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COMPUTING THE NEWTON POTENTIAL IN THE BOUNDARY INTEGRAL EQUATION FOR THE DIRICHLET PROBLEM OF THE POISSON EQUATION

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Evaluating the Newton potential is crucial for efficiently solving the boundary integral equation of the Dirichlet boundary value problem of the Poisson equation. In the context of the Fourier–Garlerkin method for solving the boundary integral equation, we propose a fast algorithm for evaluating Fourier coefficients of the Newton potential by using a sparse grid approximation. When the forcing function of the Poisson equation expressed in the polar coordinates has mth-order bounded mixed derivatives, the proposed algorithm achieves an accuracy of order $\mathcal{O}(n^{-m}\log^3 n)$, with requiring $\mathcal{O}(n\log^2 n)$ number of arithmetics for the computation, where n is the number of quadrature points used in one coordinate direction. With the help of this algorithm, the boundary integral equation derived from the Poisson equation can be efficiently solved by a fast fully discrete Fourier–Garlerkin method.

1. Introduction

Boundary value problems of the Laplace equation can be reformulated as boundary integral equations (BIEs) which are defined only on the boundaries of their domains. A main advantage of the BIE method lies in its reduction of the dimension of the spacial domains by one [27]. The reduction of the dimesion translates to efficient numerical solutions of the boundary value problems. The resulting BIE can be solved approximately by the Fourier–Garlerkin methods and other numerical methods [2; 12; 25; 26; 29; 32; 33]. The fast Fourier–Galerkin method was developed in [8; 11; 21; 18; 30], via approximating the dense coefficient matrix obtained from discretization of the BIEs using the Galerkin principle and the Fourier basis by a sparse matrix having only $\mathcal{O}(n\log n)$ number of nonzero entries. This leads to a fast method with the optimal convergence of order $\mathcal{O}(n^{-m})$, where n is the order of Fourier bases in the method and m is the degree of regularity of the exact solution.

The BIE method seems to have less advantage when it is applied to solving a boundary value problem of the Poisson equation, which has a nonzero forcing term, due to the present of the Newton potential, an integral defined on its domain, in its BIE reformulation [27]. Even though the integral operator of the resulting BIE is defined on the boundary of the domain, the Newton potential is defined on the domain, instead of the boundary. Hence, efficient evaluation of the Newton potential becomes the bottleneck of the BIE method for solving boundary value problems of the Poisson equation. When the domain has a simple geometry and its boundary is smooth, the fast multipole (FMP) method was applied to the direct evaluation of the Newton potential [14; 28]. Moreover, for the FMP method to have a high accuracy, excessive adaptive refinement near the boundary is required, which demands additional computational

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complexity. Hence, it is desirable to develop a fast method for computing the Newton potential on a general domain required only quasilinear (linear up to a logarithmic factor) computational costs, having the optimal order of accuracy.

The purpose of this paper is to develop a fast method for evaluation of the Newton potential on a general smooth boundary by using the sparse grid technique. To this end, we need to overcome certain difficulties. The Newton potential is a singular integral defined on the domain of the Poisson equation. In the context of solving the Poisson equation by the Fourier–Galerkin method, we are required to compute the Fourier coefficients of the Newton potential. They are singular integrals of three variables. The use of the sparse grid technique prefers smooth integrands. The singularity of the integrands of these integrals prevents the direct use of the sparse grid technique. To address this issue, we divide the integration domain into *graded* (noncuboid) subdomains on each of which the integrand is smooth. Aiming at designing a sparse grid approximation of the integrand on each of the subdomains, we extend each of the noncuboid subdomains to a cuboid region where a sparse grid approximation of the integrand is designed. In order to obtain the integral value on each of the noncuboid subdomains, we turn to compute the integral of the sparse grid approximation defined on the extension with restriction to the corresponding noncuboid subdomain. Thus the Fourier coefficients of the Newton potential on the boundary can be estimated by summing up these integral values. We prove that when the forcing function of the Poisson equation expressed in the polar coordinates has mth-order bounded mixed derivatives, the proposed method for the evaluation of the Fourier Coefficients of the Newton potential achieves an accuracy order $\mathbb{O}(n^{-m}\log^3 n)$ with the computational complexity of order $\mathbb{O}(n\log^2 n)$, where n is the number of the points used in one coordinate direction. In passing, we remark that graded meshes have been widely used to deal with approximation of functions with singularity. In particular, they were applied to numerical quadratures [22; 31], solutions of partial differential equations [1; 3; 4] and integral equations with weakly singular kernels [5].

This paper is organized in six sections. In Section 2, we review the BIE for the Dirichlet problem of the Poisson equation, and describe the fast Fourier–Galerkin method for solving the BIE. Section 3 is devoted to the development of the fast algorithm for computing the Fourier coefficients of the Newton potential. For the purpose of analyzing the accuracy of the proposed method we estimate in Section 4 the regularity of the integrands of the integrals for the Fourier coefficients of the Newton potential. The optimal accuracy order of the proposed fast algorithm is established in Section 5. In Section 6, we present numerical examples to confirm the theoretical estimates.

2. BIE for the Poisson equation

We review the BIE of the Dirichlet problem for the Poisson equation, describe the Fourier–Galerkin method for solving the BIE applied to the problem and identify computing the Fourier coefficients of the Newton potential as a bottleneck of its fast solution.

We begin with a description of the Dirichlet problem for the Poisson equation. Let $\Omega \subseteq \mathbb{R}^2$ denote a bounded open domain with a smooth boundary, $\partial\Omega$ denote the boundary of Ω and $\overline{\Omega}$ denote the closure of Ω. We denote by $C(\mathbb{E})$ the space of continuous functions defined on set $\mathbb{E}$, and $C^2(\mathbb{E})$ the space of functions defined on set $\mathbb{E}$ with the continuous second-order derivatives. Suppose that $f \in C(\overline{\Omega})$

and $u_D(x) \in C(\partial\Omega)$ are given. We consider the Dirichlet problem for the Poisson equation

$$\begin{cases} -\Delta u(x) = f(x), & x \in \Omega, \\ u(x) = u_D(x), & x \in \partial\Omega, \end{cases} \tag{2-1}$$

where Δ denotes the Laplace operator mapping from $C^2(\Omega)$ to $C(\Omega)$.

We describe the BIE of the boundary value problem (2-1). By the second Green identity, the solution u of problem (2-1) can be represented by the sum of the Newton potential, the double layer potential and the single layer potential (see, [2; 23; 27])

$$u(x) = (\mathcal{N} f)(x) + \int_{\partial\Omega} \left(\Phi(x-y) \frac{\partial u(y)}{\partial \boldsymbol{n}} - \frac{\partial \Phi(x-y)}{\partial \boldsymbol{n}_y} u(y) \right) \mathrm{d}s_y, \quad x \in \Omega. \tag{2-2}$$

The Newton potential is an operator $\mathcal{N}$ from $C(\overline{\Omega})$ to $C^2(\overline{\Omega})$ defined by

$$(\mathcal{N} f)(x) := \int_{\Omega} f(y)\, \Phi(x-y)\, \mathrm{d}y, \quad x \in \Omega,$$

where

$$\Phi(x) := -\frac{1}{2\pi} \log|x|, \quad x \in \mathbb{R}^2 \setminus \{0\}.$$

We comment that Φ is the fundamental solution of the two-dimensional Laplace equation (see [15]). Formula (2-2) implies that the solution u of problem (2-1) can be calculated by the boundary data $u|_{\partial\Omega}$, $(\partial u/\partial n)|_{\partial\Omega}$ and the forcing function f. Since $(\partial u/\partial n)|_{\partial\Omega}$ is unknown, formula (2-2) is not adequate to determine u directly. We then consider the case of $x \in \partial\Omega$ and obtain the BIE for the problem (2-1):

$$\int_{\partial\Omega} \Phi(x-y) \frac{\partial u(y)}{\partial \boldsymbol{n}} \mathrm{d}s_y = h(x), \quad x \in \partial\Omega, \tag{2-3}$$

where

$$h(x) := -(\mathcal{N} f)(x) + \int_{\partial\Omega} \frac{\partial \Phi(x-y)}{\partial \boldsymbol{n}_{\boldsymbol{y}}} u_D(y)\, \mathrm{d}s_y + \frac{1}{2} u_D(x), \quad x \in \partial\Omega.$$

Upon solving $\partial u(x)/\partial \boldsymbol{n}$, $x \in \partial\Omega$, from integral equation (2-3) and substituting it into the of (2-2), we obtain the solution u of problem (2-1).

