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Nonlinear resonant problems with an indefinite potential and concave boundary condition

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Abstract. We consider a nonlinear elliptic problem driven by the p-Laplacian plus and indefinite potential term. The reaction is $(p-1)$-linear and resonant and the boundary term is concave. The problem is nonparametric. Using variational tools, together with truncation and perturbation techniques and critical groups, we show that the problem has at least three nontrivial smooth solutions.

Keywords: resonant reaction, concave boundary term, critical group, nonlinear regularity, multiple solutions.

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1 Introduction

Let $\Omega \subseteq \mathbb{R}^N$ be a bounded domain with a C^2-boundary $\partial\Omega$. In this paper, we deal with the following nonlinear boundary value problem

$$\begin{cases} -\Delta_p u(z) + \xi(z)|u(z)|^{p-2}u(z) = f(z,u(z)) & \text{in } \Omega, \\ \dfrac{\partial u}{\partial n_p} = \beta(z)|u|^{q-2}u & \text{on } \partial\Omega. \end{cases} \tag{1.1}$$

with $1 < q < p$.

In this problem, Δ_p denotes the p-Laplace differential operator defined by

$$\Delta_p u = \operatorname{div}\left(|Du|^{p-2}Du\right) \quad \text{for all } u \in W^{1,p}(\Omega),\ 1 < p < N.$$

This problem has three special features which make its study interesting. The first feature is that the potential coefficient $\xi \in L^\infty(\Omega)$ is indefinite (that is, sign changing) and so the left hand side of the problem is noncoercive. The second feature is that the forcing term $f(z,x)$ which is a Carathéodory function (that is, for all $x \in \mathbb{R}$, $z \mapsto f(z,x)$ is measurable and for a.a. $z \in \Omega$, $x \mapsto f(z,x)$ is continuous) asymptotically as $x \to \pm\infty$ is resonant with respect to

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the principal eigenvalue of the differential operator $u \mapsto -\Delta_p u + \xi(z)|u|^{p-2}u$ with Neumann boundary condition. So, the problem is resonant and as it is well-known such problems are more difficult to deal with. The third feature is that combined with the resonant reaction, we have a concave boundary term (since $\beta(z) \geq 0$ for all $z \in \partial\Omega$ and $1 < q < p$). Therefore problem (1.1) is a variant of the classical *concave-convex* problem, in which the convex ($(p-1)$-superlinear) term is replaced by a resonant ($(p-1)$-linear) term and the *concave* contribution comes from the boundary condition. Problems with such *competition* phenomena, were studied recently by Abreu–Madeira [1], Hu–Papageorgiou [6], Papageorgiou–Rădulescu [9], Papageorgiou–Scapellato [12] and Sabina de Lis–Segura de Leon [14]. All these works deal with parametric problems. The presence of a parameter in the problem, makes the analysis easier, since by varying and restricting the parameter, we are able to satisfy the relevant geometry in order to apply the minimax theorems of critical point theory. In our problem there is no parameter. In addition, in all the aforementioned works the reaction is $(p-1)$-superlinear and so do not cover the resonant case treated here.

In the boundary condition, $\frac{\partial u}{\partial n_p}$ denotes the conormal derivative of $u \in W^{1,p}(\Omega)$. It is interpreted using the nonlinear Green's identity (see [11, p. 35]) and if $u \in W^{1,p}(\Omega) \cap C^{0,1}(\overline{\Omega})$, then

$$\frac{\partial u}{\partial n_p} = |Du|^{p-2}(Du, n)_{\mathbb{R}^N} = |Du|^{p-2}\frac{\partial u}{\partial n},$$

with $n(\cdot)$ being the outward unit normal on $\partial\Omega$.

Using variational tools based on the critical point theory together with critical groups (Morse theory), we show that problem (1.1) has at least three nontrivial smooth solutions.

2 Mathematical background – hypotheses

The study of problem (1.1), uses the Sobolev space $W^{1,p}(\Omega)$, the Banach space $C^1(\overline{\Omega})$ and the boundary Lebesgue spaces $L^\tau(\partial\Omega)$, $1 \leq \tau < \infty$.

By $\|\cdot\|$ we denote the norm of the Sobolev space $W^{1,p}(\Omega)$. We have

$$\|u\| = \left[\|u\|_p^p + \|Du\|_p^p\right]^{\frac{1}{p}} \quad \text{for all } u \in W^{1,p}(\Omega).$$

The Banach space $C^1(\overline{\Omega})$ is ordered using the positive (order) cone

$$C_+ = \left\{u \in C^1(\overline{\Omega}) : u(z) \geq 0 \ \text{ for all } z \in \overline{\Omega}\right\}.$$

This cone has a nonempty interior given by

$$\operatorname{int} C_+ = \left\{u \in C_+ : u(z) > 0 \ \text{ for all } z \in \overline{\Omega}\right\}.$$

Also by $\sigma(\cdot)$ we denote the $(N-1)$-dimensional Hausdorff (surface) measure on $\partial\Omega$. Using this measure, we can define the boundary Lebesgue spaces $L^\tau(\partial\Omega)$, $1 \leq \tau < \infty$. By $\gamma_0 : W^{1,p}(\Omega) \to L^p(\partial\Omega)$ we denote the *trace map*. This map is linear, compact and $\gamma_0(u) = u|_{\partial\Omega}$ for all $u \in W^{1,p}(\Omega) \cap C(\overline{\Omega})$. So, the trace map defines boundary values for all Sobolev functions. In the sequel, we drop the use of the trace map $\gamma_0(\cdot)$ and all restrictions of Sobolev functions on $\partial\Omega$, are interpreted in the sense of traces.

Let $\langle\cdot,\cdot\rangle$ denote the duality brackets for the pair $(W^{1,p}(\Omega), W^{1,p}(\Omega)^*)$ and consider the map $A : W^{1,p}(\Omega) \to W^{1,p}(\Omega)^*$ to be the nonlinear operator defined by

$$\langle A(u), h\rangle = \int_\Omega |Du|^{p-2}(Du, Dh)_{\mathbb{R}^N}\, dz \quad \text{for all } u, h \in W^{1,p}(\Omega).$$

From Gasiński–Papageorgiou [5] (p. 279), we have that this map is:

- monotone, continuous (hence maximal monotone too) and maps bounded sets to bounded sets;
- it is of type $(S)_+$, that is,

$$u_n \xrightarrow{w} u \text{ in } W^{1,p}(\Omega) \text{ and } \limsup_{n\to\infty}\langle A(u_n), u_n - u\rangle \le 0$$
$$\text{imply that}$$
$$u_n \to u \text{ in } W^{1,p}(\Omega) \text{ as } n \to \infty.$$

Let $\xi \in L^\infty(\Omega)$ and consider the following nonlinear eigenvalue problem

$$\begin{cases} -\Delta_p u(z) + \xi(z)|u(z)|^{p-2}u(z) = \widehat{\lambda}|u(z)|^{p-2}u(z) & \text{in } \Omega, \\ \dfrac{\partial u}{\partial n_p} = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.1}$$

We say that $\widehat{\lambda} \in \mathbb{R}$ is an *eigenvalue*, if (2.1) admits a nontrivial solution $\widehat{u} \in W^{1,p}(\Omega)$ known as an eigenfunction corresponding to the eigenvalue $\widehat{\lambda}$.

