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Existence, uniqueness, and global asymptotic stability of an equilibrium in a multiple unbounded distributed delay network

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Abstract. By employing the notion of M-matrices and Banach's contraction mapping principle, we provide complete characterisation of the existence and uniqueness of an equilibrium of a Cohen–Grossberg–Hopfield-type neural network endowed with multiple unbounded distributed time delays. Invoking similar arguments, and by constructing a suitable Lyapunov functional, we establish sufficient conditions for the global asymptotic stability of the equilibrium, independent of time delays.

Keywords: Cohen–Grossberg–Hopfield-type neural networks, unbounded distributed time delays, Banach's contraction mapping principle, M-matrix, equilibrium, Lyapunov functional, global asymptotic stability.

2020 Mathematics Subject Classification: 34K05, 34K06, 34K20, 34K99.

1 Introduction

The principal objective of this article is to put on a firm mathematical foundation the existence, uniqueness, and global asymptotic stability of an equilibrium of a Cohen–Grossberg–Hopfield-type neural network [9,20,21] motif endowed with multiple distributed time delays. The neural network model studied in this article falls within the class of so-called static neural network models with S-type distributed delays [31, 32]. We characterise, in a rigorous manner, the delay-independent global asymptotic stability of the unique equilibrium using only the notion of M-matrices [3, 12] and the technique of Lyapunov functionals. Let us begin by recalling that the idea of an artificial neural network equipped with signal transmission time delays was first studied by Marcus et al. [27], and since then, the research area has blossomed. Marcus et al. [27] studied a certain class of Hopfield and Cohen–Grossberg [9, 20, 21] artificial neural networks, and demonstrated that the introduction of discrete signal transmission time delays in the neuronal responses induced sustained oscillations and chaos in the emergent network dynamics. In the electronic implementation of analog artificial neural networks,

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signal transmission time delays are a consequence of the finite switching speed of individual amplifiers (neurons) in the network [8, 27]. It is well-known that time delays abound in biological neuronal networks [10, 25, 27] and in electronic artificial neural networks [27]. Discrete time delays are a good first approximation in mathematical models of simple neural network circuits comprised of only a small number of units or neurons [8, 34]. Such neural network circuits are characterised by a compact network structure, with negligible spatial extent effects. However, the undisputed biophysical reality is that biological neuronal networks are characterised by an intricate spatial structure of parallel neural pathways in the form of axons (or bundles of axons) of varying thicknesses and lengths. As these neural pathways are known to conduct signals between various neurons, it is self-evident that a biophysically reasonable mathematical modelling paradigm for neuronal networks is one that incorporates signal transmission time delays in which the time delays are distributed rather than discrete. Artificial neural networks incorporating discrete time delays have been widely studied in the literature [2, 18, 26, 30, 33]. The problem of neuronal networks endowed with distributed time delays has received some attention in the literature in recent times (see [5, 8, 11, 29, 34] and references therein). Nonetheless, the dynamics of artificial neuronal networks endowed with distributed time delays remain largely poorly understood today. In this article, much of our analysis is inspired by the work of Zhang et al. [34] and Chen [8], who studied a special class of Cohen–Grossberg–Hopfield artificial neural networks endowed with distributed time delays, and whose work in turn was a further development of the results of [13] and [14] who had previously established global asymptotic stability results for a class of additive neural networks without any time delays. Extending the results of Gopalsamy et al. [16] and Hofbauer et al. [19], Campbell [4] established delay independent local and global asymptotic stability results for a certain class of additive neural networks endowed with multiple discrete time delays using technical machinery from matrix theory and the method Lyapunov functionals. Wang et al. [32] studied the asymptotic robust stability of the static neural network model endowed with so-called S-type *finitely* distributed time delays, by employing the framework of Lebesgue–Stieltjes integrals. Oliveira [31] studied the global asymptotic stability of a general class of retarded functional differential equations using ideas from matrix theory and Lebesgue–Stieltjes integration, and avoided employing the well-known technique of Lyapunov functionals. Of particular interest, Oliveira [31] studied the existence and the global asymptotic stability of an equilibrium point in the case of two neural network models with *finitely* distributed time delays without using the technique of Lyapunov functionals, namely, the Cohen–Grossberg and the static models.