We need a parametrization of the boundary $\partial\Omega$. Suppose that $\partial\Omega$ is described by the parametrization

$$\boldsymbol{r}(t) := (r_1(t), r_2(t)), \quad t \in I_{2\pi} := [0, 2\pi), \tag{2-4}$$

where $r_1(t) := r(t)\cos(t)$, $r_2(t) := r(t)\sin(t)$, $r(t) \in C^\infty_{2\pi}(\mathbb{E})$, the space of the infinitely differentiable 2π-periodic functions defined on set $\mathbb{E}$. For function ϕ defined on $\partial\Omega$, we define $\phi \circ \boldsymbol{r}(t) := \phi(\boldsymbol{r}(t))$ and write $\boldsymbol{r}'(t) := (\frac{\mathrm{d}r_1}{\mathrm{d}t}(t), \frac{\mathrm{d}r_2}{\mathrm{d}t}(t))$, $t \in I_{2\pi}$. With the parametrization (2-4) of the boundary $\partial\Omega$, the BIE (2-3) is rewritten as

$$-\frac{1}{2\pi} \int_{I_{2\pi}} \frac{\partial u}{\partial \boldsymbol{n}} \circ \boldsymbol{r}(s)\, |\boldsymbol{r}'(s)| \log|\boldsymbol{r}(t) - \boldsymbol{r}(s)|\, \mathrm{d}s = h \circ \boldsymbol{r}(t), \quad t \in I_{2\pi}. \tag{2-5}$$

We next rewrite equation (2-5) in an operator form. For each $l \geq 0$, we denote by $H^l(I_{2\pi})$ the standard Sobolev space of functions $\phi \in L^2(I_{2\pi})$ satisfying $\sum_{k \in \mathbb{Z}} (1+k^2)^l |\hat{\phi}_k|^2 < +\infty$, where $\hat{\phi}_k := \langle \phi, e_k \rangle$ denote the coefficient of ϕ in the kth Fourier basis function $e_k(x) := e^{ikx}/(2\pi)^{1/2}$, for $x \in I_{2\pi}$. The inner product of space $H^l(I_{2\pi})$ is defined by

$$\langle \phi, \psi \rangle_l := \sum_{k\in\mathbb{Z}} (1+k^2)^l \, \hat{\phi}_k \, \bar{\hat{\psi}}_k \quad \text{for } \phi, \psi \in H^l(I_{2\pi})$$

and the norm is defined as $\|\phi\|_{l,2} := \langle \phi, \phi \rangle_{l,2}^{1/2}$. Let $X := H^0(I_{2\pi})$ and $Y := H^1(I_{2\pi})$. Let $\mathcal{A} : X \to Y$ denote the bounded linear operator defined by

$$(\mathcal{A}w)(t) := \int_{I_{2\pi}} a(t,s)\, w(s)\, \mathrm{d}s, \quad t \in I_{2\pi},$$

where

$$a(t,s) := -\frac{1}{2\pi} \log\left| 2e^{-1/2} \sin\frac{t-s}{2} \right|, \quad t, s \in I_{2\pi}.$$

The operator $\mathcal{A}$ has the Fourier basis functions e_k as its eigenfunctions and it has a bounded inverse $\mathcal{A}^{-1} : Y \to X$. Let $\mathcal{B} : X \to Y$ be a compact operator defined by

$$(\mathcal{B}w)(t) = \int_{I_{2\pi}} b(t,s)\, w(s)\, \mathrm{d}s, \quad t \in I_{2\pi},$$

where

$$b(t,s) := \begin{cases} -\frac{1}{2\pi}\left(\frac{1}{2} + \log \frac{|\boldsymbol{r}(t)-\boldsymbol{r}(s)|}{|2\sin\frac{t-s}{2}|}\right), & t-s \neq 2k\pi, \\ -\frac{1}{2\pi}\left(\frac{1}{2} + \log|\boldsymbol{r}'(t)|\right), & t-s = 2k\pi, \end{cases}$$

where $k \in \mathbb{Z}$, see [2]. Clearly, b is a smooth kernel. In the notation defined above, the integral equation (2-5) has the operator form

$$(\mathcal{A}+\mathcal{B})\, w = g, \tag{2-6}$$

where $g \in Y$ is given, $w \in X$ is unknown.

We now turn to describing the Fourier–Galerkin method for solving the BIE (2-6). Specifically, we employ the Galerkin method by using the Fourier (trigonometric polynomial) basis to discretize the BIEs. Let $\mathbb{N} := \{1, 2, \ldots\}$, $\mathbb{Z}_n^+ := \{1, 2, \ldots, n-1\}$ and $\mathbb{Z}_n := \{0\} \cup \mathbb{Z}_n^+$ for $n \in \mathbb{N}$. Let X_n denote a finite-dimensional subspace $\mathrm{span}\{e_k : |k| \in \mathbb{Z}_n\}$ and P_n denote the orthogonal projector from $L^2(I_{2\pi})$ to X_n. Then

$$P_n w = \sum_{|k|\in\mathbb{Z}_n} \hat{w}(k)\, e_k, \quad w \in L^2(I_{2\pi}).$$

The classical Fourier–Galerkin method for solving (2-6) is to seek $w_n \in X_n$ such that

$$P_n(\mathcal{A}+\mathcal{B})\, w_n = P_n g.$$

Let $\mathcal{B}_n := P_n\mathcal{B}|_{X_n}$ and $g_n := P_n g$. Since eigenfunction of $\mathcal{A}$, the above equation can be rewritten as

$$(\mathcal{A}+\mathcal{B}_n)\, w_n = g_n. \tag{2-7}$$

Next we show the matrix form of the above equation. To this end, for all $k, l \in \mathbb{Z}$, we define

$$b_{k,l} := \int_{I_{2\pi}^2} b(t, s)\, \overline{e_k(t)}\, e_l(s)\, \mathrm{d}t\, \mathrm{d}s,$$

where $I_{2\pi}^2 := I_{2\pi} \times I_{2\pi}$. For all $k, l \in \mathbb{Z}$ with $k, l > 0$, we let

$$D_{k,l} := \begin{bmatrix} b_{-k,-l} & b_{-k,l} \\ b_{k,-l} & b_{k,l} \end{bmatrix}.$$

For all $n \in \mathbb{N}$, we use $[v_{-k}, v_k : k \in \mathbb{Z}_n]$ to denote vector $[v_0, v_{-1}, v_1, \ldots, v_{1-n}, v_{n-1}]$. For all $n \in \mathbb{N}$, we define matrix blocks

$$B_1 := [b_{0,0}], \quad B_2 := [b_{0,-l}, b_{0,l} : l \in \mathbb{Z}_n], \quad B_3 := [b_{-l,0}, b_{l,0} : l \in \mathbb{Z}_n]^T, \quad B_4 := [D_{k,l} : k, l \in \mathbb{Z}_n^+].$$

Then we represent the operater $\mathcal{B}_n$ by the matrix form

$$\boldsymbol{B}_n := \begin{bmatrix} B_1 & B_2 \\ B_3 & B_4 \end{bmatrix}.$$

Since e_k is an eigenfunction of the operator $\mathcal{A}$, we likewise define the matrix

$$\boldsymbol{A}_n := \operatorname{diag}[\lambda_{-k}, \lambda_k : k \in \mathbb{Z}_n]$$

for the Dirichlet problem, where λ_k is the eigenvalue of e_k. The solution of the equation (2-7) will take the form $w_n(x) := \sum_{|k| \in \mathbb{Z}_n} \mathrm{w}_k e_k(x)$, where $x \in I_{2\pi}$ and $\mathrm{w}_k \in \mathbb{C}$. We let $\boldsymbol{w}_n := [\mathrm{w}_{-k}, \mathrm{w}_k : k \in \mathbb{Z}_n]$ and $\boldsymbol{g}_n := [g_{-k}, g_k : k \in \mathbb{Z}_n]$, $g_k := \langle g, e_k \rangle$. Thus we can obtain the matrix representation of (2-7) as

$$(\boldsymbol{A}_n + \boldsymbol{B}_n)\, \boldsymbol{w}_n = \boldsymbol{g}_n. \tag{2-8}$$

