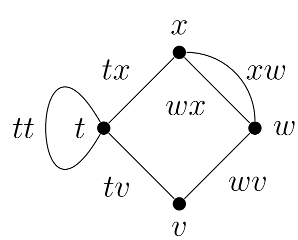
Definition 2.1.2. A **loop** is an edge that connects a vertex to itself: $e = (x, x)$ for some $x \in V(G)$. **Multiple edges** are the same edge that occur more than once in the edge set. That is, for $e_1, e_2 \in E(G)$, $e_1 = e_2$. For graphs with multiple edges, we regard $E(G)$ to be a multiset.

The ideas of loops and multiple edges have many practical uses in the study of graph theory. However, in this paper, we will restrict our view to only simple graphs.

Definition 2.1.3. A **simple graph** is a graph without loops or multiple edges.

We see can clearly see the difference between a simple graph and a graph with loops and multiple edges in Figure 2.1.1.

Graphs are often represented by a picture. We may visually represent a graph by drawing dots for the vertices and edges as lines that connect the appropriate vertices.



(a) A graph with loops and multiple edges. The edge set is $E(G_1) = \{tx, wx, xw, wv, tt, tv\}$

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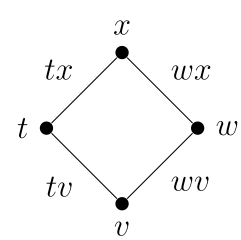


(b) A simple graph. The edge set is $E(G_2) = \{tx, wx, wv, tv\}$.

Figure 2.1.1: Two graphs, G_1 and G_2, that have the same vertex set $\{t, w, v, x\}$. Notice the graph in Figure 2.1.1a is not simple while the graph depicted in Figure 2.1.1b is a simple graph.

One of the most fundamental ways to record the interactions between vertices and edges in a graph is to count the number of edges associated with some vertex.

Definition 2.1.4. The **degree** of a vertex x is the number of edges of the form xy in $E(G)$ for some y in $V(G)$. A **pendant vertex** is a vertex with degree one. An **isolated vertex** is a vertex with degree zero.

We see a graph where its vertices are labeled by their degree in Figure 2.1.2.

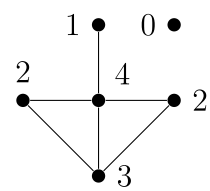


Figure 2.1.2: A graph G on six vertices. Each vertex is labeled with its degree. Notice how the vertex of degree 0 does not have any connecting edges.

Another key idea is the idea of how many "pieces" a graph is composed of. That is, can we navigate from some vertex to another along a series of edges in the graph or does no such path exist?

Definition 2.1.5. A graph is **connected** if, given any vertex $x \in V(G)$, there exists a path to any other vertex $y \in V(G)$. If for some vertices x and y there is no path from x to y in G, then G is said to be **disconnected.**

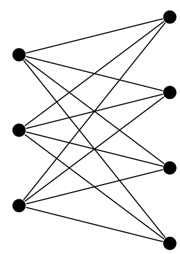
In the case that we only want to consider part of a graph, we can restrict our view to some subset of vertices and edges.

Definition 2.1.6. A **subgraph** H of a graph G is a graph whose vertex set is some subset $V(H) \subseteq V(G)$ and edge set $E(H) \subseteq E(G)$ such that $e \in E(H)$ if and only if both endpoints of e lie in $V(H)$.

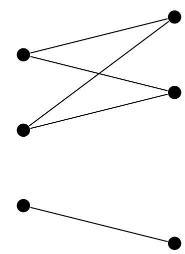
In particular, we might want to consider a combination of the two previous ideas and look at connected subgraphs.

Definition 2.1.7. A **component** of a graph G is any connected subgraph of G that is not contained in another connected subgraph of G.

That is, a component is a maximal connected subgraph: a connected subgraph that cannot be made any larger. The ideas of connectedness and components are illustrated in Figure 2.1.3.



(a) A connected graph G.



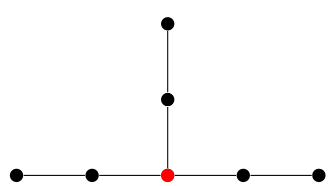
(b) A disconnected subgraph H. Note that H is made up of 2 components.

Figure 2.1.3: A graph G and one of its subgraphs, H. Notice that H does not have the same vertex set as G and, unlike G, it is not connected.

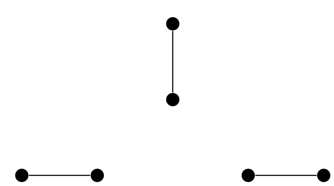
We may take a graph G and perform one or more operations on the graph to create a new graph G'.

Definition 2.1.8. The operation of **deleting a vertex** x from a graph G removes x from the vertex set $V(G)$ as well as every edge of the form xy, $y \in V(G)$, in the edge set $E(G)$.

The removal of a vertex might result in a graph that is no longer connected, as in Figure 2.1.4.



(a) A connected graph G with a vertex identifed in red.



(b) The graph G' after removing the red vertex from G. Notice that G' is composed of three components.

Figure 2.1.4: In Figure 2.1.4a we see a simple graph with a single vertex identified in red. In Figure 2.1.4b the red vertex has been deleted, which also results in the removal of any edge connected to that vertex. As a result, the remaining graph is no longer connected.

2.1.9 Families of Graphs

While the above definitions apply to all simple graphs, considering groups of graphs that have similar structure can often bring broader insight to the problem at hand. Below we will explore some of the more common groups, or families, of graphs.

A **family of graphs** is a group of graphs that all have similar structure. Often, we generate these graphs by specifying some parameter, such at the number of vertices, and use this to create a predefined visual form. As a note, the number of vertices is always defined to be any positive integer unless otherwise noted. Let us first look at one of the more simple family of graphs, path graphs.

Definition 2.1.10. A **path graph** on n vertices, denoted P_n, is a connected graph where two vertices have degree 1 and every other vertex has degree 2.

An example of a path of length five is seen in Figure 4.2.3a.

If one adds an edge between the two vertices of degree 1, then the path becomes a cycle.

Definition 2.1.11. A **cycle graph** on $n > 2$ vertices, denoted C_n, is a connected graph where every vertex has degree 2.

An example of a cycle with eight vertices is seen in Figure 4.2.4c.

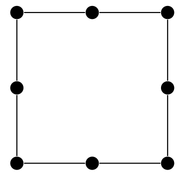
We can also take the idea of a cycle and add in a central vertex as well as a series of edges so that every original vertex is adjacent to this new vertex. This change would produce a new type of graph: a wheel graph.