Our work in this article draws much of its technical motivation from [4, 6, 16, 19, 31]. In particular, we consider the *infinitely* distributed time-delayed Hopfield-type network [6,20,21] of n artificial neurons described by the system

$$x_k'(t) = -x_k(t) + g_k\left(\sum_{j=1}^{n} a_{kj}\int_0^\infty x_j(t-u)f_{kj}(u)du\right), \qquad k = 1,\ldots,n\,, \tag{1.1}$$

where $a_{kj} \in \mathbb{R}$, $k, j = 1,\ldots,n$, and the nonlinearity g_k is responsible for modulating the activity of the k^{th} neuron. It is clear that the system (1.1) is a generalisation of the static neural network model [31, equation (3), page 82] with multiple general infinitely distributed time delays, and devoid of any external input signals. Construction of a phase space for infinitely distributed time delay systems such as (1.1) is a little nuanced and technically delicate. Let $\rho > 0$ be a fixed real number. An appropriate (see [6], and references contained therein) phase space for systems with infinite time delays, such as (1.1), is the Banach space $\mathcal{C}_n := C_{0,\rho}\left((-\infty, 0], \mathbb{R}^n\right)$

comprising of all continuous $\mathbb{R}^n$-valued functions $\psi(\theta)$ such that the function $e^{\rho\theta}\psi(\theta)$, $\theta \in (-\infty, 0]$, is bounded, uniformly continuous, and satisfies ([23, page 102], [6])

$$\lim_{\theta\to-\infty} e^{\rho\theta}\psi(\theta) = 0\,. \tag{1.2}$$

Furthermore, the Banach space $\mathcal{C}_n$ is equipped with the weighted sup-norm ([23, page 102], [6])

$$\|\psi\|_{\infty,\rho} := \sup_{\theta\in(-\infty,0]} e^{\rho\theta}|\psi(\theta)|\,. \tag{1.3}$$

We assume the following hypotheses on the nonlinearity g_k [4].

(H1) $g_k \in C^2(\mathbb{R})$, $g_k'(u) > 0$, $\sup_{u\in\mathbb{R}} g_k'(u) = g_k'(0) = 1$;

(H2) $g_k(0) = 0$, $\lim_{u\to\pm\infty} g_k(u) = \pm 1$.

Without loss of generality, we adopt throughout this article the specific g_k given by the hyperbolic tangent function

$$g_k(x) = \tanh(\gamma x), \quad \gamma > 0\,. \tag{1.4}$$

We assume that the time delay kernels $f_{kj} : [0,\infty) \mapsto [0,\infty)$, for $k, j = 1, \ldots, n$, are continuous functions satisfying the constraints

$$\int_0^\infty f_{kj}(s)ds = 1\,, \qquad \int_0^\infty s f_{kj}(s)ds < \infty\,, \quad \text{and} \quad f_{kj} = f_{jk}\,, \quad \forall k, j = 1, \ldots, n\,. \tag{1.5}$$

The usual initial conditions associated with (1.1) are given by [8,34]

$$x_k(\theta) = \phi_k(\theta)\,, \quad \theta \in (-\infty, 0]\,, \; k = 1, \ldots, n\,, \tag{1.6}$$

where the ϕ_k are bounded continuous functions on $(-\infty, 0]$. The linearisation of (1.1) about its trivial equilibrium is given by

$$x_k'(t) = -x_k(t) + \sum_{j=1}^n \ell_{kj} \int_{-\infty}^t x_j(s) f_{kj}(t-s)ds\,, \; k = 1, \ldots, n\,, \tag{1.7}$$

where $\ell_{kj} := g_k'(0)a_{kj} = a_{kj} \in \mathbb{R}$, $k, j = 1, \ldots, n$, are constants. With respect to (1.7), let $\mathbb{R}^n \ni x \mapsto (x_1, \ldots, x_n)^\mathsf{T}$ and denote the interconnection matrix by $A := (\ell_{kj}) = (a_{kj})$, $k, j = 1, \ldots, n$.