The classical Fourier–Garlerkin method yields the linear system (2-8) with the dense coefficient matrix $\boldsymbol{A}_n + \boldsymbol{B}_n$. Solving a linear system with a dense coefficient matrix requires large computational costs. To address this issue, a truncation strategy was introduced in [8] to approximate the dense coefficient matrix $\boldsymbol{A}_n + \boldsymbol{B}_n$ by a sparse one having $\mathcal{O}(n \log n)$ number of nonzero entries with preserving the convergence order of the original method. Specifically, for each $n \in \mathbb{N}$, we define an index set $\mathbb{L}_n^2 := \{(k, l) \in \mathbb{Z}_n^2 : kl \le n\}$ and define the truncation matrix of $\boldsymbol{B}_n$ by setting

$$\widetilde{B}_4 := [\widetilde{D}_{k,l} : k, l \in \mathbb{Z}_n^+], \quad \widetilde{D}_{k,l} := \begin{cases} D_{k,l} & (k, l) \in \mathbb{L}_n^2, \\ 0_{2\times 2}, & \text{otherwise}, \end{cases}$$

and letting

$$\widetilde{\boldsymbol{B}}_n := \begin{bmatrix} B_1 & B_2 \\ B_3 & \widetilde{B}_4 \end{bmatrix}.$$

Replacing the dense matrix $\boldsymbol{B}_n$ in (2-8) by the sparse matrix $\widetilde{\boldsymbol{B}}_n$, we have that

$$(\boldsymbol{A}_n + \widetilde{\boldsymbol{B}}_n)\, \widetilde{\boldsymbol{w}}_n = \boldsymbol{g}_n. \tag{2-9}$$

This truncation strategy yields the fast Fourier–Galerkin method. Its convergence order and computational complexity are analyzed in [8]. Moreover, a fast quadrature strategy was proposed in [20] for computing the nonzero entries of the truncated matrix with total number $\mathcal{O}(n \log^2 n)$ of arithmetics for solving the entire linear system and with preserving the optimal order of convergence, when this method is applied to solving the Laplace equation, in which $\mathcal{N}f = 0$.

When this method is applied to solving the Poisson equation, one has to compute the integrals for the Fourier coefficients of the Newton potential $\mathcal{N}f$, which are defined on the domain Ω, not the boundary $\partial\Omega$. Due to the fact that the domain has one dimension higher than its boundary, computing these integrals requires high computational costs. This becomes a bottleneck of applying the fast Fourier–Galerkin method to solve the BIE of the Poisson equation. The purpose of this study is to develop a fast algorithm for computing the Fourier coefficients of the Newton potential $\mathcal{N}f \circ \boldsymbol{r}$ by using graded meshes and sparse grids so that the total number of arithmetics for solving the entire BIE is bounded above by $\mathcal{O}(n \log^2 n)$ and the resulting fully discrete Fourier–Galerkin method achieves an accuracy of order $\mathcal{O}(n^{-m} \log^3 n)$, where n is the number of quadrature points used in one coordinate direction, when the forcing function of the Poisson equation expressed in the polar coordinates has mth-order bounded mixed derivatives.

3. A fast algorithm for computing the Fourier coefficients of the Newton potential

We propose a fast algorithm for computing the Fourier coefficients of Newton potential $\mathcal{N}f \circ \boldsymbol{r}$. The definition of $\mathcal{N}f$ shows that computing the Fourier coefficients of $\mathcal{N}f \circ \boldsymbol{r}$ requires to compute triple integrals with weak singularity. To develop a fast algorithm for computing these integrals, it is required to address two mathematical issues: the singularity of their integrands and their high dimensionality. To overcome the difficulty caused by the singularity of their integrands, we transform the integrands to functions which have the only singularity on a line parallel to a coordinate axis, and then design a graded mesh which splits the integration domain into subdomains. The three-dimensionality of the integrals requires high computational costs to compute them. To handle this, we develop a sparse grids to compute the triple integrals efficiently on each of the subdomains.

We now address the first issue. Recall that the kth Fourier coefficient of $\mathcal{N}f \circ \boldsymbol{r}$ has the form

$$\widehat{\mathcal{N}f \circ \boldsymbol{r}}(k) = \int_{I_{2\pi}} \int_{\Omega} f(y)\, \Phi(\boldsymbol{r}(t) - y)\, e_{-k}(t)\, \mathrm{d}y\, \mathrm{d}t \tag{3-1}$$

for each $k \in \mathbb{L}_n := \{j \in \mathbb{Z} : |j| \le n-1\}$. Due to the singularity of Φ, we see that the integrand of (3-1) has a singularity on the set $\{(t, y) : t \in I_{2\pi},\ y \in \Omega \text{ and } y = \boldsymbol{r}(t)\}$. For example, when Ω is a circular domain, the integrand of (3-1) has a singularity along the curve marked red in Figure 1 (left). We transform the integrand of (3-1) to satisfy two conditions in order to apply the graded sparse grid technique to the integrand: the integration domain is a cuboid, and its singularity points locate on a line parallel to a coordinate axis. By the parametrization (2-4) of the boundary $\partial\Omega$, the domain Ω can be described as $\{y : y = \lambda \boldsymbol{r}(t-\mu),\ \lambda \in [0,1],\ \mu \in [-\pi, \pi)\}$, for all $t \in [0, 2\pi)$. We then substitute y in the integral (3-1) by $\lambda \boldsymbol{r}(t-\mu)$ and define the function Λf on $S_0 := \{(t, \mu, \lambda) : t \in [0, 2\pi),\ \mu \in [-\pi, \pi) \text{ and } \lambda \in [0,1]\}$ as

$$\Lambda f(t, \mu, \lambda) := f(\lambda \boldsymbol{r}(t-\mu))\, \Phi(\boldsymbol{r}(t) - \lambda \boldsymbol{r}(t-\mu))\, \lambda r^2(t-\mu). \tag{3-2}$$

Thus, the integral (3-1) is rewritten as

$$\widehat{\mathcal{N}f \circ \boldsymbol{r}}(k) = \iiint_{S_0} \Lambda f(t, \mu, \lambda)\, e_{-k}(t)\, \mathrm{d}t\, \mathrm{d}\mu\, \mathrm{d}\lambda. \tag{3-3}$$

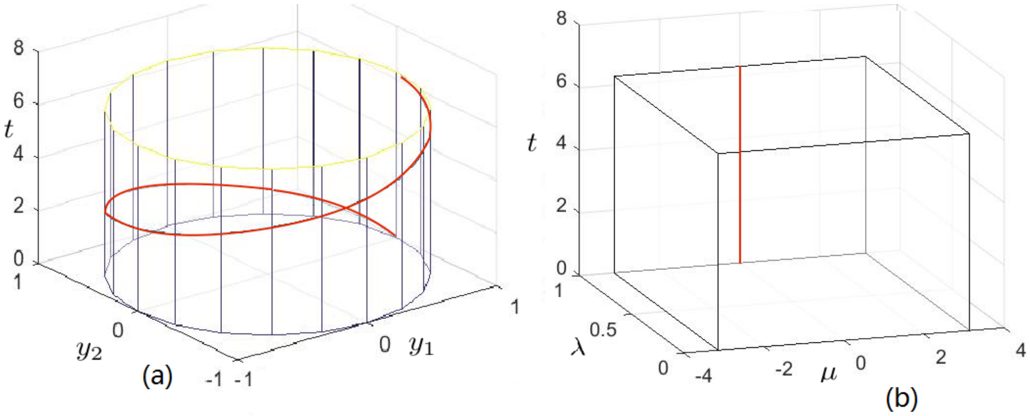

Figure 1. The integration domains of (3-1) (left) and (3-3) (right). The red curves indicate the singularity.

The integration domain of (3-3) becomes a cuboid after transformation. The integrand has a singularity on the line segment $T := \{(t, \mu, \lambda) : t \in [0, 2\pi), \mu = 0, \lambda = 1\}$, the red line shown in Figure 1 (right).