Definition 2.1.12. A **wheel graph** on $n > 3$ vertices, denoted W_n, is a connected graph where one vertex, denoted the **hub**, has degree $(n-1)$ and every other vertex has degree 3.

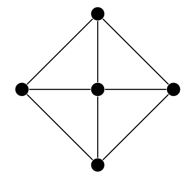
An example of a wheel on five vertices is seen in Figure 4.2.3c.

We may also consider the family of graphs where every vertex is connected to every other vertex. That is, the edge set is "complete."

(a) A path of length five, P_5.



(b) A cycle on eight vertices, C_8.



(c) A wheel on five vertices, W_5.

Figure 2.1.5: Examples of some of the common families of graphs. Each has a predefined structure with unique characteristics that makes them ideal to consider when exploring open problems.

Definition 2.1.13. A **complete graph** on n vertices, denoted K_n, is a simple graph in which every vertex has degree $n-1$. That is, there is one edge between any two vertices in K_n.

An example of a complete graph on four vertices is given in Figure 2.1.6a.

We may also separate the vertex set into two parts and connect those vertices that lie in different partitions.

Definition 2.1.14. A **multipartite graph** is a graph in which its vertices are divided into multiple partitions with no edges between vertices of different partitions. A **bipartite graph** has the vertices separated into two partitions.

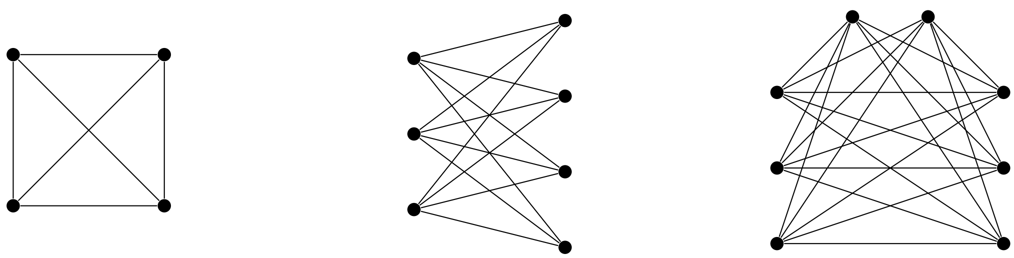
Definition 2.1.15. A **complete bipartite graph** is a simple bipartite graph such that there is an edge between two vertices if and only if they are in different partite sets. For partite sets of size $n, m \in N$ respectively, this is denoted $K_{n,m}$.

An example of a complete bipartite graph with partitions of three and four vertices is given in Figure 2.1.6b.

Generalizing this idea of splitting the vertex set into partitions, we can also define a complete multipartite graph.

Definition 2.1.16. A **complete multipartite graph** is a simple multipartite graph such that there is an edge between two vertices if an only if they are in different partite sets. For partite sets of size $t_1, t_2, \ldots, t_n \in N$, we denote this graph as $K_{t_1,t_2,\ldots,t_n}$.

An example of a complete multipartite graph with three partitions is given in Figure 2.1.6c.



(a) The complete graph on four vertices, K_4.

(b) The complete bipartite graph with partitions of size three and four, $K_{3,4}$.

(c) The complete multipartite graph with partitions of size three, three and two, $K_{3,3,2}$.

Figure 2.1.6: Examples of different types of families of complete graphs. Notice that the idea of "complete" comes from having the maximum number of edges allowed from the given vertex partitioning.

2.2 Games on Graphs

A **game** is a form of recreation that is structured. Generally, this structure includes concepts like rules, players, and teams. We can combine the ideas from Section 2.1 and the general notion of a game to discuss playing games on graphs. The definitions given below are consistent with those in [2].

2.2.1 Combinatorial Game Theory

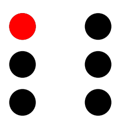
As the idea of a game covers a broad spectrum of concepts, we will narrow our focus to one specific family of games: combinatorial games.

Definition 2.2.2. A game is **combinatorial** if there are two players who take alternating turns. The **winner** under normal play is the one that makes the last legal move.

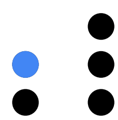
As with most work done in combinatorial games, we will focus on games involving only two players. Here we will address the two players as Player 1 and Player 2, where Player 1 plays first.

Definition 2.2.3. A game is **impartial** if, for any given position, Player 1 and Player 2 have the same options.

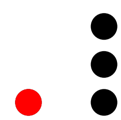
One example of an impartial game is Nim. Below, we shall consider a simplified version, wherein players take turn removing either 1 or 2 items from a single pile of items. The object of the game is to take the last object from the pile. Let's look at a simple example with a pile of size six, shown in Figure 2.2.1.



(a) Player 1 removes one item.



(b) Player 2 matches by removing one item.



(c) Player 1 removes one item.

(d) Player 2 takes two items.

(e) Player 1 wins!

Figure 2.2.1: A small game of Nim played with one pile of six items. Items in red indicate those taken by Player 1 and items in blue indicate those removed by Player 2.

Assume Player 1 takes one item, leaving Player 2 a pile of five items. In Figure 2.2.1b, Player 2 then also decides to take one item, leaving Player 1 a pile of four items. In move three, Figure 2.2.1c, we have Player 1 taking one item, leaving three items for Player 2. In Player 2's second turn, Figure 2.2.1d, they take two items and Player 1 wins by taking the final item, as seen in Figure 2.2.1e.

Let us introduce some combinatorial game theory notation to help us analyze the above example game.

Definition 2.2.4. [2] A game is in $\mathcal{P}$ **position** if the previous player has a winning strategy. A game is in $\mathcal{N}$ **position** if the next player has a winning strategy. This notation has three properties:

1. Every move from a $\mathcal{P}$ position goes to an $\mathcal{N}$ position.

2. There exists a move from each $\mathcal{N}$ position to some $\mathcal{P}$ position.

3. The terminal position for the game is a $\mathcal{P}$ position.

We should note here that if a game starts in $\mathcal{N}$ position, then Player 1 has a winning strategy, since Player 1 is the next player. Similarly, Player 2 has a winning strategy for every game in $\mathcal{P}$ position.

The game of Nim has been rigorously studied and found that, under optimal play, this game is in $\mathcal{P}$ position. Since this was not the outcome since Player 1 won, we can assume one of the two players played suboptimally. This leads us to the definition of optimal play.

Definition 2.2.5. Optimal play means that both players play perfectly. If a player has a winning strategy, they will take it.