The goal of the present article is to characterise the existence and uniqueness of an equilibrium of (1.1) on one hand, and the global asymptotic stability of this equilibrium on the other. We do so by appealing to the well-known Banach's contraction mapping principle and by constructing an appropriate Lyapunov functional, and employing arguments from the theory of M-matrices [3,12].

2 Existence and uniqueness of the equilibrium

In this section, we establish sufficient conditions for the existence and uniqueness of an equilibrium point of the system (1.1). The approach adopted here hinges on Banach's contraction mapping theorem, and is largely motivated by the inspirational work of [16] and [4].

Theorem 2.1. *If*

$$\beta := \max_{1\leq j\leq n} \left(\sum_{k=1}^{n} |a_{kj}| \right) < 1 \,, \tag{2.1}$$

then the system of algebraic equations

$$x_k = g_k \left(\sum_{j=1}^{n} a_{kj} x_j \right) \,,\; k = 1, \ldots, n \tag{2.2}$$

admits a unique solution.

Proof. For calculational convenience, let $v_k := x_k$, $k = 1, \ldots, n$, so that (2.2) becomes

$$v_k = g_k \left(\sum_{j=1}^{n} a_{kj} v_j \right) := G_k(v_1, \ldots, v_n) \,,\; k = 1, \ldots, n \,. \tag{2.3}$$

Our goal is to establish the existence of fixed points of the map $\mathbf{G} : \mathbb{R}^n \mapsto \mathbb{R}^n$ defined by $\mathbf{G} := (G_1(\mathbf{v}), \ldots, G_n(\mathbf{v}))$, with $\mathbf{v} := (v_1, \ldots, v_n)$. From the hypotheses (H1) and (H2), we have that

$$-1 \leq g_k \left(\sum_{j=1}^{n} a_{kj} v_j \right) \leq 1 \,, \quad k = 1, \ldots, n \,. \tag{2.4}$$

This observation implies that the set D defined by

$$D := \{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid -1 \leq x_k \leq 1,\; k = 1, \ldots, n\} \tag{2.5}$$

is invariant with respect to the mapping $\mathbf{G}$ [4, 16]. In what follows, we establish that $\mathbf{G}$ is a contraction mapping on D. By Banach's contraction mapping principle, it will follow that $\mathbf{G}$ has a unique fixed point. First, let $\mathbf{v} := (v_1, \ldots, v_n)$ and $\mathbf{u} := (u_1, \ldots, u_n)$. We begin by noting from (2.3) that

$$\begin{aligned}
\|\mathbf{G}(\mathbf{v}) - \mathbf{G}(\mathbf{u})\| &= \sum_{k=1}^{n} |G_k(\mathbf{v}) - G_k(\mathbf{u})| \\
&= \sum_{k=1}^{n} \left| g_k \left(\sum_{j=1}^{n} a_{kj} v_j \right) - g_k \left(\sum_{j=1}^{n} a_{kj} u_j \right) \right| \\
&\leq \sum_{k=1}^{n} |g_k'(\theta_k)| \sum_{j=1}^{n} |a_{kj}| \, |v_j - u_j| \\
&= \sum_{k=1}^{n} c_k \sum_{j=1}^{n} |a_{kj}| \, |v_j - u_j| \\
&= \sum_{j=1}^{n} \left(\sum_{k=1}^{n} c_k \, |a_{kj}| \right) |v_j - u_j| \\
&\leq \beta \sum_{j=1}^{n} |v_j - u_j| \\
&= \beta \, \|\mathbf{v} - \mathbf{u}\| \;,
\end{aligned} \tag{2.6}$$

where

$$\sum_{j=1}^{n} a_{kj} u_j \leq \theta_k \leq \sum_{j=1}^{n} a_{kj} v_j \,, \quad k = 1, \ldots, n \,, \quad c_k := g_k'(\theta_k) \in (0, 1] \,,$$