The singularity of the integrand of (3-3) on T prevents a direct use of the sparse grid technique to evaluate the integral (3-3), since the use of the sparse grid technique requires smooth functions. To overcome this problem, we use a graded mesh by dividing the integration domain $S_0 = [0, 2\pi) \times [-\pi, \pi) \times [0, 1]$ into a sequence of subdomains. We observe that the derivatives of Λf increase as it closes to the singularity. This observation leads to a radial refinement towards the singularity in the μ-λ plane by adapting the idea in [22] to this setting (see, also [24]). Specifically, for all $q \in \mathbb{N}_0 := \{0\} \cup \mathbb{N}$, we define

$$S_q := \{(t, \mu, \lambda) : t \in [0, 2\pi), \mu \in [-\mu_q, \mu_q], \lambda \in [\lambda_q, 1]\} \quad \text{and} \quad D_q := S_q - S_{q+1},$$

where $\mu_q := \pi/2^q$, $\lambda_q := 1 - 1/2^q$. It is clear that for any fixed $\tau \in \mathbb{N}$,

$$S_0 = S_\tau \cup D_0 \cup \cdots \cup D_{\tau-1},$$

see, Figure 2. With this partition, the integral (3-3) can be expressed as a summation of the integrals over the subdomains S_τ and D_q, $q \in \mathbb{Z}_\tau$. That is

$$\widehat{\mathcal{N} f \circ \boldsymbol{r}}(k) = \iiint_{S_\tau} \Lambda f(t, \mu, \lambda)\, e_{-k}(t)\, \mathrm{d}t\, \mathrm{d}\mu\, \mathrm{d}\lambda + \sum_{q=0}^{\tau-1} \widehat{(\Lambda f)}_q(k), \tag{3-4}$$

where

$$\widehat{(\Lambda f)}_q(k) := \iiint_{D_q} (\Lambda f)(t, \mu, \lambda)\, e_{-k}(t)\, \mathrm{d}t\, \mathrm{d}\mu\, \mathrm{d}\lambda. \tag{3-5}$$

Since the integral (3-3) is a weakly singular integral, when τ is sufficiently large, we drop the first term of (3-4) and the integral can be approximated well by

$$\widetilde{\mathfrak{Q}}_\tau f(k) := \sum_{q=0}^{\tau-1} \widehat{(\Lambda f)}_q(k), \quad k \in \mathbb{L}_n. \tag{3-6}$$

An appropriate choice of τ will be specified later. On passing, we comment that when $k \gg 1$, the integral $\widetilde{\mathfrak{Q}}_\tau f(k)$ involves an oscillation factor. We construct an approximation of Λf and then employ the methods originated in [16; 24] so that the oscillatory integrals are evaluated efficiently.

Computing (3-6) requires to approximate Λf on the noncuboid domains D_q. We will develop the sparse grid method to approximate Λf. The sparse grid method requires a cube domain while the domains D_q are noncuboid. To overcome this problem, we extend the function Λf to the cuboid domains S_q. We next construct an extension of function $(\Lambda f)|_{\overline{D}_q}$ from the noncuboid domain $\overline{D}_q$ to the cuboid domain S_q such that the high-order derivative of the extension is continuous on S_q, where $\overline{D}_q$ is the closure of D_q.

We construct below an extension $\mathscr{E}_q\omega$ of a smooth function ω from the noncuboid domain $\overline{D}_q$ to the cuboid domain S_q by employing the Hermite interpolation. For all $\boldsymbol{\alpha} := [\alpha_0, \alpha_1, \alpha_2] \in \mathbb{N}_0^3$, we let $|\boldsymbol{\alpha}| := \sum_{i\in\mathbb{Z}_3} |\alpha_i|$ and

$$\omega^{(\boldsymbol{\alpha})}(t, \mu, \lambda) := \left(\frac{\partial^{|\boldsymbol{\alpha}|}}{\partial t^{\alpha_0}\, \partial \mu^{\alpha_1}\, \partial \lambda^{\alpha_2}}\omega\right)(t, \mu, \lambda).$$

For a fixed $m \in \mathbb{N}$, we require the high-order derivatives $(\mathscr{E}_q\omega)^{(\boldsymbol{\alpha})}$, for all $|\boldsymbol{\alpha}|_\infty < m$, are continuous on S_q. To this end, we define the polynomial function

$$q_j(\mu) := \frac{\mu^j}{j!}(1-\mu)^{m+1} \sum_{s=0}^{m-j} \binom{m+s}{s} \mu^s. \tag{3-7}$$

For a smooth function ω defined on $\overline{D}_q$, we construct its extension $\mathscr{E}_q\omega$ by the Hermite interpolation with respect to the variables λ and μ having the form

$$\mathscr{E}_q\omega(t, \mu, \lambda) := \begin{cases} \omega(t, \mu, \lambda), & (t, \mu, \lambda) \in D_q, \\ \omega_{q,1}(t, \mu, \lambda) + \omega_{q,2}(t, \mu, \lambda) - \omega_{q,\text{corner}}(t, \mu, \lambda), & (t, \mu, \lambda) \in S_{q+1}, \end{cases} \tag{3-8}$$

where

$$\omega_{q,1}(t,\mu,\lambda) := \sum_{j=0}^{m} \frac{\partial^j \omega}{\partial \lambda^j}(t,\mu,\lambda_{q+1}) \frac{1}{j!}(\lambda-\lambda_{q+1})^j,$$

$$\omega_{q,2}(t,\mu,\lambda) := \sum_{j=0}^{m} (2\mu_{q+1})^j \left[\frac{\partial^j \omega}{\partial \mu^j}(t,-\mu_{q+1},\lambda)\, q_j\left(\frac{\mu+\mu_{q+1}}{2\mu_{q+1}}\right) + (-1)^j \frac{\partial^j \omega}{\partial \mu^j}(t,\mu_{q+1},\lambda)\, q_j\left(\frac{\mu_{q+1}-\mu}{2\mu_{q+1}}\right)\right],$$

and

$$\begin{aligned}\omega_{q,\text{corner}}(t, \mu, \lambda) := \sum_{j=0}^{m}\sum_{k=0}^{m} (2\mu_{q+1})^j \frac{1}{k!}(\lambda-\lambda_{q+1})^k \Bigg[&\frac{\partial^{j+k}\omega}{\partial \mu^j\, \partial \lambda^k}(t, -\mu_{q+1}, \lambda_{q+1})\, q_j\left(\frac{\mu+\mu_{q+1}}{2\mu_{q+1}}\right) \\ &+ (-1)^j \frac{\partial^{j+k}\omega}{\partial \mu^j\, \partial \lambda^k}(t, \mu_{q+1}, \lambda_{q+1})\, q_j\left(\frac{\mu_{q+1}-\mu}{2\mu_{q+1}}\right)\Bigg].\end{aligned}$$

Clearly, $\mathscr{E}_q\omega$ is an extension of ω. In the following lemma, we verify that the high-order derivatives $(\mathscr{E}_q\omega)^{(\boldsymbol{\alpha})}$, for all $|\boldsymbol{\alpha}|_\infty < m$, are continuous on S_q.

Lemma 3.1. Let $q \in \mathbb{N}_0$, $m \in \mathbb{N}$ and $\boldsymbol{\alpha} \in \mathbb{N}_0^3$. If $\omega^{(\boldsymbol{\alpha})} \in C(\overline{D}_q)$, for all $|\boldsymbol{\alpha}|_\infty < m$, then $(\mathscr{E}_q\omega)^{(\boldsymbol{\alpha})} \in C(S_q)$.

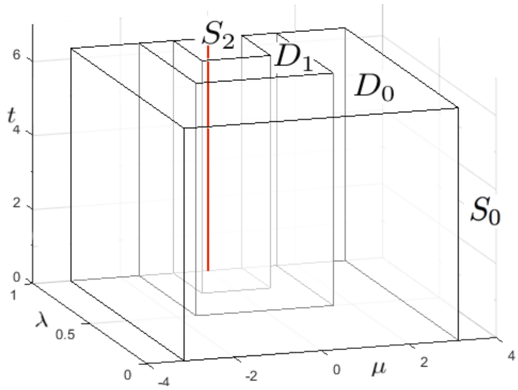

Figure 2. The integration domain S_0 consists of the domains D_0, D_1 and S_2.