Problem (2.1) was studied by Fragnelli–Mugnai–Papageorgiou [3] (Robin problem) and Mugnai–Papageorgiou [8] (Neumann problem), who proved that there is a smallest eigenvalue $\widehat{\lambda}_1 \in \mathbb{R}$ with the following properties:

(a) $\widehat{\lambda}_1$ is isolated, that is, if $\widehat{\sigma}(p)$ denotes the spectrum of (2.1), then we can find $\epsilon > 0$ small such that $(\widehat{\lambda}_1, \widehat{\lambda}_1 + \epsilon) \cap \widehat{\sigma}(p) = \varnothing$.

(b) $\widehat{\lambda}_1$ is simple, that is, if $\widehat{u}, \widehat{v} \in W^{1,p}(\Omega)$ are eigenfunctions corresponding to $\widehat{\lambda}_1$, then $\widehat{u} = \vartheta\widehat{v}$ for some $\vartheta \in \mathbb{R} \setminus \{0\}$.

(c) If $\gamma(u) = \|Du\|_p^p + \int_\Omega \xi(z)|u|^p\,dz$ for all $u \in W^{1,p}(\Omega)$, then

$$\widehat{\lambda}_1 = \inf\left[\frac{\gamma(u)}{\|u\|_p^p} : u \in W^{1,p}(\Omega),\ u \neq 0\right]. \tag{2.2}$$

In (2.2) the infimum is realized on the corresponding one dimensional eigenspace (see (b)). Then, it follows that the elements of this eigenspace have fixed sign. By $\widehat{u}_1 \in W^{1,p}(\Omega)$ we denote the positive, L^p-normalized (that is, $\|\widehat{u}_1\|_p = 1$) eigenfunction corresponding to $\widehat{\lambda}_1$. The nonlinear regularity theory of Lieberman [7] and the nonlinear maximum principle (see, for example, Gasiński–Papageorgiou [4], p. 738), imply that $\widehat{u}_1 \in \mathrm{int}C_+$. We mention that $\widehat{\lambda}_1$ is the only eigenvalue with eigenfunctions of constant sign. All other eigenvalues have nodal (that is, sign changing) eigenfunctions. Note that using the Ljusternik–Schnirelmann minimax scheme, we can generate a whole strictly increasing sequence $\{\widehat{\lambda}_k\}_{k\ge1}$ of eigenvalues such that $\widehat{\lambda}_k \to +\infty$ as $k \to +\infty$. We do not know if this sequence exhausts $\widehat{\sigma}(p)$.

Let X be a Banach space and $\varphi \in C^1(X, \mathbb{R})$, $c \in \mathbb{R}$. We introduce the following two sets

$$\varphi^c = \{u \in X : \varphi(u) \le c\},$$
$$K_\varphi = \{u \in X : \varphi'(u) = 0\} \quad \text{(the critical set of } \varphi).$$

Let (Y_1, Y_2) be a topological pair such that $Y_2 \subseteq Y_1 \subseteq X$ and $k \in \mathbb{N}_0$. By $H_k(Y_1, Y_2)$ we denote the k^{th}-relative singular homology group for the pair (Y_1, Y_2) with integer coefficients. If $u \in K_\varphi$ is isolated and $c = \varphi(u)$, then the critical groups of φ at u are defined by

$$C_k(\varphi, u) = H_k(\varphi^c \cap U, \varphi^c \cap U \setminus \{u\}) \quad \text{for all } k \in \mathbb{N}_0,$$

with U being a neighborhood of u such that $\varphi^c \cap U \cap K_\varphi = \{u\}$. The excision property of singular homology, implies that the above definition is independent of the isolating neighborhood.

Finally, we fix some basic notation. Given $x \in \mathbb{R}$, we set $x^\pm = \max\{\pm x, 0\}$. Then, for $u \in W^{1,p}(\Omega)$, we define $u^\pm(z) = u(z)^\pm$ for all $z \in \Omega$. We have

$$u^\pm \in W^{1,p}(\Omega), \quad u = u^+ - u^-, \quad |u| = u^+ + u^-.$$

If $u, v \in W^{1,p}(\Omega)$ and $u \le v$, then

$$[u, v] = \left\{ h \in W^{1,p}(\Omega) : u(z) \le h(z) \le v(z) \ \text{ for a.a. } z \in \Omega \right\}.$$

Our hypotheses on the data of problem (1.1) are the following:

H_0: $\xi \in L^\infty(\Omega)$, $\beta \in C^{0,\alpha}(\partial\Omega)$ with $\alpha \in (0,1)$ and $\beta(z) > 0$ for all $z \in \partial\Omega$.

H_1: $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function such that $f(z, 0) = 0$ for a.a. $z \in \Omega$ and

(i) $|f(z,x)| \le a(z)[1 + |x|^{p-1}]$ for a.a. $z \in \Omega$, all $x \in \mathbb{R}$, with $a \in L^\infty(\Omega)$;

(ii) if $F(z,x) = \int_0^x f(z,s)\,\mathrm{d}s$, then $\lim\limits_{x \to \pm\infty} \dfrac{pF(z,x)}{|x|^p} \le \widehat{\lambda}_1(p)$ uniformly for a.a. $z \in \Omega$;

(iii) there exists $\tau \in (q, p)$ such that

$$0 < \gamma_0 \le \liminf_{x \to \pm\infty} \frac{f(z,x)x - pF(z,x)}{|x|^\tau} \quad \text{uniformly for a.a. } z \in \Omega;$$

(iv) there exist $\delta_0 > 0$, $\widehat{c} > \|\xi^-\|_\infty$ and $\mu \in [q, p)$ such that

$$\widehat{c}|x|^p \le F(z,x) \quad \text{for a.a. } z \in \Omega, \text{ all } |x| \le \delta_0$$

and

$$\mu F(z,x) - f(z,x)x \ge 0 \quad \text{for a.a. } z \in \Omega, \text{ all } |x| \le \delta_0.$$

Remarks 2.1. Hypotheses H_1(i),(ii), imply that the reaction $f(z, \cdot)$ is $(p-1)$-linear as $x \to \pm\infty$ and the problem is resonant with respect to $\widehat{\lambda}_1(p)$. Note that the resonance condition (hypothesis H_1(ii)) is formulated in terms of the primitive $F(z,x)$ which is more general.

3 Solutions of constant sign

In this section, we prove the existence of two nontrivial smooth solutions of constant sign (one positive and the other negative).