Unsurprisingly, we assume optimal play when we study games like the one posed above. For this version of Nim, every pile which has a size that is a multiple of 3 is in $\mathcal{P}$ position. Every other pile is $\mathcal{N}$ position. Playing optimally, Player 2 can always use their move to give player 1 a pile of size n where $n \equiv 0 \pmod 3$. They do this by taking one item when Player 1 takes two and vice versa. It is simple enough to see this strategy. You can see in Figure 2.2.2b that Player 2 can take two items to leave Player 1 with pile of size three.

Some games are not as clear cut in determining an optimal strategy for play. For example, what if we played a game on a graph where, at each turn, we removed a vertex of degree at least one and the winner was the player who could make the last

(a) Player 1 starts. (b) Player 2 moves. (c) Player 1 moves. (d) Player 2 wins!

Figure 2.2.2: A game of Nim played optimally. This game consists of one pile of six objects. While a similar game was played in Figure 2.2.1, this game is played optimally by both players, making Player 2 the winner.

such move. Is there a strategy for such play? This leads us to the game that is the primary focus of this research, Grim.

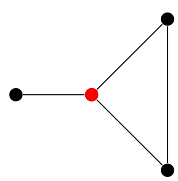
2.2.6 Grim

As noted above, Grim is a game played on a graph where each player chooses a non-isolated vertex and deletes it from the graph. Let us formally define such a game.

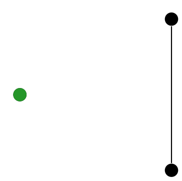
Definition 2.2.7. **Grim** is an impartial graph game. It is played with two players where each player deletes a vertex from the current graph on their turn. If any vertices became isolated due to a vertex deletion, these vertices are also deleted. A **legal move** for Grim is the deletion of a vertex. The player to make the last legal move wins the game.

Definition 2.2.8. Given the current state of the game G, the **follower** H is the state of the game after a move. In Grim, the state of the game is a graph, and the follower is a subgraph.

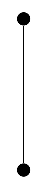
We note that this was not optimal play on Grim. As stated before, we require that game be played optimally. Thus it is necessary to look at some known results on Grim.



(a) Opening graph on which Grim is played.



(b) The isolated vertex deletion rule is applied.



(c) The follower graph after one move.

Figure 2.2.3: A game of Grim. We see that the first player deletes the red vertex. This move also removes the green vertex because it becomes an isolated vertex. The second player chooses either of the remaining vertices and wins the game.

Chapter 3

Known Results

The graph game Grim has been explored in two papers, by Adam et al. [1] and Barretto, Basi, and Miyaki [3]. Here we explore these known results before introducing our new research on an adapted turn order. Given this adaptation, we will also considered a paper by Benjamin Gaines and James Welsh [8], which explores, in a general sense, the idea of variations on turn orders when playing a game.

3.1 Known Results for Grim

First, in [1], Adams et al. find that playing Grim on a weighted graph would be identical to playing it on an unweighted graph.

Theorem 3.1.1. *[1] Let G be a weighted graph with $V(G) = v_1, \ldots, v_n$ and with t_i being the weight of vertex v_i, for all $i = 1, \ldots, n$. We denote by $G(v_1, v_2, \ldots, v_n)$ the $(t_1, t_2, \ldots, t_n)$-blowup of the graph obtained by reducing (if possible) the weight of each vertex of G to one. Then, the outcome of playing Grim on G or on $G(v_1, \ldots, v_n)$ is the same.*

The above result extends to all weighted graphs, allowing us to restrict our view to simple graphs.

Adams et al. also used principles of automorphisms to create a series of corollaries aimed at some common families of graphs as well as a union of two graphs.

Corollary 3.1.2. *[1] Let* $n \in \mathbb{N}$.

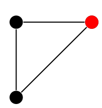
1. *If* n *is odd, and* $n \geq 3$, *then* P_n *is an* $\mathcal{N}$ *position.*

2. *If* n *is even, and* $n \geq 4$, *then* C_n *is a* $\mathcal{P}$ *position.*

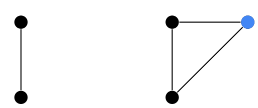
3. *If* n *is odd, and* $n \geq 5$, *then* W_n *is an* $\mathcal{N}$ *position.*

4. $G \cup G$ *is a* $\mathcal{P}$ *position, for all non-empty graphs* G.

Notice that three out of the four results depend on the vertex parity in the graph family. If one could control the parity while they retaining the family of the graph, they could guarantee a win on every game. An example of controlling the parity is given in Figure 3.1.1.

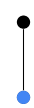


(a) The opening graph.



(b) Player 2 mimics Player 1's move.

(c) Player 1 moves.



(d) Player 2 wins!

Figure 3.1.1: An example of $G \cup G$. Player 2 simply has to mimic Player 1's move to win the game.

Moreover, Adams et al. also developed many results on families of complete graphs, both the general complete graph as well as bipartite and tripartite complete graphs.

Lemma 3.1.3. *[1] Let* $m, n \in \mathbb{N}$.

1. K_n *is in* $\mathcal{N}$ *position if and only if* n *is even.*

2. $K_{1,n}$ *is in* $\mathcal{N}$ *position for all* $|V|$.

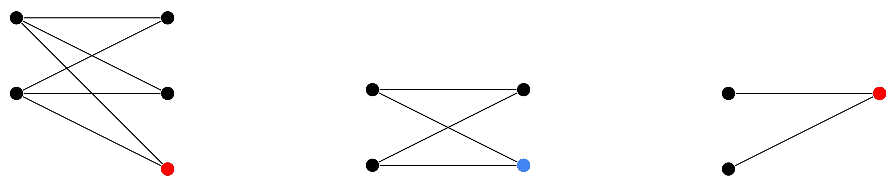
3. *Assume $m, n > 1$. $K_{m,n}$ is in $\mathcal{N}$ position if and only if $|V|$ is odd.*

4. *$K_{1,1,n}$ is in $\mathcal{N}$ position.*

5. *Assume $n \geq 2$ then $K_{1,2,n}$ is in $\mathcal{N}$ position.*

6. *Assume $m, n \geq 3$ then $K_{1,m,n}$ is in $\mathcal{N}$ position if and only if $|V|$ is odd.*

We see an example of part 3 of Lemma 3.1.3 in Figure 3.1.2.



(a) The opening graph. (b) Player 2 moves. (c) Player 1 wins!

Figure 3.1.2: A game of Grim on $K_{2,3}$. As per Lemma 3.1.3, this graph is in $\mathcal{N}$ position.

The ideas used in the proofs above were extendable, and in [1] the following general statement on playing Grim on complete multipartite graphs was given.