Proof. By the definition (3-8), $\mathscr{E}_q\omega$ is a polynomial on S_{q+1}. It suffices to verify that on the junction between D_q and S_{q+1}, $\mathscr{E}_q\omega^{(\boldsymbol{\alpha})}$ are continuous. That is, we shall prove that $(\mathscr{E}_q\omega)^{(\boldsymbol{\alpha})}(t,\mu,\lambda)=\omega(t,\mu,\lambda)$ on the boundaries $[0,2\pi)\times\{-\mu_{q+1},\mu_{q+1}\}\times[\lambda_{q+1},1]$ and $[0,2\pi)\times[-\mu_{q+1},\mu_{q+1}]\times\{\lambda_{q+1}\}$. Specifically, it remains to show that the extension $\mathscr{E}_q\omega$ satisfies for all $|\boldsymbol{\alpha}|_\infty<m$ that

$$(\mathscr{E}_q\omega)^{(\boldsymbol{\alpha})}(t,\mu,\lambda_{q+1})=\omega^{(\boldsymbol{\alpha})}(t,\mu,\lambda_{q+1}), \tag{3-9}$$

$$(\mathscr{E}_q\omega)^{(\boldsymbol{\alpha})}(t,\mu_{q+1},\lambda)=\omega^{(\boldsymbol{\alpha})}(t,\mu_{q+1},\lambda), \tag{3-10}$$

$$(\mathscr{E}_q\omega)^{(\boldsymbol{\alpha})}(t,-\mu_{q+1},\lambda)=\omega^{(\boldsymbol{\alpha})}(t,-\mu_{q+1},\lambda). \tag{3-11}$$

We next verify (3-9), (3-10) and (3-11). For this purpose, we rewrite (3-8) as a summation of an Hermite interpolation of the function ω with respect to λ and an Hermite interpolation of the function ω with respect to μ. Firstly, we define the one-point Hermit interpolating extension $\mathscr{G}_q\omega$ of the function ω with respect to the variable λ for each $(t,\mu)\in[0,2\pi)\times[-\mu_{q+1},\mu_{q+1}]$ by

$$\mathscr{G}_q\omega(t,\mu,\lambda):=\begin{cases}\omega(t,\mu,\lambda), & (t,\mu,\lambda)\in D_q,\\ \sum_{j=0}^{m}\frac{\partial^j\omega}{\partial\lambda^j}(t,\mu,\lambda_{q+1})\frac{1}{j!}(\lambda-\lambda_{q+1})^j, & (t,\mu,\lambda)\in S_{q+1}.\end{cases}$$

This extension satisfies that for all $|\boldsymbol{\alpha}|_\infty<m$,

$$(\mathscr{G}_q\omega)^{(\boldsymbol{\alpha})}(t,\mu,\lambda_{q+1})=\omega^{(\boldsymbol{\alpha})}(t,\mu,\lambda_{q+1}). \tag{3-12}$$

The extension $\mathscr{G}_q\omega$ is smooth with respect to the variables t and λ. However, it has a discontinuity in the variable μ on $\{(t,\mu,\lambda):t\in[0,2\pi),\ \mu=-\mu_{q+1}\text{ or }\mu=\mu_{q+1},\ \lambda\in[\lambda_{q+1},1]\}$. We then construct the two-point Hermit interpolating extension $\mathscr{H}_q\nu$ of the function $\nu:=\omega-\mathscr{G}_q\omega$ with respect to variable μ to overcome the discontinuity of $\mathscr{G}_q\omega$. The two-point Hermit interpolating extension $\mathscr{H}_q\nu$ is defined by

$$\mathscr{H}_q\nu(t,\mu,\lambda):=\begin{cases}0, & (t,\mu,\lambda)\in D_q,\\ \sum_{j=0}^{m}\{(2\mu_{q+1})^j\frac{\partial^j\nu}{\partial\mu^{j,-}}(t,-\mu_{q+1},\lambda)q_j(\frac{\mu+\mu_{q+1}}{2\mu_{q+1}}) & \\ \quad+(-2\mu_{q+1})^j\frac{\partial^j\nu}{\partial\mu^{j,+}}(t,\mu_{q+1},\lambda)q_j(\frac{\mu_{q+1}-\mu}{2\mu_{q+1}})\}, & (t,\mu,\lambda)\in S_{q+1}.\end{cases}$$

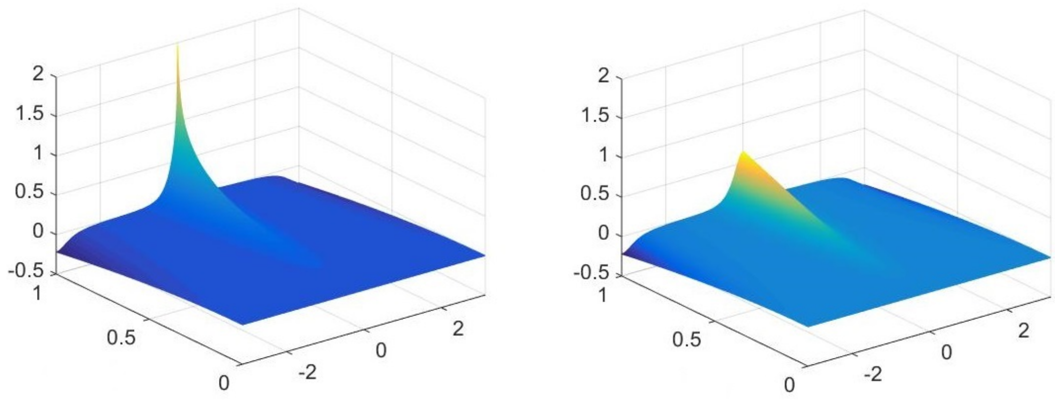

Figure 3. Left: the graph of a singular function ω on a cross section of S_0. Right: the graph of the function $\mathscr{E}_0\omega$ on the same cross section of S_0, where $\mathscr{E}_0\omega$ is the extension of ω from D_0 to S_0.

It can be verified that $\mathscr{H}_q\nu$ satisfies the conditions for all $|\boldsymbol{\alpha}|_\infty < m$ that

$$(\mathscr{H}_q\nu)^{(\boldsymbol{\alpha})}(t, \mu, \lambda_{q+1}) = 0, \tag{3-13}$$

$$(\mathscr{H}_q\nu)_-^{(\boldsymbol{\alpha})}(t, -\mu_{q+1}, \lambda) = \nu_-^{(\boldsymbol{\alpha})}(t, -\mu_{q+1}, \lambda), \tag{3-14}$$

$$(\mathscr{H}_q\nu)_+^{(\boldsymbol{\alpha})}(t, \mu_{q+1}, \lambda) = \nu_+^{(\boldsymbol{\alpha})}(t, \mu_{q+1}, \lambda), \tag{3-15}$$

where for a smooth enough function h on S_q, $h_-^{(\boldsymbol{\alpha})}(t, -\mu_{q+1}, \lambda)$ is the left partial derivative of h with respect to μ on $(t, -\mu_{q+1}, \lambda)$, and $h_+^{(\boldsymbol{\alpha})}(t, \mu_{q+1}, \lambda)$ is the right partial derivative of h with respect to μ on (t, μ_{q+1}, λ). With the above one- and two-point Hermit interpolating extensions, we rewrite (3-8) as

$$\mathscr{E}_q\omega = \mathscr{G}_q\omega + \mathscr{H}_q(\omega - \mathscr{G}_q\omega).$$

where

$$\mathscr{G}_q\omega(t, \mu, \lambda) = \omega_{q,1}(t, \mu, \lambda) \quad \text{and} \quad \mathscr{H}_q(\omega - \mathscr{G}_q\omega) = \omega_{q,2}(t, \mu, \lambda) - \omega_{q,\text{corner}}(t, \mu, \lambda).$$

By equations (3-12), (3-13), (3-14) and (3-15), it can be verified for all $|\boldsymbol{\alpha}|_\infty < m$ that the equations (3-9), (3-10) and (3-11) hold, completing the proof. □

We perform the above extension procedure to $(\Lambda f)|_{\overline{D}_q}$ from $\overline{D}_q$ to S_q so that the extension $\mathscr{E}_q(\Lambda f) := \mathscr{E}_q((\Lambda f)|_{\overline{D}_q})$ has enough degree of smoothness to construct a sparse grid approximation. Since the extension $\mathscr{E}_q(\Lambda f)$ is equal to Λf on D_q, it follows from (3-6) that

$$\widetilde{\mathfrak{D}}_\tau f(k) = \sum_{q=0}^{\tau-1} \iiint_{D_q} \mathscr{E}_q(\Lambda f)(t, \mu, \lambda)\, e_{-k}(t)\, \mathrm{d}t\, \mathrm{d}\mu\, \mathrm{d}\lambda, \qquad k \in \mathbb{L}_n. \tag{3-16}$$

We shall construct a sparse grid approximation of $\mathscr{E}_q(\Lambda f)$ on the cuboid domain S_q and use it to replace the function $\mathscr{E}_q(\Lambda f)$ in the integrand of (3-16). This will lead to a fast algorithm for computing an approximate value of $\widetilde{\mathfrak{D}}_\tau f(k)$.