To this end, let $\eta > \|\xi\|_\infty$ and consider the following two C^1-functionals $\varphi_\pm : W^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\varphi_+(u) = \frac{1}{p}\|Du\|_p^p + \frac{1}{p}\int_\Omega [\xi(z)+\eta]|u|^p\,\mathrm{d}z - \frac{1}{q}\int_{\partial\Omega}\beta(z)(u^+)^q\,\mathrm{d}\sigma - \int_\Omega \left[F(z,u^+) + \frac{\eta}{p}(u^+)^p\right]\mathrm{d}z,$$
$$\varphi_-(u) = \frac{1}{p}\|Du\|_p^p + \frac{1}{p}\int_\Omega [\xi(z)+\eta]|u|^p\mathrm{d}z + \frac{1}{q}\int_{\partial\Omega}\beta(z)(u^-)^q\mathrm{d}\sigma - \int_\Omega \left[F(z,-u^-) - \frac{\eta}{p}(u^-)^p\right]\mathrm{d}z,$$

for all $u \in W^{1,p}(\Omega)$.

We show that these functionals are coercive.

Proposition 3.1. *If hypotheses* H_0, H_1 *hold, then the functionals* $\varphi_\pm(\cdot)$ *are both coercive.*

Proof. We do the proof for $\varphi_+(\cdot)$, the proof for $\varphi_-(\cdot)$ being similar.

We argue by contradiction. So, suppose that $\varphi_+(\cdot)$ is not coercive. Then we can find a sequence $\{u_n\}_{n\geq 1} \subseteq W^{1,p}(\Omega)$ such that

$$\|u_n\| \to \infty \quad \text{and} \quad \varphi(u_n) \leq M_1 \quad \text{for some } M_1 > 0, \text{ all } n \in \mathbb{N}. \tag{3.1}$$

Then we have

$$\begin{aligned} M_1 &\geq \varphi_+(u_n) \\ &= \frac{1}{p}\left[\|Du_n^+\|_p^p + \int_\Omega \xi(z)(u_n^+)^p\,\mathrm{d}z\right] + \frac{1}{p}\left[\|Du_n^-\|_p^p + \int_\Omega [\xi(z)+\eta](u_n^-)^p\,\mathrm{d}z\right] - \\ &\quad - \frac{1}{q}\int_{\partial\Omega}\beta(z)(u_n^+)^q\,\mathrm{d}\sigma - \int_\Omega F(z,u_n^+)\,\mathrm{d}z \\ &\geq \frac{1}{p}\left[\|Du_n^+\|_p^p + \int_\Omega \xi(z)(u_n^+)^p\,\mathrm{d}z\right] - \frac{1}{q}\int_{\partial\Omega}\beta(z)(u_n^+)^q\,\mathrm{d}\sigma - \int_\Omega F(z,u_n^+)\,\mathrm{d}z \\ &\quad (\text{since } \eta > \|\xi\|_\infty). \end{aligned} \tag{3.2}$$

We will use (3.2) to show that $\{u_n^+\}_{n\geq 1} \subseteq W^{1,p}(\Omega)$ is bounded. We proceed indirectly. So, suppose that at least for a subsequence, we have

$$\|u_n^+\| \to \infty \quad \text{as } n \to \infty. \tag{3.3}$$

Let $y_n = \frac{u_n^+}{\|u_n^+\|}$, $n \in \mathbb{N}$. We have $\|y_n\| = 1$, $y_n \geq 0$ for all $n \in \mathbb{N}$ and so we may assume that

$$y_n \xrightarrow{w} y \quad \text{in } W^{1,p}(\Omega) \quad \text{and} \quad y_n \to y \quad \text{in } L^p(\Omega) \text{ and in } L^p(\partial\Omega); \quad y \geq 0. \tag{3.4}$$

From (3.2) we have

$$\frac{1}{p}\left[\|Dy_n\|_p^p + \int_\Omega \xi(z)y_n^p\,\mathrm{d}z\right] - \frac{1}{q\|u_n^+\|^{p-q}}\int_{\partial\Omega}\beta(z)y_n^q\,\mathrm{d}\sigma - \int_\Omega \frac{F(z,u_n^+)}{\|u_n^+\|^p}\,\mathrm{d}z \leq \frac{M_1}{\|u_n^+\|^p}, \tag{3.5}$$

for all $n \in \mathbb{N}$

Hypothesis H_1(i) implies that

$$\begin{aligned} &|F(z,x)| \leq c_1[1+|x|^p] \quad \text{for a.a. } z \in \Omega, \text{ all } x \in \mathbb{R}, \text{ some } c_1 > 0, \\ \Rightarrow \quad &\left\{\frac{F(\cdot,u_n^+(\cdot))}{\|u_n^+\|^p}\right\}_{n\geq 1} \subseteq L^1(\Omega) \quad \text{is uniformly integrable.} \end{aligned}$$

Then, by the Dunford–Pettis theorem (see Papageorgiou–Winkert [13], Theorem 4.1.18, p. 289), we have that $\{\frac{F(\cdot,u_n^+(\cdot))}{\|u_n^+\|^p}\}_{n\geq 1} \subseteq L^1(\Omega)$ is relatively weakly compact. Then, by the Eberlein–Smulian theorem and by passing to a subsequence if necessary, we have

$$\frac{F(\cdot,u_n^+(\cdot))}{\|u_n^+\|^p} \xrightarrow{w} \frac{1}{p}\vartheta(\cdot)y(\cdot)^p \quad \text{in } L^1(\Omega) \text{ as } n\to\infty \tag{3.6}$$

with $\vartheta \in L^\infty(\Omega)$, $\vartheta(z) \leq \widehat{\lambda}_1(p)$ for a.a. $z\in\Omega$

(see hypothesis H_1(ii) and Aizicovici–Papageorgiou–Staicu [2], proof of Proposition 16).

We return to (3.5), pass to the limit as $n\to\infty$ and use (3.4), (3.3), (3.6) and the fact that $q<p$, to obtain

$$\|Dy\|_p^p + \int_\Omega \xi(z)y^p\,\mathrm{d}z \leq \int_\Omega \vartheta(z)y^p\,\mathrm{d}z. \tag{3.7}$$

First suppose that $\vartheta \not\equiv \widehat{\lambda}_1(p)$ (see (3.6)). Then from (3.7) and Mugnai–Papageorgiou [8] (Lemma 4.11), we have

$$\begin{aligned} & c_2\|y\|^p \leq 0 \quad \text{for some } c_2>0,\\ \Rightarrow\quad & y=0. \end{aligned} \tag{3.8}$$

Then from (3.5), (3.7), (3.8), (3.4) and (3.6), we obtain

$$\begin{aligned} & \|Du_n\|_p \to 0,\\ \Rightarrow\quad & y_n\to 0 \quad \text{in } W^{1,p}(\Omega), \end{aligned}$$

a contradiction since $\|y_n\|=1$ for all $n\in\mathbb{N}$.