Lemma 3.1.4. *[1] Let $G = K_{n_1,n_2,\ldots,n_t}$, where $n_i \in \mathbb{N}$ for all $i = 1, \ldots, t$.*

1. *Assume $t \geq 3$ and $n_i \geq 2$, for all $i = 2, \ldots, t$. Then G is in $\mathcal{N}$ position if and only if $|V|$ is odd.*

2. *Assume $n_1 = 1$, $t \geq 4$ and $n_i \geq 3$, for all $i = 2, \ldots, t$. Then G is in $\mathcal{N}$ position if and only if $|V|$ is odd.*

We will see in the next chapter that the idea of vertex parity in these results will prove useful in proving similar results on a new turn order.

Barreto et al. [3] used many of the above results to develop theorems on four-partite graphs.

Lemma 3.1.5. *[3] Let $m, n \in \mathbb{N}$. Then,*

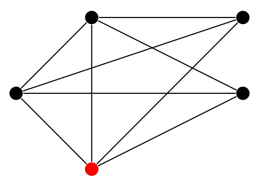
1. *$K_{1^3,n}$ where $n \geq 1$, is an $\mathcal{N}$ position if and only if $|V|$ is even.*

2. *$K_{1^2,2,n}$ where $n \geq 2$, is an $\mathcal{N}$ position if and only if $|V|$ is even.*

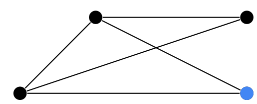
3. *$K_{1^2,3,n}$ is an $\mathcal{N}$ position for all $n \geq 3$.*

4. *$K_{1^2,m,n}$ where $m, n \geq 4$, is an $\mathcal{N}$ position if and only if $|V|$ is odd.*

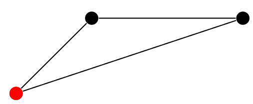
An example of part 1 of Lemma 3.1.5 on $K_{1^3,2}$ can be seen in Figure 3.1.3.



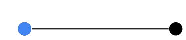
(a) The opening graph, $K_{1^3,2}$.



(b) Player 2 takes a non-singleton vertex.



(c) Player 1 has to make a move on K_3.



(d) Player 2 wins!

Figure 3.1.3: A game of Grim on $K_{1^3,2}$. As per Lemma 3.1.5, this graph is in $\mathcal{P}$ position.

We end this section with two more results on multipartite graphs.

Lemma 3.1.6. *[3] For $n, t \in \mathbb{N}$, where $n, t \geq 2$, $K_{1^t,2,n}$ is in $\mathcal{N}$ position if and only if $|V|$ is even.*

Lemma 3.1.7. *[3] For $n \in \mathbb{N}$, where $n \geq 2$, $K_{1^2,2^2,n}$ is in $\mathcal{N}$ position if and only if $|V|$ is even.*

3.2 Known Results on an Alternate Turn Order for an Impartial Game

The typical turn order in a two person impartial game can be written symbolically as $1, 2, 1, 2, 1, 2, 1, 2, 1, 2, \ldots$ Has it ever seemed like the first person to play in any given game has an unfair advantage? For comparison, think when you are choosing teams, the first person has an advantage because they can choose the very best person for the activity. They have more options than every person after them. The same is true in Grim and in other impartial games. The first player has more options in terms of removing a vertex than the second player, allowing Player 1 to win more games. However, this seems unfair and some have studied the effects of alternate turn orders in order to compensate for this, including Gaines and Welsh in [8].

Definition 3.2.1. [8] In the **compensation turn order**, Player 1 takes the first move, then Player 2 gets both the second and third moves. The game then returns to normal alternating order on the fourth move, with Player 1's turn. The order is $1, 2, 2, 1, 2, 1, 2, \ldots$, symbolically.

This idea was introduced as a way to make up for the disadvantage Player 2 has because they go second. The question becomes, does this work? That is, by giving Player 2 two moves in a row at the start of the game, can this change the overall outcome of the game? In [8], they showed it had a less balanced effect.

Theorem 3.2.2. *[8] Let G be an impartial game, played with the compensation variant. If the first player can on their first turn go to either a terminal position, or to some $\mathcal{N}$ position with no possible moves to other $\mathcal{N}$ positions or terminal positions, then the first player will win. Otherwise, the second player will win.*

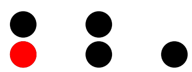
Gaines and Welsh go on to use this theorem on the impartial games Chomp and Nim that show the distinct advantage Player 2 has under the compensation variant. The theorems below show that point clearly.

Chomp is a combinatorial game where each player takes turns removing a square from a rectangle of squares. When removing a square, the player also removes every square above and to the right. The person that takes the last remaining square loses.

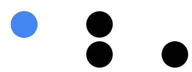
Theorem 3.2.3. *[8] Under the compensation variant, for every starting position of Chomp with at least 3 rows and 3 columns, the second player will win.*

Theorem 3.2.4. *[8] Under the compensation variant, for every starting position of Nim with $n \geq 3$ heaps where at least two heaps have more than one object, the second player will win.*

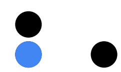
An example of Theorem 3.2.4 on three heaps is seen in Figure 3.2.1.



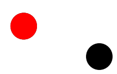
(a) The opening game: three piles with sizes 2, 2, and 1.



(b) Player 2's first move.



(c) Player 2's second move.



(d) Player 1 can only remove one item.

(e) Player 2 wins!

Figure 3.2.1: A game of Nim on 3 heaps of size 2,2 and 1. Because of the compensation variant, Player 2 can leave Player 1 with 2 heaps of size 1, guaranteeing themselves the victory.

The cases described in these theorems make up the majority of possible games of Nim and Chomp. This leads us to believe the compensation turn order will affect the game of Grim in a similar way.

Chapter 4

New Results

In this section, we will look at some new results on Grim using the compensation variant. We hoped that this turn order might produce a fairer game of Grim. However, it seems that the new turn order does the opposite, as we will show.

4.1 New Results on Standard Turn Order

We first note some properties of games under normal rules which are helpful in proving results about our compensation turn order.

Theorem 4.1.1. *Under normal rules of Grim, if a graph G is the disjoint union of two graphs, G_1 and G_2, where G_1 and G_2 are in $\mathcal{P}$ position, then G is in $\mathcal{P}$ position.*

Proof. By Definition 2.2.4, every move from a $\mathcal{P}$ position goes to an $\mathcal{N}$ position. Without loss of generality, we say that Player 1's move creates a follower of G_2 that is in $\mathcal{N}$ position. Thus, the follower of G is the disjoint union of a graph in $\mathcal{P}$ position and a graph in $\mathcal{N}$ position.