We next describe the multiscale Lagrange interpolation for the construction of the sparse grid approximation which will be used to compute an approximate value of the integral (3-16). We recall the multiscale Lagrange interpolation on $I := [0, 1]$, originally developed in [10] and the multiscale piecewise polynomial interpolation formula on sparse grids presented in [9; 17; 19]. The piecewise polynomial interpolating functions have a recursive structure and they are generated efficiently by the sequences of interpolation functionals with a multiscale structure. This requires the concept of the refinable set relative to a family of contractive mappings on I, which is used to generate the interpolation functionals. We define two contractive mappings $\epsilon_k : I \to I,\ k = \{0, 1\}$, by $\epsilon_0(t) := t/2$ and $\epsilon_1(t) := t + 1/2,\ t \in I$, and let $\boldsymbol{\epsilon} := \{\epsilon_k : k \in \mathbb{Z}_2\}$. For a finite set $V \subseteq I$, we let $\boldsymbol{\epsilon}(V) := \{\epsilon_k(t) : t \in V, k \in \mathbb{Z}_2\}$ be the image of V under the family $\boldsymbol{\epsilon}$ of contractive mappings. According to [10; 12], a subset V of I is said refinable relative to the mapping $\boldsymbol{\epsilon}$ if $V \subseteq \boldsymbol{\epsilon}(V)$. We then define the Lagrange polynomials with the interpolating points on a refinable set $V := \{v_r : 0 \le v_0 < v_1 < \cdots < v_{m-1} < 1\}$. The Lagrange polynomials ℓ_r of degree $m-1$ are defined as

$$\ell_r(t) := \prod_{q=0, q\neq r}^{m-1} \frac{t - v_q}{v_r - v_q}$$

for $t \in I,\ r \in \mathbb{Z}_m$. We next describe the multiscale Lagrange polynomials associate with the refinable set V. To this end, for the family $\boldsymbol{\epsilon}$ of contractive mappings, we define linear operators $\mathcal{J}_k : L^\infty(I) \to L^\infty(I)$, $k \in \mathbb{Z}_2$ for $\omega \in L^\infty(I)$ by

$$(\mathcal{J}_k\omega)(x) := \begin{cases} (\omega \circ \epsilon_k^{-1})(x), & x \in \epsilon_k(I), \\ 0, & x \notin \epsilon_k(I). \end{cases}$$

For all $N \in \mathbb{N}_0$ and $r \in \mathbb{Z}_{m2^N}$, we define the multiscale Lagrange polynomials as

$$\ell_{N,r} := \begin{cases} \ell_r, & N = 0, \\ \mathcal{J}_{\boldsymbol{p}}\, \ell_{r_0}, & N > 0, \end{cases}$$

where $\boldsymbol{p} := [p_k : k \in \mathbb{Z}_N] \in \mathbb{Z}_2^N$ and $\mathcal{J}_{\boldsymbol{p}} := \mathcal{J}_{p_{N-1}} \circ \cdots \circ \mathcal{J}_{p_0}$. The multiscale interpolation points of $\ell_{N,r}$ are given by

$$v_{N,r} := \begin{cases} v_r, & N = 0, \\ \epsilon_{\boldsymbol{p}}\, v_{r_0}, & N > 0, \end{cases}$$

where $\boldsymbol{p} \in \mathbb{Z}_2^N,\ \epsilon_{\boldsymbol{p}} := \epsilon_{p_{N-1}} \circ \cdots \circ \epsilon_{p_0},\ r_0 = r - m\mathcal{K}(\boldsymbol{p}) \in \mathbb{Z}_m$, and $\mathcal{K}(\boldsymbol{p}) := \sum_{k \in \mathbb{Z}_N} p_k 2^k$. It is easy to see that the multiscale Lagrange polynomials satisfy $\ell_{N,r}(v_{N,r'}) = 1$, if $r = r'$, and 0 otherwise. The set of Nth-level intepolation points V_N is defined by $V_N := \{v_{N,r} : r \in \mathbb{Z}_{m2^N}\}$ for all $N \in \mathbb{N}_0$. Thus, the Nth-level Lagrange interpolation operator $\mathcal{P}_N : C(I) \to \mathbb{P}_N$ associate with the polynomials $\ell_{N,r}$ is defined as

$$\mathcal{P}_N\omega := \sum_{r \in \mathbb{Z}_{m2^N}} \chi_{N,r}(\omega)\, \ell_{N,r}, \quad \omega \in C(I), \tag{3-17}$$

where $\chi_{N,r}(\omega) := \omega(v_{N,r})$, and $\mathbb{P}_N := \operatorname{span}\{\ell_{N,r} : r \in \mathbb{Z}_{m2^N}\}$.

We next rewrite the interpolation (3-17) in a multiscale form in order to describe the Lagrange interpolation on sparse grids. We define the index set on each level as

$$\mathbb{W}_0 := \mathbb{Z}_m \quad \text{and} \quad \mathbb{W}_j := \{r \in \mathbb{Z}_{m2^j} : v_{j,r} \in V_j \setminus V_{j-1}\}$$

for $j \in \mathbb{N}$. With the notation above, for all $N \in \mathbb{N}$ and $\omega \in C(I)$ we have

$$\mathcal{P}_N \omega = \sum_{j=0}^{N} \sum_{r \in \mathbb{W}_j} \eta_{j,r}(\omega)\, \ell_{j,r}. \tag{3-18}$$

where

$$\eta_{0,r}(\omega) := \omega(v_{0,r}), \quad \eta_{j,r}(\omega) := \omega(v_{j,r}) - \sum_{q \in \mathbb{Z}_m} \omega(v_{j-1,m\lfloor \frac{r}{2m} \rfloor + q})\, a_{q,r \mod 2m},$$

and

$$a_{r,k} := \ell_{0,r}(v_{1,k}), \quad r \in \mathbb{Z}_m \quad \text{and} \quad k \in \mathbb{Z}_{2m}.$$

The formula (3-18) is called the multiscale Lagrange interpolation on I.

We next describe the construction of the tensor product type multiscale Lagrange interpolation formula on I^3. To this end, we introduce the tensor product of linear functionals as below. For $d \in \mathbb{N}$, we denote by $\mathcal{L}(C(I^d))$ the set of linear functionals in $C(I^d)$ which satisfy the condition: for each $\kappa \in \mathcal{L}(C(I^d))$, there exist $a_j \in \mathbb{R}$ and $\boldsymbol{x}_j \in I^d$ such that for all $f \in C(I^d)$,

$$\kappa(f) = \sum_{j \in \mathbb{Z}_n} a_j f(\boldsymbol{x}_j).$$

Let $d_1, d_2, n_1, n_2 \in \mathbb{N}$. For all $\kappa_1 \in \mathcal{L}(C(I^{d_1}))$ and $\kappa_2 \in \mathcal{L}(C(I^{d_2}))$, we define the tensor product $\kappa_1 \otimes \kappa_2$, for all $\omega \in C(I^{d_1+d_2})$, by

$$(\kappa_1 \otimes \kappa_2)(\omega) := \sum_{j_1 \in \mathbb{Z}_{n_1}} \sum_{j_2 \in \mathbb{Z}_{n_2}} a^1_{j_1} a^2_{j_2}\, \omega(\boldsymbol{x}^1_{j_1}, \boldsymbol{x}^2_{j_2}),$$

where

$$\kappa_1(\omega_1) = \sum_{j_1 \in \mathbb{Z}_{n_1}} a^1_{j_1} \omega_1(\boldsymbol{x}^1_{j_1})$$

for all $\omega_1 \in C(I^{d_1})$, and

$$\kappa_2(\omega_2) = \sum_{j_2 \in \mathbb{Z}_{n_2}} a^2_{j_2} \omega_2(\boldsymbol{x}^2_{j_2}),$$