Next we suppose that $\vartheta(z)=\widehat{\lambda}_1(p)$ for a.a. $z\in\Omega$. Then from (3.7) and (2.2), we have that

$$y=\mu\widehat{u}_1(p) \quad \text{with } \mu\geq 0 \text{ (recall that } y\geq 0).$$

If $\mu=0$, then $y=0$ and as above, we show that

$$y_n\to 0 \quad \text{in } W^{1,p}(\Omega),$$

a contradiction to the fact that $\|y_n\|=1$ for all $n\in\mathbb{N}$. So, suppose $\mu>0$. Then $y\in \operatorname{int} C_+$. This implies that

$$u_n^+(z)\to+\infty \quad \text{for a.a. } z\in\Omega. \tag{3.9}$$

From (3.2) we have

$$M_1 \geq \frac{1}{p}\int_\Omega \left[\widehat{\lambda}_1(p)(u_n^+)^p - pF(z,u_n^+)\right]\mathrm{d}z - \frac{1}{q}\int_{\partial\Omega}\beta(z)(u_n^+)^q\,\mathrm{d}\sigma$$

(see (3.7), (2.2) and recall that $\eta>\|\eta\|_\infty$),

$$\Rightarrow\quad \frac{M_1}{\|u_n^+\|^\tau} \geq \frac{1}{p}\int_\Omega \frac{\left[\widehat{\lambda}_1(p)(u_n^+)^p - pF(z,u_n^+)\right]}{(u_n^+)^p}y_n^p\,\mathrm{d}z - \frac{1}{q\|u_n^+\|^{p-q}}\int_{\partial\Omega}\beta(z)y_n^q\,\mathrm{d}\sigma, \tag{3.10}$$

for all $n\in\mathbb{N}$.

On $\mathring{\mathbb{R}}_+=(0,\infty)$ we have

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{F(z,x)}{x^p}\right] = \frac{f(z,x)x^p - px^{p-1}F(z,x)}{x^{2p}} = \frac{f(z,x)x-pF(z,x)}{x^{p+1}}.$$

On account of hypothesis H_1(iii), we can find $\gamma_1 \in (0, \gamma_0)$ and $M_2 > 0$ such that

$$\begin{aligned}
&\frac{f(z,x)x - pF(z,x)}{x^{p+1}} \geq \frac{\gamma_1}{x^{p+1-\tau}} \quad \text{for a.a. } z \in \Omega, \text{ all } x \geq M_2, \\
\Rightarrow\ & \frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{F(z,x)}{x^p}\right] \geq \frac{\gamma_1}{x^{p+1-\tau}} \quad \text{for a.a. } z \in \Omega, \text{ all } x \geq M_2, \\
\Rightarrow\ & \frac{F(z,v)}{v^p} - \frac{F(z,x)}{x^p} \geq \frac{\gamma_1}{p-\tau}\left[\frac{1}{x^{p-\tau}} - \frac{1}{v^{p-\tau}}\right] \quad \text{for a.a. } z \in \Omega, \text{ all } v \geq x \geq M_2.
\end{aligned}$$

Passing to the limit as $v \to \infty$ and since $\frac{F(z,v)}{v^p} \to \frac{1}{p}\widehat{\lambda}_1(p)$ as $v \to +\infty$, we obtain

$$\begin{aligned}
&\frac{\widehat{\lambda}_1(p)}{p} - \frac{F(z,x)}{x^p} \geq \frac{\gamma_1}{p-\tau} \cdot \frac{1}{x^{p-\tau}} \quad \text{for a.a. } z \in \Omega, \text{ all } x \geq M_2, \\
\Rightarrow\ & \frac{\widehat{\lambda}_1(p)x^p - pF(z,x)}{x^\tau} \geq \frac{p\gamma_1}{p-\tau} \quad \text{for a.a. } z \in \Omega, \text{ all } x \geq M_2, \\
\Rightarrow\ & \liminf_{x\to+\infty} \frac{\widehat{\lambda}_1(p)x^p - pF(z,x)}{x^\tau} \geq \frac{p\gamma_1}{p-\tau} > 0 \quad \text{uniformly for a.a. } z \in \Omega. \qquad (3.11)
\end{aligned}$$

Returning to (3.10), passing to the limit as $n \to \infty$ and using (3.9), (3.11) and Fatou's lemma, we obtain

$$0 \geq \liminf_{n\to\infty} \int_\Omega \frac{\left[\widehat{\lambda}_1(p)(u_n^+)^p - pF(z,u_n^+)\right]}{(u_n^+)^p} y_n^p \,\mathrm{d}z > 0$$
(recall that $q < p$ and see (3.3)),

a contradiction. We infer that

$$\{u_n^+\}_{n\geq 1} \subseteq W^{1,p}(\Omega) \quad \text{is bounded.} \qquad (3.12)$$

From (3.1) and (3.12), we have

$$\begin{aligned}
&\frac{1}{p}\left[\|Du_n^-\|_p^p + \int_\Omega [\xi(z)+\eta](u_n^-)^p \,\mathrm{d}z\right] \leq M_3 \quad \text{for some } M_3 > 0, \text{ all } n \in \mathbb{N}, \\
\Rightarrow\ & c_3\|u_n^-\|^p \leq M_3 \quad \text{for some } c_3 > 0, \text{ all } n \in \mathbb{N}, \\
\Rightarrow\ & \{u_n^-\} \subseteq W^{1,p}(\Omega) \quad \text{is bounded.} \qquad (3.13)
\end{aligned}$$

From (3.12) and (3.13) it follows that

$$\{u_n\}_{n\geq 1} \subseteq W^{1,p}(\Omega) \quad \text{is bounded,}$$

which contradicts (3.1). This proves that $\varphi_+(\cdot)$ is coercive.
In a similar fashion we show that $\varphi_-(\cdot)$ is coercive too. □

Now we are ready to produce the two constant sign solutions.

Proposition 3.2. *If hypotheses* H_0, H_1 *hold, then problem* (1.1) *has at least two constant sign smooth solutions*

$$u_0 \in \mathrm{int}C_+ \quad \text{and} \quad v_0 \in -\mathrm{int}C_+.$$

Proof. From Proposition 3.1 we know that $\varphi_+(\cdot)$ is coercive. Also by the Sobolev embedding theorem and the compactness of the trace map, we see that $\varphi_+(\cdot)$ is sequentially weakly lower semicontinuous. So, by the Weierstrass-Tonelli theorem, we can find $u_0 \in W^{1,p}(\Omega)$ such that

$$\varphi_+(u_0) = \min \left[\varphi_+(u) : u \in W^{1,p}(\Omega)\right]. \tag{3.14}$$

Since $\widehat{u}_1(p) \in \operatorname{int} C_+$, we can choose $t \in (0,1)$ small such that

$$0 < t\widehat{u}_1(p)(z) \le \delta_0 \quad \text{for all } z \in \overline{\Omega},$$

with $\delta_0 > 0$ as in hypothesis H_1(iv). We have

$$0 \le F(z, t\widehat{u}_1(p)(z)) \quad \text{for a.a. } z \in \Omega. \tag{3.15}$$

Then we have

$$\varphi_+(t\widehat{u}_1(p)) \le \frac{t^p}{p}\widehat{\lambda}_1(p) - \frac{t^q}{q}\int_{\partial\Omega} \beta(z)\widehat{u}_1(p)^q \,d\sigma \quad \text{(see (3.15))}.$$

Since $q < p$, choosing $t \in (0,1)$ even smaller if necessary, we have

$$\begin{aligned} & \varphi_+(t\widehat{u}_1(p)) < 0, \\ \Rightarrow \quad & \varphi_+(u_0) < 0 = \varphi_+(0) \quad \text{(see (3.14))}, \\ \Rightarrow \quad & u_0 \neq 0. \end{aligned}$$