and

$$\beta := \max_{1\le j\le n}\left(\sum_{k=1}^{n} c_k \left|a_{kj}\right|\right) = \max_{1\le j\le n}\left(\sum_{k=1}^{n} \left|a_{kj}\right|\right) < 1 \tag{2.7}$$

by hypothesis. Without loss of generality, and by recourse to hypothesis (H1), we have here set $c_k = 1$, $\forall k = 1,\dots,n$. Consequently, $\mathbf{G}$ is a contraction on D, and by Banach's contraction mapping principle, it has a unique fixed point, say $\mathbf{v}^* := (v_1^*,\dots,v_n^*)$, such that

$$v_k^* = g_k\left(\sum_{j=1}^{n} a_{kj}v_j^*\right), \quad k = 1,\dots,n\ .$$

Thus, (1.1) has a unique equilibrium point. This completes the proof. □

3 Global asymptotic stability of the equilibrium

We now establish the global asymptotic stability of the equilibrium $\mathbf{x}^* := (x_1^*,\dots,x_n^*)$ of (1.1) by recourse to the theory of M-matrices, and by constructing an appropriate Lyapunov functional. Let $y_k(t) := x_k(t) - x_k^*$, $k = 1,\dots,n$. From the hypothesis (H1) and Lagrange's Mean Value Theorem, there exists

$$\vartheta_k \in \left(\sum_{j=1}^{n} a_{kj}x_j^*\ ,\ \sum_{j=1}^{n} a_{kj}\int_0^\infty y_j(t-u)f_{kj}(u)du + \sum_{j=1}^{n} a_{kj}x_j^*\right), \quad k = 1,\dots,n\ , \tag{3.1}$$

such that

$$g_k\left(\sum_{j=1}^{n} a_{kj}\int_0^\infty y_j(t-u)f_{kj}(u)du + \sum_{j=1}^{n} a_{kj}x_j^*\right) - g_k\left(\sum_{j=1}^{n} a_{kj}x_j^*\right)$$
$$= g_k'\left(\vartheta_k\right)\sum_{j=1}^{n} a_{kj}\int_0^\infty y_j(t-u)f_{kj}(u)du\ , \tag{3.2}$$

for $k = 1,\dots,n$. It is important to stress the fact that ϑ_k identified in (3.1) is not a constant – it depends on the solution y_j, $j = 1,\dots,n$, and the time t. By virtue of the coordinate translation $y_k(t) := x_k(t) - x_k^*$, $k = 1,\dots,n$, and (3.2), the system (1.1) transforms to

$$y_k'(t) = -y_k(t) + g_k\left(\sum_{j=1}^{n} a_{kj}\int_0^\infty \left(y_j(t-u) + x_j^*\right) f_{kj}(u)du\right) - g_k\left(\sum_{j=1}^{n} a_{kj}x_j^*\right),$$
$$k = 1,\dots,n, \tag{3.3}$$

which subsequently leads to the linearisation

$$y_k'(t) = -y_k(t) + c_k\sum_{j=1}^{n} a_{kj}\int_0^\infty y_j(t-u)f_{kj}(u)du, \quad k = 1,\dots,n\ , \tag{3.4}$$

where $c_k := g_k'(\vartheta_k) \in (0,1]$, $\forall k = 1,\dots,n$, by the hypothesis (H1). We note that c_k depends on t, and this observation has some consequential ramifications as will be shown in the analysis to come. Now, borrowing some of the notation of [4], let $A := (a_{kj})$, $|A| := (|a_{kj}|)$, $K := -I + A$, and $\widehat{K} := -I + |A|$, where I is the $n \times n$ identity matrix. Sufficient conditions for the local asymptotic stability of the equilibrium $\mathbf{x}^* := (x_1^*,\dots,x_n^*)$ of (1.1) can be established in a manner analogous to that presented in [4, Theorem 2.6 and Corollary 2.7, page 6], and are given in [6]. To prepare the groundwork for the analysis to follow, we note that the off-diagonal entries of the matrix $-\widehat{K}$ are less than or equal to zero, which means that it is a Z-matrix. The matrix $-\widehat{K}$ is expressible in the form