Again, by Definition 2.2.4, there exists a move from each $\mathcal{N}$ position to some $\mathcal{P}$ position. Player 2 should take this move on G_2, resulting in a follower that is the disjoint union of 2 graphs both in $\mathcal{P}$ position.

Since this is how the game started, Player 2 can continue to follow this pattern. By Definition 2.2.4, the terminal position for a graph is in $\mathcal{P}$ position. Eventually, Player 2 will make a move that completely eliminates one of the disjoint graphs, since they are the only ones that are able to create followers in $\mathcal{P}$ position. The follower of this move is a graph in $\mathcal{P}$ position. Since it is now Player 1's turn and the remaining graph is $\mathcal{P}$ position, Player 2 will win. Since Player 2 has a strategy to win G, G is in $\mathcal{P}$ position. □

Corollary 4.1.2. *Under normal rules of Grim, if a graph G is the union of 2 graphs, G_1 and G_2, where G_1 is in $\mathcal{P}$ position and G_2 is in $\mathcal{N}$ position, then G is in $\mathcal{N}$ position.*

Proof. By Definition 2.2.4, there exists a move from each $\mathcal{N}$ position to some $\mathcal{P}$ position. Player 1 should take this move on G_2. Thus, the follower is the disjoint union of two graphs both in $\mathcal{P}$ position. By Theorem 4.1.1, this follower is in $\mathcal{P}$ position. By Definition 2.2.4, every move from a $\mathcal{P}$ position must go to an $\mathcal{N}$ position. Thus, G is not in $\mathcal{P}$ position and must then be in $\mathcal{N}$ position. □

4.2 Results on the Compensation Turn Order

From here we will be looking strictly at games under the compensation turn order. Recall that this allows Player 2 two moves after Player 1's initial move, then returns to alternating between Player 1 and Player 2 taking moves until someone is declared the winner. This leads us to Observation 4.2.1 below.

Observation 4.2.1. *By Player 2's second turn, we can apply known results on $\mathcal{P}$ or $\mathcal{N}$ position to the graph at hand, since the turn order is back to the normal alternating order.*

This observation is the basis of proof for the following lemma.

Lemma 4.2.2. *If Player 2 is given a graph in $\mathcal{P}$ position on their first turn, then Player 2 has a winning strategy.*

Proof. By the definition of $\mathcal{P}$ position, Player 2's first move creates a follower in $\mathcal{N}$ position. Since the game reverts to standard rules by Player 2's second turn, Player 2 has a winning strategy. □

We will use these ideas to obtain results on multiple families of graphs.

Observation 4.2.3. *Note that P_4 and P_5 can be won in two turns. A player creates the follower P_2 or P_3 then finishes the game on the second turn.*

Theorem 4.2.4. *For all n, Player 2 has a strategy to win C_n.*

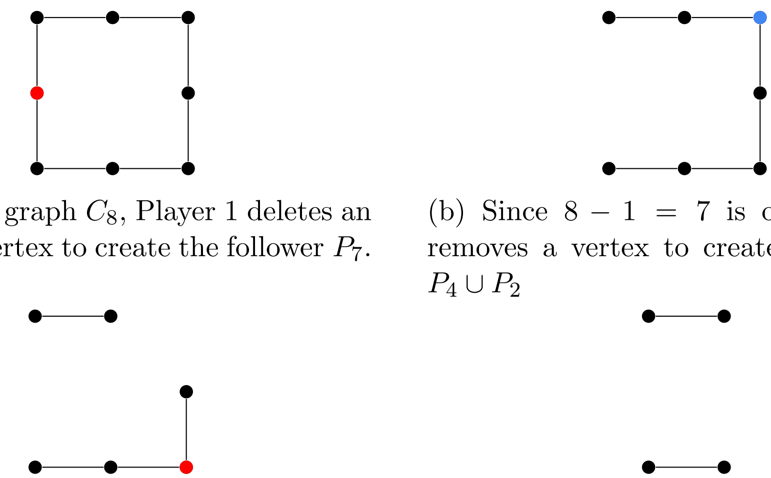
Proof. For $n = 3$ or 4, Player 1 creates a path of length 2 or 3, respectively, which can be won by Player 2 on their first move. For a cycle on 5 or 6 vertices, Player 1 creates P_4 or P_5, respectively, as followers. These are won by Player 2 on their second turn as seen by Observation 4.2.3.

Consider C_n with $n > 6$. Without loss of generality, Player 1 creates the follower P_{n-1}. We show that Player 2 has a winning strategy in two moves. If $n-1$ is even, Player 2 removes a vertex to create the follower $P_{\frac{n-1}{2}} \cup P_{\frac{n-3}{2}}$. On their second turn they remove a vertex of degree 1 from the larger path, creating as a follower two copies of $P_{\frac{n-3}{2}}$.

If $n-1$ is odd, then Player 2 removes a vertex to create the follower $P_{\frac{n}{2}} \cup P_{\frac{n-4}{2}}$. On Player 2's second turn, they remove a vertex of degree 2 from the larger path such that the resulting follower is two copies of $P_{\frac{n-4}{2}}$. By Corollary 3.1.2, a graph made up of two disjoint copies of the same graph is in $\mathcal{P}$ position. Thus, Player 2 has a strategy to win C_n.

□

An example of Theorem 4.2.4 on C_8 can be seen in Figure 4.2.1.



(a) Given a graph C_8, Player 1 deletes an arbitrary vertex to create the follower P_7.

(b) Since $8 - 1 = 7$ is odd, Player 2 removes a vertex to create the follower $P_4 \cup P_2$

(c) Player 2 then removes a vertex of degree 2 such that the resulting follower is $P_2 \cup P_2$.

(d) Player 1 must make a move on a graph in $\mathcal{P}$ position and thus loses.

Figure 4.2.1: An example of the compensation game played on C_8 using the strategy of Theorem 4.2.4.