From (3.14), we have

$$\begin{aligned} & \varphi_+'(u_0) = 0, \\ \Rightarrow \quad & \langle A(u_0), h\rangle + \int_\Omega [\xi(z) + \eta]|u_0|^{p-2}u_0 h \,dz \\ & \quad = \int_{\partial\Omega} \beta(z)(u_0^+)^{q-1} h \,d\sigma + \int_\Omega [f(z, u_0^+) + \eta(u_0^+)^{p-1}] h \,dz \quad \text{for all } h \in W^{1,p}(\Omega). \end{aligned} \tag{3.16}$$

In (3.16) we choose $h = -u_n^- \in W^{1,p}(\Omega)$ and obtain

$$\begin{aligned} & \|Du_0^-\|_p^p + \int_\Omega [\xi(z) + \eta](u_0^-)^p \,dz = 0, \\ \Rightarrow \quad & c_4\|u_0^-\|^p \le 0 \quad \text{for some } c_4 > 0 \text{ (since } \eta > \|\xi\|_\infty), \\ \Rightarrow \quad & u_0 \ge 0, \ u_0 \neq 0. \end{aligned}$$

Then from (3.16) we have

$$\begin{cases} -\Delta_p u_0(z) + \xi(z)u_0(z)^{p-1} = f(z, u_0(z)) & \text{for a.a. } z \in \Omega, \\ \dfrac{\partial u_0}{\partial n_p} = \beta(z)u_0^{q-1} & \text{on } \partial\Omega. \end{cases} \tag{3.17}$$

From (3.17) and Proposition 2.10 of Papageorgiou–Rădulescu [10] (see also Theorem 4.1 of Winkert [15]), we have that $u_0 \in L^\infty(\Omega)$. Then Theorem 2 of Lieberman [7], implies that $u_0 \in C_+ \setminus \{0\}$.

Let $\rho = \|u_0\|_\infty$. We can find $\widehat{\xi}_\rho > 0$ such that $f(z,x)x + \widehat{\xi}_\rho |x|^p \geq 0$ for a.a. $z \in \Omega$, all $|x| \leq \rho$. Then from (3.17) we have

$$
\begin{aligned}
& -\Delta_p u_0(z) + \left[\xi(z) + \widehat{\xi}_\rho\right] u_0(z)^{p-1} \geq 0 \quad \text{for a.a. } z \in \Omega \text{ (see hypothesis } H_1\text{(iv))},\\
\Rightarrow \quad & \Delta_p u_0(z) \leq \left[\|\xi\|_\infty + \widehat{\xi}_\rho\right] u_0(z)^{p-1} \quad \text{for a.a. } z \in \Omega,\\
\Rightarrow \quad & u_0 \in \operatorname{int} C_+ \quad \text{(by the nonlinear maximum principle; see [4, p. 738]).}
\end{aligned}
$$

Similarly working this time with the functional $\varphi_-(\cdot)$, we obtain a negative solution $v_0 \in -\operatorname{int} C_+$ for problem (1.1). □

It is easy to check that

$$
K_{\varphi_+} \subseteq \operatorname{int} C_+ \cup \{0\} \quad \text{and} \quad K_{\varphi_-} \subseteq (-\operatorname{int} C_+) \cup \{0\}.
$$

So, we may assume that

$$
K_{\varphi_+} = \{0, u_0\} \quad \text{and} \quad K_{\varphi_-} = \{0, v_0\}, \tag{3.18}
$$

or otherwise we already have a third nontrivial smooth solution which in fact has fixed sign. So, we are done. In the next section we produce a third nontrivial smooth solution for problem (1.1).

4 Three nontrivial solutions

Starting from (3.18), we introduce the following truncation-perturbation of $f(z,\cdot)$ (as before $\eta > \|\xi\|_\infty$):

$$
\widehat{f}(z,x) = \begin{cases} f(z, v_0(z)) + \eta |v_0(z)|^{p-2} v_0(z) & \text{if } x < v_0(z),\\ f(z,x) + \eta |x|^{p-2} x & \text{if } v_0(z) \leq x \leq u_0(z),\\ f(z, u_0(z)) + \eta u_0(z)^{p-1} & \text{if } u_0(z) < x. \end{cases} \tag{4.1}
$$

We also consider the positive and negative truncations of $\widehat{f}(z,x)$, namely the functions

$$
\widehat{f}_\pm(z,x) = \widehat{f}(z, \pm x^\pm). \tag{4.2}
$$

It is clear that $\widehat{f}$ and $\widehat{f}_\pm$ are all three Carathéodory functions. We see that

$$
\widehat{F}(z,x) = \int_0^x \widehat{f}(z,s)\,ds \quad \text{and} \quad \widehat{F}_\pm(z,x) = \int_0^x \widehat{f}_\pm(z,s)\,ds.
$$

We also introduce similar truncations of the boundary term:

$$
\widehat{g}(z,x) = \begin{cases} \beta(z)|v_0(z)|^{q-2} v_0(z) & \text{if } x < v_0(z),\\ \beta(z)|x|^{q-2} x & \text{if } v_0(z) \leq x \leq u_0(z),\\ \beta(z) u_0(z)^{q-1} & \text{if } u_0(z) < x, \end{cases} \quad \text{for all } (z,x) \in \partial\Omega \times \mathbb{R}. \tag{4.3}
$$

We also consider the positive and negative truncations of $g(z,\cdot)$, namely the functions

$$
\widehat{g}_\pm(z,x) = \widehat{g}(z, \pm x^\pm). \tag{4.4}
$$

Evidently $\widehat{g}$ and $\widehat{g}_\pm$ are all three Carathéodory functions on $\partial\Omega \times \mathbb{R}$. We set

$$\widehat{G}(z,x) = \int_0^x \widehat{g}(z,s)\,\mathrm{d}s \quad \text{and} \quad \widehat{G}_\pm(z,x) = \int_0^x \widehat{g}_\pm(z,s)\,\mathrm{d}s.$$

We introduce the C^1-functionals $\widehat{\psi}, \widehat{\psi}_\pm : W^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\widehat{\psi}(u) = \frac{1}{p}\|Du\|_p^p + \frac{1}{p}\int_\Omega [\xi(z)+\eta]|u|^p\,\mathrm{d}z - \int_\Omega \widehat{F}(z,u)\,\mathrm{d}z - \int_{\partial\Omega} \widehat{G}(z,u)\,\mathrm{d}\sigma,$$
$$\widehat{\psi}_\pm(u) = \frac{1}{p}\|Du\|_p^p + \frac{1}{p}\int_\Omega [\xi(z)+\eta]|u|^p\,\mathrm{d}z - \int_\Omega \widehat{F}_\pm(z,u)\,\mathrm{d}z - \int_{\partial\Omega} \widehat{G}_\pm(z,u)\,\mathrm{d}\sigma,$$

for all $u \in W^{1,p}(\Omega)$.