for all $\omega_2 \in C(I^{d_2})$. Let $\mathbb{W}^3_{\boldsymbol{j}} := \mathbb{W}_{j_0} \times \mathbb{W}_{j_1} \times \mathbb{W}_{j_2}$. With the notation above, for $\boldsymbol{j} := [j_k : k \in \mathbb{Z}_3] \in \mathbb{N}_0^3$, $\boldsymbol{r} \in \mathbb{W}^3_{\boldsymbol{j}}$, we define the linear functional $\eta_{\boldsymbol{j},\boldsymbol{r}} := \eta_{j_0,r_0} \otimes \eta_{j_1,r_1} \otimes \eta_{j_2,r_2}$, and the 3D Lagrange polynomial $\ell_{\boldsymbol{j},\boldsymbol{r}}(\boldsymbol{x}) := \ell_{j_0,r_0}(x_0)\, \ell_{j_1,r_1}(x_1)\, \ell_{j_2,r_2}(x_2)$, where $\boldsymbol{x} := [x_0, x_1, x_2] \in I^3$. For $\omega \in C(I^3)$ and $N \in \mathbb{N}$, the multiscale Lagrange interpolation of ω on full grids is defined by

$$\mathcal{P}^3_N \omega := \sum_{\boldsymbol{j} \in \mathbb{Z}^3_N} \sum_{\boldsymbol{r} \in \mathbb{W}^3_{\boldsymbol{j}}} \eta_{\boldsymbol{j},\boldsymbol{r}}(\omega)\, \ell_{\boldsymbol{j},\boldsymbol{r}}. \tag{3-19}$$

This interpolation involves $\mathcal{O}(2^{3N})$ interpolation points.

Employing the multiscale Lagrange interpolation on I, we introduce the sparse approximations of functions in $C(I^3)$ by using the sparse grid technique. For $\omega \in C(I^3)$ and $N \in \mathbb{N}$, the multiscale Lagrange interpolation of ω on sparse grids is defined by

$$\mathcal{S}_N\,\omega := \sum_{\boldsymbol{j}\in\mathbb{S}_N^3}\sum_{\boldsymbol{r}\in\mathbb{W}_{\boldsymbol{j}}^3} \eta_{\boldsymbol{j},\boldsymbol{r}}(\omega)\,\ell_{\boldsymbol{j},\boldsymbol{r}}, \quad \text{where} \quad \mathbb{S}_N^3 := \Big\{\boldsymbol{j}\in\mathbb{Z}_{N+1}^3 : \sum_{k\in\mathbb{Z}_3} j_k \le N\Big\}. \tag{3-20}$$

Unlike the full grid interpolation $\mathcal{P}_N^3\,\omega$ which uses $\mathbb{O}(2^{3N})$ number of interpolation points, the interpolation on the sparse grid uses only $\mathbb{O}(N^2 2^N)$ number of interpolation points.

We apply the interpolation scheme (3-20) to approximate the extension $\mathscr{E}_q(\Lambda f)$. To this end, we transform $\mathscr{E}_q(\Lambda f)$ so that the resulting function is defined on the domain I^3. We denote by $\boldsymbol{\phi}_q := (\phi_0, \phi_{1,q}, \phi_{2,q})$ the mapping from the cuboid D_q to the cube I^3 where

$$\phi_0(t) := \frac{1}{2\pi}t, \quad \phi_{1,q}(t) := \frac{2^q}{2\pi}t + \frac{1}{2}, \quad \phi_{2,q}(t) := 2^q t + 1 - 2^q.$$

Let $\boldsymbol{\phi}_q^{-1} := (\phi_0^{-1}, \phi_{1,q}^{-1}, \phi_{2,q}^{-1})$. For $N_q \in \mathbb{N}$, we construct the approximation of $\mathscr{E}_q(\Lambda f)$ by

$$\widetilde{\mathcal{S}}_{N_q}(\mathscr{E}_q(\Lambda f)) := \mathcal{S}_{N_q}\big(\mathscr{E}_q(\Lambda f)\circ\boldsymbol{\phi}_q^{-1}\big)\circ\boldsymbol{\phi}_q.$$

We now return to computing the integral (3-16). Replacing the extension $\mathscr{E}_q(\Lambda f)$ in the integral (3-16) by the sparse approximation $\widetilde{\mathcal{S}}_{N_q}(\mathscr{E}_q(\Lambda f))$ leads to the following quadrature formula for computing the Fourier coefficients of the Newton potential $\mathcal{N}f\circ\boldsymbol{r}$,

$$\mathcal{Q}_{\tau,\boldsymbol{N}} f(k) := \sum_{q\in\mathbb{Z}_\tau} \iiint_{D_q} \widetilde{\mathcal{S}}_{N_q}(\mathscr{E}_q(\Lambda f))(t,\mu,\lambda)\, e_{-k}(t)\,\mathrm{d}t\,\mathrm{d}\mu\,\mathrm{d}\lambda, \quad k\in\mathbb{L}_n, \tag{3-21}$$

where $\tau\in\mathbb{N}$ and $\boldsymbol{N} := [N_0,\dots,N_{\tau-1}]\in\mathbb{N}^\tau$. We define the multiscale Lagrange polynomials on $[0,2\pi)$, $[-\mu_q,\mu_q)$ and $[\lambda_q,1)$, respectively, by

$$\ell_{0,j,r} := \ell_{j,r}\circ\phi_0, \quad \ell_{1,q,j,r} := \ell_{j,r}\circ\phi_{1,q}, \quad \ell_{2,q,j,r} := \ell_{j,r}\circ\phi_{2,q}.$$

The partition $\{D_q : q\in\mathbb{N}\}$ of S_0 leads to a partition $\{\widetilde{D}_q : q\in\mathbb{N}\}$ of $[-\pi,\pi)\times[0,1)$ in the μ-λ plane, where $\widetilde{D}_q$ is defined by

$$\widetilde{D}_q := ([-\mu_q,\mu_q]\times[\lambda_q,1])\setminus([-\mu_{q+1},\mu_{q+1}]\times[\lambda_{q+1},1]).$$

By the definition of $\widetilde{\mathcal{S}}_{N_q}(\mathscr{E}_q(\Lambda f))$, the formula (3-21) can be rewritten as

$$\mathcal{Q}_{\tau,\boldsymbol{N}} f(k) = \sum_{q\in\mathbb{Z}_\tau}\sum_{\boldsymbol{j}\in\mathbb{S}_{N_q}^3}\sum_{\boldsymbol{r}\in\mathbb{W}_{\boldsymbol{j}}^3} \tilde{\eta}_{\boldsymbol{j},\boldsymbol{r}}(\mathscr{E}_q(\Lambda f))\,A_{j_0,r_0}(k)\,L_{q,\boldsymbol{j},\boldsymbol{r}}, \quad k\in\mathbb{L}_n, \tag{3-22}$$

where

$$\tilde{\eta}_{\boldsymbol{j},\boldsymbol{r}}(\mathscr{E}_q(\Lambda f)) := \eta_{\boldsymbol{j},\boldsymbol{r}}(\mathscr{E}_q(\Lambda f)\circ\boldsymbol{\phi}_q^{-1}), \quad A_{j_0,r_0}(k) := \int_0^{2\pi} \ell_{0,j_0,r_0}(t)\,e_{-k}(t)\,\mathrm{d}t,$$

and

$$L_{q,\boldsymbol{j},\boldsymbol{r}} := \iint_{\widetilde{D}_q} \ell_{1,q,j_1,r_1}(\mu)\,\ell_{2,q,j_2,r_2}(\lambda)\,\mathrm{d}\mu\,\mathrm{d}\lambda.$$

The formula (3-22) serves a basis for computing the kth Fourier coefficients of $\mathcal{N}f\circ\boldsymbol{r}$.