Theorem 4.2.5. *If n is odd and $n \geq 5$, then Player 2 has a strategy to win on W_n.*

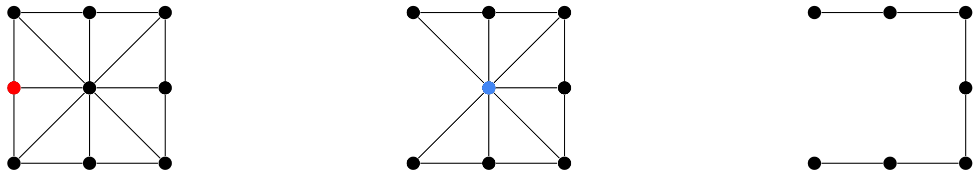
Proof. At the start of the game, Player 1 has two options: (1) they either remove the hub or (2) they remove a vertex on the exterior cycle. If Player 1 chooses to remove the original hub, then Player 2 will remove a vertex on the exterior cycle. If Player 1 removes a vertex on the exterior cycle, then Player 2 should remove the hub. Regardless, the follower of Player 2's first turn is a path of length $n-2$. Since n was odd and $n \geq 5$, $n-2$ is odd and $n-2 \geq 3$. By Corollary 3.1.2, P_{n-2} is in $\mathcal{N}$ position. Since it is Player 2's turn, Player 2 has a strategy to win the game. □

We see an example of Theorem 4.2.5 on W_9 can be seen in Figure 4.2.2.

Theorem 4.2.6. *For all $n > 3$, Player 2 has a winning strategy on P_n.*

Proof. On Player 1's first turn, they create one of two scenarios for Player 2: either a path, or the disjoint union of two paths.

We first address the case where Player 1 creates a single path as a follower. To play optimally, Player 2 should remove a pendant vertex or a vertex adjacent to a



(a) Given a graph W_9, Player 1 can delete the hub or a vertex on the exterior cycle. In this case, Player 1 deletes an exterior vertex.

(b) Since Player 1 did not remove the hub, Player 2 removes the hub to create the follower P_7.

(c) Since 7 is odd, P_7 is in $\mathcal{N}$ by Theorem 3.1.2. Thus Player 2 has a winning strategy.

Figure 4.2.2: Here is an example of the Compensation game played on W_9 using the strategy of Theorem 4.2.5.

pendant vertex. Specifically, Player 2 should use their first move to create a path of odd length k as a follower. By Corollary 3.1.2, P_k is now in $\mathcal{N}$ position. Since the remaining graph is in $\mathcal{N}$ position on Player 2's second turn, Player 2 has a strategy to win.

Next, we consider the case where Player 1's initial move created a follower that is the disjoint union of two paths. We call the disjoint graphs H and Q respectively. There are three possible cases: Either H and Q are both in $\mathcal{P}$ position, H and Q are both in $\mathcal{N}$ position, or one of H and Q is in $\mathcal{P}$ position while the other is in $\mathcal{N}$ position.

Case 1: By Theorem 4.1.1, the union of two graphs that are both in $\mathcal{P}$ position is in $\mathcal{P}$ position. Since it is Player 2's first turn, Player 2 has a strategy to win by Lemma 4.2.2

Case 2: By Definition 2.2.4, there exists a move from each $\mathcal{N}$ position to some $\mathcal{P}$ position. Player 2 should take this move on H. Thus the follower is the disjoint union of two graphs, where one is in $\mathcal{P}$ position and the other is in $\mathcal{N}$ position. By Corollary 4.1.2, this follower is in $\mathcal{N}$ position. Since the remaining graph is in $\mathcal{N}$ position on Player 2's second turn, Player 2 has a strategy to win.

Third case: Without loss of generality, assume H is in $\mathcal{P}$ position and Q is in $\mathcal{N}$

position. If the length of Q is greater than 3, then Player 2 should remove either 1 or 2 vertices on Q to create the follower of Q, P_k, where k is odd. By Corollary 3.1.2, the follower of Q is still in $\mathcal{N}$ position. If $Q = P_3$, then Player 2 should remove one vertex to create P_2 as the follower of Q, since P_2 is in $\mathcal{N}$ position. Thus the follower of the entire graph is the union of a path in $\mathcal{N}$ position and a path in $\mathcal{P}$ position. By Corollary 4.1.2, this follower is in $\mathcal{N}$ position. Since it is Player 2's turn, Player 2 has a winning strategy.

The scenario does exist where $Q = P_2$. If $H = P_4$, then Player 2 should create the followers $P_2 \cup P_3$, $P_2 \cup P_2$ on his next two turns respectively. Thus Player 1 is given $P_2 \cup P_2$ which is in $\mathcal{P}$ position by Corollary 3.1.2, which means that Player 2 has a winning strategy.

If $H = P_n$ where $n > 4$, then Player 2 should create the follower $P_2 \cup P_2 \cup P_{n-3}$. We should quickly note that since H was in $\mathcal{P}$ position, n was even by Corollary 3.1.2 and thus $(n-3)$ is odd. By part 4 of Corollary 3.1.2, $P_2 \cup P_2$ is in $\mathcal{P}$ position and by part 1 of Corollary 3.1.2, P_{n-3} is in $\mathcal{N}$ position. By Corollary 4.1.2, this follower is in $\mathcal{N}$ position. Since it is Player 2's turn, Player 2 has a winning strategy.

We have shown that Player 2 has a winning strategy in every case. Therefore, Player 2 has a winning strategy for all P_n where $n > 3$. □

An example of Theorem 4.2.6 is seen in Figure 4.2.3.

(a) Player 1 removes a non-pendant vertex from P_8.

(b) Since P_4 is in $\mathcal{P}$ position, Player 1 removes a pendant vertex from P_3.

(c) P_2 is in $\mathcal{N}$ position and P_4 is still in $\mathcal{P}$ position, so the entire graph is in $\mathcal{N}$ position on Player 2's second turn.

Figure 4.2.3: Here is an example of the Compensation game played on P_8 using the strategy of Theorem 4.2.6, case 3.

4.2.7 Complete Graphs and Complete Multipartite Graphs

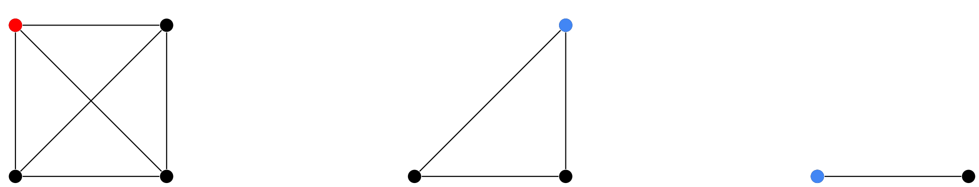
We now move on to complete graphs and complete multipartite graphs. These graphs were very dependent on the parity of $|V|$ under normal rules. This dependency on parity also arises under the compensation variant.

Theorem 4.2.8. *For $n \geq 4$, Player 2 has a winning strategy on K_n if and only if $|V|$ is even.*

Proof. Assume that $n = |V|$ is even. By the nature of complete graphs, regardless of which vertex Player 1 removes, the follower is K_{n-1}. Similarly, Player 2 must create the follower K_{n-2}. Since n was even, $n-2$ is even. By Lemma 3.1.3, this graph is in $\mathcal{N}$ position. Since it is Player 2's turn, Player 2 has a winning strategy.