Finally, let $\varphi : W^{1,p}(\Omega) \to \mathbb{R}$ be the energy (Euler) functional for problem (1.1) defined by

$$\varphi(u) = \frac{1}{p}\|Du\|_p^p + \frac{1}{p}\int_\Omega \xi(z)|u|^p\,\mathrm{d}z - \int_\Omega F(z,u)\,\mathrm{d}z - \frac{1}{q}\int_{\partial\Omega} \beta(z)|u|^q\,\mathrm{d}\sigma \quad \text{for all } u \in W^{1,p}(\Omega).$$

We have that $\varphi \in C^1(W^{1,p}(\Omega))$. Also

$$K_\varphi = \text{set of solutions of problem (1.1)}, \tag{4.5}$$

while from (4.3), (4.4) and the nonlinear regularity theory [7], we have

$$K_{\widehat{\psi}} \subseteq [v_0, u_0] \cap C^1(\overline{\Omega}), \quad K_{\widehat{\psi}_+} \subseteq [0, u_0] \cap C_+, \quad K_{\widehat{\psi}_-} \subseteq [v_0, 0] \cap C_+. \tag{4.6}$$

Note that

$$\varphi|_{[v_0,u_0]} = \widehat{\psi}|_{[v_0,u_0]} \quad \text{and} \quad \varphi'|_{[v_0,u_0]} = \widehat{\psi}'|_{[v_0,u_0]}, \tag{4.7}$$

$$\varphi|_{[0,u_0]} = \varphi_+|_{[0,u_0]} = \widehat{\psi}_+|_{[0,u_0]} \quad \text{and} \quad \varphi'|_{[0,u_0]} = \varphi'_+|_{[0,u_0]} = \widehat{\psi}'_+|_{[0,u_0]}, \tag{4.8}$$

$$\varphi|_{[v_0,0]} = \varphi_-|_{[v_0,0]} = \widehat{\psi}_-|_{[v_0,0]} \quad \text{and} \quad \varphi'|_{[v_0,0]} = \varphi'_-|_{[v_0,0]} = \widehat{\psi}'_-|_{[v_0,0]}. \tag{4.9}$$

From (4.5) we see that we may assume that K_φ is finite or otherwise we already have an infinity of nontrivial smooth solutions for problem (1.1) and so we are done. Combining this fact with (4.6) and (4.7), we see that $K_{\widehat{\psi}}$ is finite too. Moreover, from (3.18), (4.6), (4.8), (4.9) we infer that

$$K_{\widehat{\psi}} \subseteq [v_0, u_0] \cap C^1(\overline{\Omega}) \text{ is finite}, \quad K_{\widehat{\psi}_+} = \{0, u_0\}, \quad K_{\widehat{\psi}_-} = \{0, v_0\}. \tag{4.10}$$

These observations permit the consideration of the critical groups of φ and $\widehat{\psi}$ at $u = 0$ and for these groups we have the following result.

Proposition 4.1. *If hypotheses* H_0, H_1 *hold, then* $C_k(\varphi, 0) = C_k(\widehat{\psi}, 0)$ *for all* $k \in \mathbb{N}_0$.

Proof. Recall that we assume that K_φ is finite. We consider the homotopy $\widehat{h}(t,u)$ defined by

$$\widehat{h}(t,u) = t\widehat{\psi}(u) + (1-t)\varphi(u) \quad \text{for all } (t,u) \in [0,1] \times W^{1,p}(\Omega).$$

Suppose we could find $\{t_n\}_{n\geq 1} \subseteq [0,1]$ and $\{u_n\}_{n\geq 1} \subseteq W^{1,p}(\Omega)$ such that

$$t_n \to t \in [0,1], \quad u_n \to 0 \text{ in } W^{1,p}(\Omega), \quad \widehat{h}'_u(t_n, u_n) = 0 \text{ for all } n \in \mathbb{N}. \tag{4.11}$$

From the equation in (4.11), we have

$$\begin{cases} -\Delta_p u_n(z) + [\xi(z) + t_n\eta]|u_n(z)|^{p-2}u_n(z) \\ = t_n\widehat{f}(z,u_n(z)) + (1-t_n)f(z,u_n(z)) & \text{for a.a. } z \in \Omega, \\ \dfrac{\partial u_n}{\partial n_p} = t_n\widehat{g}(z,u_n) + (1-t_n)\beta(z)|u_n|^{q-2}u_n & \text{on } \partial\Omega. \end{cases} \tag{4.12}$$

From (4.12) and Proposition 2.10 of Papageorgiou-Rădulescu [10], we can find $c_5 > 0$ such that

$$\|u_n\|_\infty \le c_5 \quad \text{for all } n \in \mathbb{N}.$$

Then from Theorem 2 of Lieberman [7], we see that there exist $\alpha_0 \in (0,1)$ and $c_6 > 0$ such that

$$u_n \in C^{1,\alpha_0}(\overline{\Omega}) \quad \text{and} \quad \|u_n\|_{C^{1,\alpha_0}(\overline{\Omega})} \le c_6 \quad \text{for all } n \in \mathbb{N}. \tag{4.13}$$

From (4.13), the compact embedding of $C^{1,\alpha_0}(\overline{\Omega})$ into $C^1(\overline{\Omega})$ and (4.11) we infer that

$$u_n \to 0 \quad \text{in } C^1(\overline{\Omega}) \text{ as } n \to \infty. \tag{4.14}$$

Then, on account of (4.14), we can find $n_0 \in \mathbb{N}$ such that

$$\begin{aligned} &u_n \in [v_0, u_0], \quad \text{for all } n \ge n_0, \\ \Rightarrow \quad &\{u_n\}_{n\ge n_0} \subseteq K_\varphi \quad (\text{see (4.7) and (4.10)}), \end{aligned}$$

which contradicts our assumption that K_φ is finite. Therefore (4.11) can not occur and then the homotopy invariance property of critical groups (see Papageorgiou–Rădulescu–Repovš [11, Theorem 6.3.6, p. 505]), implies that

$$C_k(\varphi, 0) = C_k(\widehat{\psi}, 0) \quad \text{for all } k \in \mathbb{N}_0. \qquad \square$$

Next we compute the critical groups of φ at $u = 0$.

Proposition 4.2. *If hypotheses* H_0, H_1 *hold, then* $C_k(\varphi, 0) = 0$ *for all* $k \in \mathbb{N}_0$.