$$\begin{aligned} -\widehat{K} &:= \begin{pmatrix} 1-|a_{11}| & -|a_{12}| & -|a_{13}| & \cdots & -|a_{1n}| \\ -|a_{21}| & 1-|a_{22}| & -|a_{23}| & \cdots & -|a_{2n}| \\ -|a_{31}| & -|a_{32}| & 1-|a_{33}| & \cdots & -|a_{3n}| \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -|a_{n1}| & -|a_{n2}| & -|a_{n3}| & \cdots & 1-|a_{nn}| \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix} - \begin{pmatrix} |a_{11}| & |a_{12}| & |a_{13}| & \cdots & |a_{1n}| \\ |a_{21}| & |a_{22}| & |a_{23}| & \cdots & |a_{2n}| \\ |a_{31}| & |a_{32}| & |a_{33}| & \cdots & |a_{3n}| \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ |a_{n1}| & |a_{n2}| & |a_{n3}| & \cdots & |a_{nn}| \end{pmatrix} \\ &:= sI - B\,, \end{aligned} \tag{3.5}$$

where B is the non-negative matrix given by

$$B := \begin{pmatrix} |a_{11}| & |a_{12}| & |a_{13}| & \cdots & |a_{1n}| \\ |a_{21}| & |a_{22}| & |a_{23}| & \cdots & |a_{2n}| \\ |a_{31}| & |a_{32}| & |a_{33}| & \cdots & |a_{3n}| \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ |a_{n1}| & |a_{n2}| & |a_{n3}| & \cdots & |a_{nn}| \end{pmatrix}, \tag{3.6}$$

$s := 1 > 0$, and I is the $n \times n$ identity matrix. The following lemma will be instrumental in the proof of our main result in the present Section.

Lemma 3.1. *If $-\widehat{K}$ is a Z-matrix and $\rho(B) < 1$, then $-\widehat{K}$ is a non-singular M-matrix.*

Proof. That $-\widehat{K}$ is a Z-matrix is trivial. Suppose that $\rho(B) < 1$. Since $-\widehat{K} = I - B$, the result follows [12, page 129, Theorem 5.1.1.]. □

As an example to amplify the implication of Lemma 3.1, consider $n = 2$ populations of artificial neurons, with $a_{11} = a_{22} = 0$, $a_{12} = 2$, and $a_{21} = 1$. Then, we have that

$$\begin{aligned} A = (a_{kj}) = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \begin{pmatrix} 0 & 2 \\ 1 & 0 \end{pmatrix} \\ \Longrightarrow -\widehat{K} = \begin{pmatrix} 1 & -2 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 0 & 2 \\ 1 & 0 \end{pmatrix} := I - B\,, \end{aligned} \tag{3.7}$$

where $\rho(B) = \sqrt{2} > 1$. Hence, the matrix $-\widehat{K}$ in this example is not a non-singular M-matrix for the simple reason that it does not satisfy at least one of the hypotheses stipulated in Lemma 3.1. In the view of Lemma 3.1, we arrive at our main result in the present Section.

Theorem 3.2. *If $-\widehat{K}$ is a non-singular M-matrix, then the system (1.1) has a unique globally asymptotically stable equilibrium.*

Proof. Assume that $-\widehat{K}$ is a non-singular M-matrix. That is, assume that $-\widehat{K} = I - B$ is a Z-matrix and that $\rho(B) < 1$ [12, page 129, Theorem 5.1.1.]. It is well-known that if the spectral radius of a matrix is less than 1, then the matrix has a norm which is less than 1 [22, page 347, Lemma 5.6.10]. Since $B = (|a_{ij}|)$, $i, j = 1, \ldots, n$, the maximum column sum matrix norm of B is given by [22]

$$|||B|||_1 = \max_{1 \le j \le n} \left(\sum_{i=1}^{n} |a_{ij}| \right) < 1\,, \tag{3.8}$$

which is identical to the hypothesis of Theorem 2.1.