Assume that Player 2 has a winning strategy on K_n. By the nature of complete graphs, Player 1 must create the follower K_{n-1}. Similarly, Player 2 must create the follower K_{n-2}. Player 2 still has a winning strategy on K_{n-2}. Since we have gone past our compensation variant and it is player 2's turn, K_{n-2} is in $\mathcal{N}$ position. By Lemma 3.1.3, $n-2$ is even. Thus, n is even.

Hence, for $n \geq 4$, Player 2 has a winning strategy on K_n if and only if $|V|$ is even. ☐



(a) K_4 (b) K_3 (c) K_2

Figure 4.2.4: A compensation Game of Grim on K_4.

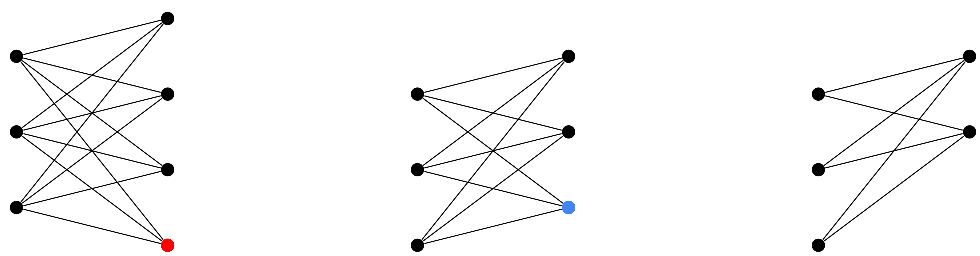
An example of Theorem 4.2.8 can be seen in Figure 4.2.4.

Theorem 4.2.9. *Assume $m, n > 2$. Then Player 2 has a winning strategy on $K_{m,n}$ if $|V|$ is odd. Moreover, for $m, n > 3$, if Player 2 has a winning strategy, $|V|$ is odd.*

Proof. Assume that $|V| = m+n$ is odd. Without loss of generality, Player 1 removes a vertex to create the follower $K_{m-1,n}$. Player 2, playing optimally, removes a vertex in the other partition to create the follower $K_{m-1,n-1}$. Since $m, n > 2$, then $m-1, n-1 > 1$. Also, $|V| = (m-1) + (n-1) = m + n - 2$ is odd because $(m+n)$ is odd. By Lemma 3.1.3, $K_{m-1,n-1}$ is in $\mathcal{N}$ position. Since it is Player 2's turn, Player 2 has a winning strategy.

Assume Player 2 has a winning strategy. Then, without loss of generality, Player 1 creates the follower $K_{m-1,n}$. Player 2 then creates either follower $H = K_{m-1,n-1}$ or $H = K_{m-2,n}$, which in either case has $|V(H)| = m + n - 2$. Since Player 2 had a winning strategy, the graph is now in $\mathcal{N}$ position. By Lemma 3.1.3, this is only true if $|V(H)| = m + n - 2$ is odd. Thus $m + n = |V(G)|$ is odd. □

An example of Theorem 4.2.9 can be seen in Figure 4.2.5.



(a) The opening graph. Player 1 creates the follower $K_{3,3}$.

(b) Player 2 creates the follower $K_{3,2}$.

(c) The game reverts back to original rules and Player 2 has a winning strategy on this graph.

Figure 4.2.5: A game of Grim on $K_{3,4}$ using the compensation variant. As per Theorem 4.2.9, Player 2 has a winning strategy.

Theorem 4.2.10. *Assume $n \geq 4$. Then, Player 2 has a winning strategy on $K_{1,2,n}$.*

Proof. The three possible followers Player 1 can create after their first turn are $K_{2,n}$, $K_{1,1,n}$, and $K_{1,2,n-1}$. For the first two of these cases, Player 2 wins on their second turn by removing either the pair of vertices or the two singleton vertices, respectively.

Now, assume Player 1 creates the follower $K_{1,2,n-1}$. On Player 2's first turn, they should create the follower $K_{1,2,n-2}$. Since $n \geq 4$, then $n-2 \geq 2$. By Lemma 3.1.3, this is in $\mathcal{N}$ position. Since it is Player 2's turn, Player 2 has a strategy to win. $\square$

Theorem 4.2.11. *Assume $m, n \geq 4$. Then Player 2 has a winning strategy on $K_{1,m,n}$ if $|V|$ is odd.*

Proof. Assume $|V| = 1+m+n$ is odd. The two possible followers Player 1 can create after their first turn are $K_{m,n}$ and, without loss of generality, $K_{1,m-1,n}$.

Regardless, Player 2 creates the follower $K_{m-1,n}$ on their first turn. Since $1+m+n$ was odd, $1+m+n-2 = (m-1)+n$ is odd. Also $m-1, n \geq 1$ since $m, n \geq 4$. By Lemma 3.1.3, the graph is in $\mathcal{N}$ position. Since it is Player 2's turn, Player 2 has a strategy to win.

$\square$

Theorem 4.2.12. *Let $G = K_{n_1,n_2,\ldots,n_t}$, where $n_i \in \mathbb{N}$ for all $i = 1, \ldots, t$. Assume $t \geq 3$ and $n_i \geq 3$, for all $i = 2, \ldots, t$. Then Player 2 has a winning strategy on G if $|V|$ is odd.*

Proof. Without loss of generality, assume Player 1 removes a vertex from the partition n_1. This leaves the follower $G_1 = K_{(n_1-1),n_2,\ldots,n_t}$. Under optimal play, Player 2 should remove a vertex from any partition except $n_1 - 1$. Without loss of generality, Player 2 removes a vertex from the partition of size n_2 to create the follower $G_2 = K_{(n_1-1),(n_2-1),\ldots,n_t}$. Since $n_1, n_2 \geq 3$ and $|V|$ is odd, then $n_1 - 1, n_2 - 1 \geq 2$ and $|V| - 1$ is odd. Thus G_2 satisfies Lemma 3.1.4 and is in $\mathcal{N}$ position. Since it is Player 2's turn, Player 2 has a strategy to win. $\square$

Theorem 4.2.13. *Let $G = K_{n_1,n_2,\ldots,n_t}$, where $n_i \in \mathbb{N}$ for all $i = 1, \ldots, t$. Assume $n_1 = 1$, $t \geq 4$ and $n_i \geq 3$, for all $i = 2, \ldots, t$. Then Player 2 has a winning strategy on G if $|V|$ is odd.*