Proof. On account of hypotheses H_1(i),(iv), we have

$$F(z,x) \ge -c_7|x|^r \quad \text{for a.a. } z \in \Omega, \text{ all } x \in \mathbb{R}, \tag{4.15}$$

with $c_7 > 0$ and $r > p$. Then, using (4.15), for every $u \in W^{1,p}(\Omega)$ and every $t > 0$, we have

$$\varphi(tu) \le t^p c_8\|u\|^p + t^r c_9\|u\|^r - t^q \int_{\partial\Omega} \beta(z)|u|^q \, d\sigma \quad \text{for some } c_8, c_9 > 0.$$

Note that $\int_{\partial\Omega} \beta(z)|u|^q \, d\sigma > 0$. Therefore since $q < p < r$, we can find $t^* = t^*(u) \in (0,1)$ such that

$$\varphi(tu) < 0 \quad \text{for all } t \in (0, t^*). \tag{4.16}$$

Let $u \in W^{1,p}(\Omega)$ with $0 < \|u\| \le 1$, $\varphi(u) = 0$ and $\vartheta \in (\mu, p)$. We have

$$\begin{aligned} \frac{d}{dt}\varphi(tu)\Big|_{t=1} &= \langle \varphi'(u), u\rangle \quad (\text{by the chain rule}) \\ &= \langle A(u), u\rangle + \int_\Omega \xi(z)|u|^p \, dz - \int_\Omega f(z,u)u \, dz - \int_{\partial\Omega} \beta(z)|u|^q \, d\sigma \\ &= \left[1 - \frac{\vartheta}{p}\right] \|Du\|_p^p + \left[1 - \frac{\vartheta}{p}\right] \int_\Omega \xi(z)|u|^p \, dz + \left[\frac{\vartheta}{q} - 1\right] \int_{\partial\Omega} \beta(z)|u|^q \, d\sigma \\ &\quad + (\vartheta - \mu) \int_\Omega F(z,u) \, dz + \int_\Omega [\mu F(z,u) - f(z,u)u] \, dz \\ &\quad (\text{since } \varphi(u) = 0). \end{aligned} \tag{4.17}$$

By hypothesis H_1(iv), we have that

$$F(z,x) \geq \widehat{c}|x|^p \quad \text{for a.a. } z \in \Omega, \text{ all } |x| \leq \delta_0. \tag{4.18}$$

Combining (4.18) with hypothesis H_1(i) we have that

$$F(z,x) \geq \widehat{c}|x|^p - c_{10}|x|^r \quad \text{for a.a. } z \in \Omega, \text{ all } x \in \mathbb{R}, \tag{4.19}$$

for some $c_{10} > 0$.

In addition, hypotheses H_1(i),(iv) imply that

$$\mu F(z,x) - f(z,x)x \geq -c_{11}|x|^r \quad \text{for a.a. } z \in \Omega, \text{ all } x \in \mathbb{R}, \text{ some } c_{11} > 0. \tag{4.20}$$

We return to (4.17) and use (4.18), (4.19), (4.20) and obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\varphi(tu)\bigg|_{t=1} \geq c_{12}\|Du\|_p^p + \left[\widehat{c} - \|\xi^-\|_\infty\right]\|u\|_p^p - c_{13}\|u\|^r$$

for some $c_{12}, c_{13} > 0$ (recall that $q < \mu < \vartheta$).

But by hypothesis H_1(iv) we have that $\widehat{c} > \|\xi^-\|_\infty$. So, from the above inequality, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\varphi(tu)\bigg|_{t=1} \geq c_{14}\|u\|^p - c_{13}\|u\|^r \quad \text{for some } c_{14} > 0.$$

Since $p < r$, we can find $\rho \in (0,1)$ small such that

$$\frac{\mathrm{d}}{\mathrm{d}t}\varphi(tu)\bigg|_{t=1} > 0 \quad \text{for all } u \in W^{1,p}(\Omega) \text{ with } < \|u\| \leq \rho,\ \varphi(u) = 0. \tag{4.21}$$

Consider a $u \in W^{1,p}(\Omega)$ as in (4.21), namely that

$$0 < \|u\| < \rho \quad \text{and} \quad \varphi(u) = 0.$$

We show that

$$\varphi(tu) \leq 0 \quad \text{for all } t \in [0,1]. \tag{4.22}$$

Suppose that (4.22) is not true. Then we can find $t_0 \in (0,1)$ such that $\varphi(t_0 u) > 0$. Since $\varphi(u) = 0$ and $\varphi(\cdot)$ is continuous, by Bolzano's theorem, we can find $\widehat{t} \in (t_0, 1]$ such that $\varphi(\widehat{t}u) = 0$. We set

$$t^* = \min\left\{\widehat{t} \in (t_0,1] : \varphi(tu) = 0\right\} > t_0 > 0.$$

We have

$$\varphi(tu) > 0 \quad \text{for all } t \in [t_0, t^*). \tag{4.23}$$

If $v = t^* u$, then $0 < \|v\| \leq \rho$ and $\varphi(v) = 0$. So, from (4.21) we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\varphi(tu)\bigg|_{t=1} > 0. \tag{4.24}$$

On the other hand

$$\frac{\mathrm{d}}{\mathrm{d}t}\varphi(tu)\bigg|_{t=1} = t^*\frac{\mathrm{d}}{\mathrm{d}t}\varphi(tu)\bigg|_{t=t^*} = t^* \lim_{t\to(t^*)^-}\frac{\varphi(tu)}{t-t^*} \leq 0 \quad \text{(see (4.23))}. \tag{4.25}$$

Comparing (4.24) and (4.25), we have a contradiction. This proves (4.22).

Recall that K_φ is finite. So, we can always choose $\rho \in (0,1)$ small so that $K_\varphi \cap \overline{B_\rho} = \{0\}$ (recall that $B_\rho = \{u \in W^{1,p}(\Omega) : \|u\| < \rho\}$). Consider the continuous deformation $h_0 : [0,1] \times (\varphi^0 \cap \overline{B_\rho}) \to \varphi^0 \cap \overline{B_\rho}$ defined by

$$h_0(t,u) = (1-t)u \quad \text{for all } (t,u) \in [0,1] \times (\varphi^0 \cap \overline{B_\rho}).$$

On account of (4.22) this deformation is well-defined and shows that $\varphi^0 \cap \overline{B_\rho}$ is contractible in itself.

Let $u \in \overline{B_\rho}$ with $\varphi(u) > 0$. We claim that there is a unique $t(u) \in (0,1)$ such that

$$\varphi(t(u)u) = 0. \tag{4.26}$$

The existence of such $t(u) \in (0,1)$ follows from (4.16) and Bolzano's theorem. For the uniqueness, suppose we could find $0 < t_1 < t_2 < 1$ such that

$$\varphi(t_1 u) = \varphi(t_2 u) = 0. \tag{4.27}$$

Consider the function

$$\eta(t) = \varphi(t t_2 u) \quad \text{for all } t \in [0,1].$$

From (4.27) and (4.22), it follows that that $t = \frac{t_1}{t_2} \in (0,1)$ is a maximizer of $\eta(\cdot)$. Therefore we have

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}\eta(t)\Big|_{t=\frac{t_1}{t_2}} = 0, \\ \Rightarrow \quad &\frac{\mathrm{d}}{\mathrm{d}t}\varphi(t t_1 u)\Big|_{t=1} = 0, \end{aligned}$$

which contradicts (4.21). So, $t(u) \in (0,1)$ satisfying (4.26) is unique. Therefore we have

$$\varphi(tu) < 0 \text{ for } t \in (0, t(u)) \quad \text{and} \quad \varphi(tu) > 0 \text{ if } t \in (t(u), 1]. \tag{4.28}$$