Now, since $-\widehat{K} := I - |A|$ is a non-singular M-matrix from Lemma 3.1, it follows [3, 12] that $\exists\, \xi_j > 0$, $j = 1, \ldots, n$, such that

$$-\xi_j + \sum_{k=1}^{n} |a_{kj}| \xi_k < 0\,, \quad j = 1, \ldots, n\,. \tag{3.9}$$

Consider the Lyapunov functional $V(t) = V(y)(t)$ defined by [4, 8, 32, 34]

$$V(y)(t) := \sum_{k=1}^{n} \xi_k \left\{ |y_k(t)| + \sum_{j=1}^{n} |a_{kj}| \int_0^\infty f_{kj}(s) \left(\int_{t-s}^{t} |y_j(\tau)| d\tau \right) ds \right\}\,. \tag{3.10}$$

Computing the upper Dini derivative of (3.10) along the solutions of the nonlinear system (3.3) yields

$$\begin{aligned}
D^+V(t) &= \sum_{k=1}^{n} \xi_k \left\{ \operatorname{sgn}(y_k(t)) y_k'(t) + \sum_{j=1}^{n} |a_{kj}| \int_0^\infty f_{kj}(s) \left(|y_j(t)| - |y_j(t-s)| \right) ds \right\} \\
&= \sum_{k=1}^{n} \xi_k \left\{ -\operatorname{sgn}(y_k(t)) y_k(t) + \operatorname{sgn}(y_k(t)) g_k \left(\sum_{j=1}^{n} a_{kj} \int_0^\infty \left(y_j(t-u) + x_j^* \right) f_{kj}(u) du \right) \right. \\
&\quad \left. -\operatorname{sgn}(y_k(t)) g_k \left(\sum_{j=1}^{n} a_{kj} x_j^* \right) + \sum_{j=1}^{n} |a_{kj}| \int_0^\infty f_{kj}(s) \left(|y_j(t)| - |y_j(t-s)| \right) ds \right\} \\
&\le \sum_{k=1}^{n} \xi_k \left\{ -|y_k(t)| + \left| g_k \left(\sum_{j=1}^{n} a_{kj} \int_0^\infty \left(y_j(t-u) + x_j^* \right) f_{kj}(u) du \right) - g_k \left(\sum_{j=1}^{n} a_{kj} x_j^* \right) \right| \right. \\
&\quad \left. + \sum_{j=1}^{n} |a_{kj}| \int_0^\infty |y_j(t)| f_{kj}(s) ds - \sum_{j=1}^{n} |a_{kj}| \int_0^\infty |y_j(t-s)| f_{kj}(s) ds \right\} \\
&\le \sum_{k=1}^{n} \xi_k \left\{ -|y_k(t)| + \sum_{j=1}^{n} |a_{kj}| \int_0^\infty |y_j(t-u)| f_{kj}(u) du + \sum_{j=1}^{n} |a_{kj}| |y_j(t)| \right. \\
&\quad \left. - \sum_{j=1}^{n} |a_{kj}| \int_0^\infty |y_j(t-u)| f_{kj}(u) du \right\} \\
&= \sum_{k=1}^{n} \xi_k \left\{ -|y_k(t)| + \sum_{j=1}^{n} |a_{kj}| |y_j(t)| \right\} \\
&= \sum_{k=1}^{n} \left(-\xi_k |y_k(t)| \right) + \sum_{k=1}^{n} \sum_{j=1}^{n} \left(|a_{jk}| \xi_j |y_k(t)| \right) \\
&= \sum_{k=1}^{n} \left(-\xi_k + \sum_{j=1}^{n} |a_{jk}| \xi_j \right) |y_k(t)| \le \mu \sum_{k=1}^{n} |y_k(t)| < 0\,,
\end{aligned}$$

where, by virtue of the condition (3.9),

$$\mu := \max_{1\le k\le n}\left\{-\xi_k + \sum_{j=1}^{n} |a_{jk}|\xi_j\right\} < 0\,. \tag{3.11}$$

Hence, the trivial equilibrium of (3.3) is globally asymptotically stable [24, corollary 5.2, page 30]. Therefore, the equilibrium $\mathbf{x}^* := (x_1^*, \dots, x_n^*)$ of (1.1) is globally asymptotically stable (see [1,4,17] and [15, pages 4-5]). This completes the proof. □