Proof. Player 1 has two options to start the game. Player 1 will either remove the singleton vertex or a vertex from one of the other partitions. Regardless, Player 2 should take the move that Player one didn't, either removing the singleton vertex or removing a vertex from one of the other partitions. In either case, and assuming without loss of generality the non-singleton vertex removed was from the partition of size n_2, Player 2's first move creates the follower $G_1 = K_{n_2-1,\ldots,n_t}$. Since $t \geq 4$, $n_i \geq 3$, and $|V|$ is odd, then $t-1 \geq 3$, $n_2 - 1 \geq 2$, and $|V| - 1$ is odd. Thus G_1 satisfies Lemma 3.1.4 and is in $\mathcal{N}$ position. Since it is Player 2's turn, Player 2 has a strategy to win. $\square$

Theorem 4.2.14. *Player 2 has a winning strategy on $K_{1^3,n}$, where $n \geq 1$, if $|V|$ is even.*

Proof. First, consider the case where $n = 1$. Then $G = K_4$ and Player 1's initial move must create the follower K_3. By Lemma 3.1.3, this is in $\mathcal{P}$ position. By Lemma 4.2.2 Player 2 has a winning strategy on K_4.

Next, let $n \geq 3$ and recall $|V|$ is even. To start the game, Player 1 can either create the follower $K_{1^2,n}$ or $K_{1^3,n-1}$. Now, $K_{1^2,n}$ is in $\mathcal{P}$ position by Lemma 3.1.3 and $K_{1^3,n-1}$ is in $\mathcal{P}$ position by Lemma 3.1.5 since $n - 1 \geq 1$ and $|V|$ is now odd. By Lemma 4.2.2 Player 2 has a winning strategy on $K_{1^3,n}$, where $n \geq 3$.

Since we have proved both cases where $|V|$ is even, then Player 2 has a winning strategy on $K_{1^3,n}$, where $n \geq 1$, if $|V|$ is even. $\square$

Theorem 4.2.15. *Player 2 has a winning strategy on $K_{1^2,2,n}$, where $n \geq 3$, if $|V|$ is even.*

Proof. Player 1 can create three different followers: $K_{1^2,2,n-1}$, $K_{1^3,n}$, or $K_{1,2,n}$. Note that $|V|$ is odd for all of these followers since initially there were an even number of vertices. For $K_{1^2,2,n-1}$, this graph is in $\mathcal{P}$ position by Lemma 3.1.5. Similarly, $K_{1^3,n}$ is also in $\mathcal{P}$ position by Lemma 3.1.5. By Lemma 4.2.2, Player 2 wins in both of these cases.

For the third possible follower that Player 1 made, $K_{1,2,n}$, Player 2 can create the follower $K_{1,2,n-1}$ from this graph. By Lemma 3.1.3, this is in $\mathcal{N}$ position on Player 2's second turn, which means Player 2 has a winning strategy. Thus, Player 2 has a winning strategy on $K_{1^2,2,n}$. □

Theorem 4.2.16. *Player 2 has a winning strategy on $K_{1^2,3,n}$ for all $n \geq 5$.*

Proof. On their first move, Player 1 can create three different followers; $K_{1,3,n}$, $K_{1^2,2,n}$, or $K_{1^2,3,n-1}$.

Assume Player 1 creates as a follower the graph $K_{1,3,n}$ or the graph $K_{1^2,2,n}$. Then Player 2 can create the follower $K_{1,2,n}$. By Lemma 3.1.3, $K_{1,2,n}$ is in $\mathcal{N}$ position. Since it is Player 2's second turn, Player 2 has a strategy to win.

Alternatively, assume that Player 1 creates the follower $K_{1^2,3,n-1}$. Then Player 2 will create the follower $K_{1^2,3,n-2}$ when playing optimally. Since $n-2 \geq 3$, this is in $\mathcal{N}$ position by Lemma 3.1.5. Since it is Player 2's second turn, Player 2 has a strategy to win.

Since Player 2 has a strategy to win on all cases, Player 2 has a strategy to win on $K_{1^2,3,n}$. □

Theorem 4.2.17. *Player 2 has a winning strategy on $K_{1^2,m,n}$ for all $n \geq 5$ if $|V|$ is odd.*

Proof. At the opening of the game, Player 1 can create two different followers; $K_{1,m,n}$ or $K_{1^2,m-1,n}$. In both cases $|V|$ is even. By Lemma 3.1.3, $K_{1,m,n}$ is in $\mathcal{P}$ position. By Lemma 3.1.5, $K_{1^2,m-1,n}$ is in $\mathcal{P}$ position. Thus, by Lemma 4.2.2, Player 2 has a winning strategy on both cases and therefore Player 2 has a winning strategy on $K_{1^2,m,n}$. □

An example of Theorem 4.2.17 can be seen in Figure 4.2.6.

Theorem 4.2.18. *For $n \in N$, where $n \geq 3$, Player 2 has a winning strategy on $K_{1^2,2^2,n}$ if $|V|$ is even.*

(a) In this case, Player 1 removes one of the singleton vertices, creating the follower $K_{1,4,5}$.

(b) By Lemma 3.1.5, this is in $\mathcal{P}$ position, so Player 2 wins.

Figure 4.2.6: A game of Grim on $K_{1^2,4,5}$ using the compensation variant. As per Theorem 4.2.17, Player 2 has a winning strategy.

Proof. Assume $|V|$ is even. With their first move Player 1 can create three possible followers; $K_{1^3,2,n}$, $K_{1^2,2^2,n-1}$ or $K_{1^2,2^2,n}$. Since $|V|-1$ is odd, $K_{1^3,2,n}$ is in $\mathcal{P}$ position by Lemma 3.1.6 . Additionally, $K_{1^2,2,n-1}$ is in $\mathcal{P}$ position by Lemma 3.1.7 since $|V|-1$ is odd and $(n-1) \geq 3$. Player 2 has a winning strategy on both these followers by Lemma 4.2.2. Thus Player 1 should create the follower $K_{1^2,2^2,n}$. However, from this move, Player 2 can create the follower $K_{1^2,2,n}$. By Lemma 3.1.6 this graph is in $\mathcal{N}$ position since $|V|-2$ is even. Since it is Player 2's second turn, Player 2 has a strategy to win. Thus, Player 2 has a strategy to win on $K_{1^2,2^2,n}$ because Player 2 has a strategy to win on all possible followers of $K_{1^2,2^2,n}$. $\square$

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