Then we introduce the function $\lambda : \overline{B_\rho} \setminus \{0\} \to [0,1]$ defined by

$$\lambda(u) = \begin{cases} 1 & \text{if } u \in \overline{B_\rho} \setminus \{0\}, \ \varphi(u) \le 0, \\ t(u) & \text{if } u \in \overline{B_\rho} \setminus \{0\}, \ \varphi(u) > 0. \end{cases}$$

It is easy to see that $\lambda(\cdot)$ is continuous. So, if we consider the map $k : \overline{B_\rho} \setminus \{0\} \to (\varphi^0 \cap \overline{B_\rho}) \setminus \{0\}$ defined by

$$k(u) = \begin{cases} u & \text{if } u \in \overline{B_\rho} \setminus \{0\}, \ \varphi(u) \le 0, \\ \lambda(u)u & \text{if } u \in \overline{B_\rho} \setminus \{0\}, \ \varphi(u) > 0, \end{cases}$$

then $k(\cdot)$ is continuous and $k|_{(\varphi^0 \cap \overline{B_\rho}) \setminus \{0\}} = \text{identity}$. It follows that $(\varphi^0 \cap \overline{B_\rho}) \setminus \{0\}$ is a retract of $\overline{B_\rho} \setminus \{0\}$, which is contractible. Therefore $(\varphi^0 \cap \overline{B_\rho}) \setminus \{0\}$ is contractible and so we have

$$\begin{aligned} &H_k(\varphi^0 \cap \overline{B_\rho}, (\varphi^0 \cap \overline{B_\rho}) \setminus \{0\}) = 0 \quad \text{for all } k \in \mathbb{N}_0 \text{ (see [11], p. 469)}, \\ \Rightarrow \quad &C_k(\varphi, 0) = 0 \quad \text{for all } k \in \mathbb{N}_0. \end{aligned}$$

□

Corollary 4.3. *If hypotheses* H_0, H_1 *hold, then* $C_k(\widehat{\psi}, 0) = 0$ *for all* $k \in \mathbb{N}_0$.

Now we are ready for the multiplicity theorem. It is interesting to point out that the solutions we produce are ordered.

Theorem 4.4. *If hypotheses* H_0, H_1 *hold, then problem* (1.1) *has at least three nontrivial smooth solutions*

$$u_0 \in \operatorname{int} C_+, \quad v_0 \in -\operatorname{int} C_+ \quad \textit{and} \quad y_0 \in C^1(\overline{\Omega}),\ v_0 \le y_0 \le u_0.$$

Proof. From Proposition 3.2 we already have two nontrivial constant sign solutions

$$u_0 \in \operatorname{int} C_+ \quad \text{and} \quad v_0 \in -\operatorname{int} C_+.$$

Claim: $u_0 \in \operatorname{int} C_+$ *and* $v_0 \in -\operatorname{int} C_+$ *are local minimizers of* $\widehat{\psi}(\cdot)$.

From (4.1), (4.2), (4.3) and (4.4), we see that $\widehat{\psi}_+(\cdot)$ is coercive. Also, it is sequentially weakly lower semicontinuous. So, we can find $\widetilde{u}_0 \in W^{1,p}(\Omega)$ such that

$$\widehat{\psi}_+(\widetilde{u}_0) = \min\left[\widehat{\psi}_+(u) : u \in W^{1,p}(\Omega.)\right] \tag{4.29}$$

Let $u \in \operatorname{int} C_+$. Since $u_0 \in \operatorname{int} C_+$, we can find $t \in (0,1)$ small such that

$$0 \le tu \le \min\{u_0, \delta_0\}$$

(see Papageorgiou–Rădulescu–Repovš [11], Proposition 4.1.22, p. 274). Then, since $\mu < p$, we have

$$\begin{aligned} & \widehat{\psi}_+(tu) < 0 \quad \text{for } t \in (0,1) \text{ small}, \\ \Rightarrow\ & \widehat{\psi}_+(\widetilde{u}_0) < 0 = \widehat{\psi}_+(0) \quad (\text{see } (4.29)), \\ \Rightarrow\ & \widetilde{u}_0 \ne 0, \\ \Rightarrow\ & \widetilde{u}_0 = u_0 \quad (\text{see } (4.10) \text{ and } (4.29)). \end{aligned}$$

Note that $\widehat{\psi}\big|_{C_+} = \widehat{\psi}_+\big|_{C_+}$. Since $u_0 \in \operatorname{int} C_+$, it follows that

$$\begin{aligned} & u_0 \text{ is a local } C^1(\overline{\Omega})\text{-minimizer of } \widehat{\psi}(\cdot), \\ \Rightarrow\ & u_0 \text{ is a local } W^{1,p}(\Omega)\text{-minimizer of } \widehat{\psi}(\cdot) \\ & (\text{see Papageorgiou–Rădulescu [10, Proposition 2.12]}). \end{aligned}$$

Similarly for $v_0 \in -\operatorname{int} C_+$ using this time the functional $\psi_-(\cdot)$.
This proves the claim.

Without any loss of generality we may assume that

$$\widehat{\psi}(v_0) \le \widehat{\psi}(u_0).$$

From (4.10), the Claim and Theorem 5.7.6, p. 449, of Papageorgiou–Rădulescu–Repovš [11], we know that we can find $\rho \in (0,1)$ small such that

$$\widehat{\psi}(v_0) \le \widehat{\psi}(u_0) < \inf\left[\widehat{\psi}(u) : \|u - u_0\| = \rho\right] = \widehat{m}_\rho, \quad \|v_0 - u_0\| > \rho. \tag{4.30}$$

Since $\widehat{\psi}(\cdot)$ is coercive, from Proposition 5.1.15, p. 369, of Papageorgiou–Rădulescu–Repovš [11], we have that

$$\widehat{\psi}(\cdot) \text{ satisfies the Palais–Smale condition.} \tag{4.31}$$

Then (4.30) and (4.31) permit the use of the mountain pass theorem. So, we can find $y_0 \in W^{1,p}(\Omega)$ such that

$$y_0 \in K_{\widehat{\psi}} \subseteq [v_0, u_0] \cap C^1(\overline{\Omega}) \text{ (see (4.10))} \quad \text{and} \quad \widehat{m}_\rho \leq \widehat{\psi}(y_0) \text{ (see (4.30))}. \tag{4.32}$$

From (4.30) and (4.32), we have that

$$y_0 \neq u_0 \quad \text{and} \quad y_0 \neq v_0.$$

Moreover, since y_0 is a critical point of $\widehat{\psi}$ of mountain pass type, from Corollary 6.6.9, p. 533, of Papageorgiou–Rădulescu–Repovš [11], we have

$$C_1(\widehat{\psi}, y_0) \neq 0. \tag{4.33}$$

On the other hand, from Corollary 4.3, we have

$$C_k(\widehat{\psi}, 0) = 0 \quad \text{for all } k \in \mathbb{N}_0. \tag{4.34}$$

Comparing (4.33) and (4.34) we infer that $y_0 \neq 0$. Therefore $y_0 \in C^1(\overline{\Omega})$ is the third nontrivial solution of (1.1) and $v_0(z) \leq y_0(z) \leq u_0(z)$ for all $z \in \overline{\Omega}$. □

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