4 A numerical example

We give a numerical example to illustrate an application of Theorem 3.2. Consider $n = 2$ populations of artificial neurons, with $a_{11} = a_{22} = \frac{1}{2}$, $a_{12} = \frac{1}{16}$, and $a_{21} = 1$. Thus, we have that

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & \frac{1}{16} \\ 1 & \frac{1}{2} \end{pmatrix}$$
$$\Rightarrow -\widehat{K} = I - |A| = \begin{pmatrix} \frac{1}{2} & -\frac{1}{16} \\ -1 & \frac{1}{2} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} \frac{1}{2} & \frac{1}{16} \\ 1 & \frac{1}{2} \end{pmatrix} := I - B\,, \tag{4.1}$$

with $\rho(B) = \frac{3}{4} < 1$. That $-\widehat{K}$ is a Z-matrix is trivial. This observation, in conjunction with the fact that $\rho(B) < 1$, implies that $-\widehat{K}$ is a non-singular M-matrix by Lemma 3.1. For the specified interconnection matrix A, the system (1.1) condenses to

$$\begin{cases} x_1'(t) = -x_1(t) + g_1\left(\sum_{j=1}^{2} a_{1j}\int_0^\infty x_j(t-u)f_{1j}(u)du\right)\,, \\ x_2'(t) = -x_2(t) + g_2\left(\sum_{j=1}^{2} a_{2j}\int_0^\infty x_j(t-u)f_{2j}(u)du\right)\,, \end{cases} \tag{4.2}$$

with the initial conditions given in (1.6) for $n = 2$. Since $-\widehat{K}$ is a non-singular M-matrix, we are guaranteed by Theorem 3.2 that the system (4.2) admits a unique globally asymptotically stable equilibrium. For the sake of completeness, we establish the existence and uniqueness of an equilibrium of (4.2). Now, since $\rho(B) < 1$, it follows that there exists a matrix norm such that $|||A||| < 1$ [22, page 347, Lemma 5.6.10]. To characterise such a norm, we proceed in the manner adumbrated below. Let $J := P^{-1}AP = \mathrm{diag}\left(\frac{1}{4}, \frac{3}{4}\right)$ be the Jordan form of A, with $P := \begin{pmatrix} -\frac{1}{4} & \frac{1}{4} \\ 1 & 1 \end{pmatrix}$, and let $D := I$ be the 2×2 identity matrix. Note that the matrix A has eigenvalues $\lambda_1 := \frac{1}{4}$ and $\lambda_2 := \frac{3}{4}$. The two columns of P are the eigenvectors of A. The eigenspace for $\lambda_1 = \frac{1}{4}$ is spanned by $\mathbf{u} := \begin{pmatrix} -\frac{1}{4} \\ 1 \end{pmatrix}$ whilst that for $\lambda_2 = \frac{3}{4}$ is spanned by $\mathbf{v} := \begin{pmatrix} \frac{1}{4} \\ 1 \end{pmatrix}$. Now, define a norm by $|||A||| := |||D^{-1}P^{-1}APD|||_p = |||P^{-1}AP|||_p$, where $|||\cdot|||_p$ denotes the induced p-norm. In other words,

$$|||A||| := \sup_{\mathbf{x}\neq\mathbf{0}} \frac{\|A\mathbf{x}\|_p}{\|\mathbf{x}\|_p}\,, \tag{4.3}$$

where the two norms $\|\cdot\|_p$ on the right hand side denote the usual p-norm for vectors. When $p = 1$, $|||A|||$ is identical to the maximum column sum of the entrywise absolute value of

A. For the matrix P in this example, we have that $P^{-1}AP = \operatorname{diag}\left(\frac{1}{4}, \frac{3}{4}\right)$, and consequently, $|||A||| = \max_{1\leq j\leq 2}\left(\sum_{i=1}^{2}|a_{ij}|\right) = \frac{3}{4} < 1$; this last inequality matches the hypothesis (2.1) of Theorem 2.1. Hence, the existence and uniqueness of an equilibrium of the system (4.2) is guaranteed.

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