We next describe a fast algorithm for computing $\mathcal{Q}_{\tau,N}f(k)$ based on (3-22) by eliminating repeated computations of $A_{j_0,r_0}(k)$ when computing the $2n-1$ different Fourier coefficients corresponding to $k\in\mathbb{L}_n$. To this end, for fixed $s\in\mathbb{Z}_{m2^u}$, $u\in\mathbb{Z}_{N_q}$, we let

$$\Gamma_{q,u,s}:=\sum_{\boldsymbol{j}\in\mathbb{S}^3_{N_q},\,j_0=u}\ \sum_{\boldsymbol{r}\in\mathbb{W}^3_{\boldsymbol{j}},\,r_0=s}\tilde{\eta}_{\boldsymbol{j},\boldsymbol{r}}(\mathcal{E}_q(\Lambda f))\,L_{q,\boldsymbol{j},\boldsymbol{r}},\quad q\in\mathbb{Z}_\tau.$$

By changing the order of summations in (3-22), we obtain that

$$\mathcal{Q}_{\tau,N}f(k)=\sum_{q\in\mathbb{Z}_\tau}\sum_{u\in\mathbb{Z}_{N_q}}\sum_{s\in\mathbb{Z}_{m2^u}}\Gamma_{q,u,s}\,A_{u,s}(k),\quad k\in\mathbb{L}_n. \tag{3-23}$$

We consider the computation of the partial sum

$$\mathcal{G}_{q,u}(k):=\sum_{s\in\mathbb{Z}_{m2^u}}\Gamma_{q,u,s}\,A_{u,s}(k),\quad k\in\mathbb{L}_n. \tag{3-24}$$

Computing these partial sums for $k\in\mathbb{L}_n$ requires $(2n-1)m2^u$ number of multiplications. In order to reduce the computational complexity, we define the discrete Fourier coefficients of the vector $\Gamma_{q,u}:=[\Gamma_{q,u,s}:s\in\mathbb{Z}_{m2^u}]$ by

$$\widehat{\Gamma}_{q,u}:=\Bigg[\sum_{s\in\mathbb{Z}_{m2^u}}\Gamma_{q,u,s}\,e^{-i2\pi k(s/m2^u)}:k\in\mathbb{L}_n\Bigg]. \tag{3-25}$$

We then rewrite the formula (3-24) as

$$\mathcal{G}_{q,u}(k)=t_u(k)(\widehat{\boldsymbol{\Gamma}}_{q,u})_k,\quad k\in\mathbb{L}_n, \tag{3-26}$$

where

$$t_u(k):=\frac{1}{2^u\sqrt{2\pi}}\int_0^{2\pi}\ell_{0,0,0}(t)\,e^{-ikt/2^u}\,\mathrm{d}t.$$

By employing the periodicity of the discrete Fourier transform, we obtain that

$$(\widehat{\boldsymbol{\Gamma}}_{q,u})_k=(\widehat{\boldsymbol{\Gamma}}_{q,u})_{\mathcal{L}_u(k)},\quad k\in\mathbb{L}_n,$$

where $\mathcal{L}_u:\mathbb{Z}\to\mathbb{Z}_{m2^u}$ is a modulo operation defined by

$$\mathcal{L}_u(k):=\begin{cases}k-m2^u\lfloor k/m2^u\rfloor, & k\ge 0,\\ k+m2^u\lceil -k/m2^u\rceil, & k<0.\end{cases}$$

By (3-26), we have that

$$\mathcal{G}_{q,u}(k)=t_u(k)(\widehat{\boldsymbol{\Gamma}}_{q,u})_{\mathcal{L}_u(k)},\quad k\in\mathbb{L}_n. \tag{3-27}$$

The vector $\widehat{\Gamma}_{q,u}$ defined in (3-25) can be computed by applying the fast Fourier transform to $\Gamma_{q,u}$, which costs $(m\log m)\cdot u2^u$ number of multiplications. Thus, the number of multiplications in the partial sums (3-27) is $(2n-1)+(m\log m)\cdot u2^u$ for $2n-1$ different Fourier coefficients, which is less than

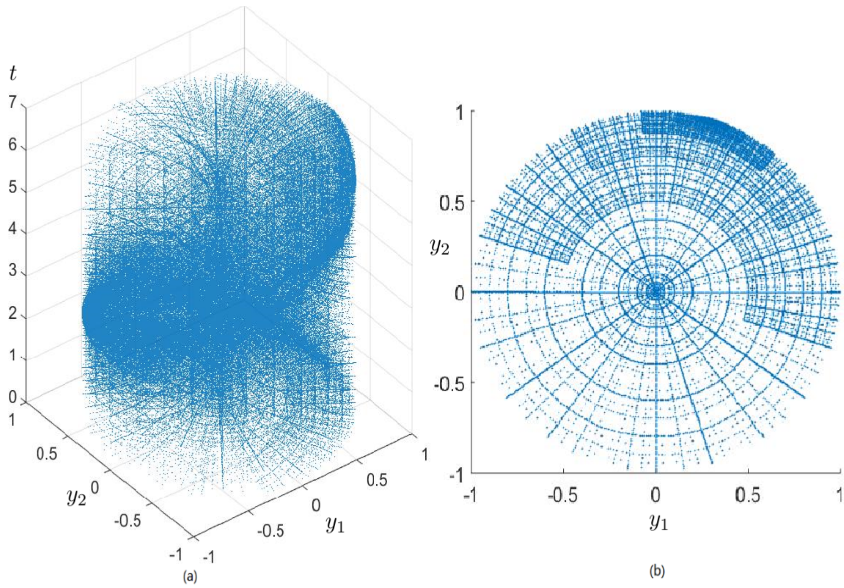

Figure 4. The sample points of Algorithm 1 (left) and its cross section (right).

that required for computing the partial sums (3-26) when $u < n$. By substituting the partial sums (3-27) into (3-23), we have that

$$\mathcal{Q}_{\tau,\boldsymbol{N}} f(k) = \sum_{q\in\mathbb{Z}_\tau} \sum_{u\in\mathbb{Z}_{N_q}} t_u(k)(\widehat{\boldsymbol{\Gamma}}_{q,u})_{\mathcal{L}_u(k)}, \quad k \in \mathbb{L}_n. \tag{3-28}$$

We summarize the procedure of computing $\mathcal{Q}_{\tau,\boldsymbol{N}} f(k)$ for all $k \in \mathbb{L}_n$ in the following algorithm.

Algorithm 1. Given $n, \tau, m \in \mathbb{N}$, $\boldsymbol{N} \in \mathbb{N}^\tau$. Compute $\mathcal{Q}_{\tau,\boldsymbol{N}} f(k)$ for all $k \in \mathbb{L}_n$.
Step 1 For each $q \in \mathbb{Z}_\tau$, compute $\tilde{\eta}_{\boldsymbol{j},\boldsymbol{r}}(\mathscr{E}_q(\Lambda f))$ for all $\boldsymbol{j} \in \mathbb{S}^3_{N_q}$, $\boldsymbol{r} \in \mathbb{W}^3_{\boldsymbol{j}}$.
Step 2 For each $q \in \mathbb{Z}_\tau$, compute the vector $\Gamma_{q,u} = [\Gamma_{q,u,s} : s \in \mathbb{Z}_{m2^u}]$ for all $u \in \mathbb{Z}_{N_q}$,
Step 3 For each $q \in \mathbb{Z}_\tau$ and $u \in \mathbb{Z}_{N_q}$, compute $\widehat{\Gamma}_{q,u}$, by applying the fast Fourier transform to the vector $\Gamma_{q,u}$.
Step 4 Compute $\mathcal{Q}_{\tau,\boldsymbol{N}} f(k)$ for all $k \in \mathbb{L}_n$ by (3-28).

To close this section, we present an estimate on the computational costs of Algorithm 1. Let $\mathcal{M}_{n,\tau,\boldsymbol{N}}$ denote the number of multiplications used in Algorithm 1. In the next proposition, we estimate the bound on $\mathcal{M}_{n,\tau,\boldsymbol{N}}$.

Proposition 3.2. Let $m \in \mathbb{N}$ be fixed. If there exists a positive constant c_1 such that for each $x \in \Omega$, $t \in \mathbb{R}$, the numbers of multiplications used in computing $f(x)$ and $r(t)$ are less than c_1, then there exists a positive constant c_2 such that for all $n, \tau \in \mathbb{N}$ and $\boldsymbol{N} := [N_q : q \in \mathbb{Z}_\tau, N_q \in \mathbb{N}]$,

$$\mathcal{M}_{n,\tau,\boldsymbol{N}} \le c_2 \biggl(\sum_{q\in\mathbb{Z}_\tau} (N_q^2 2^{N_q}) + n \sum_{q\in\mathbb{Z}_\tau} N_q \biggr).$$

Proof. We prove this proposition by estimating the number of multiplications used in each step of Algorithm 1. As a preparation, we estimate the number $\sum_{\boldsymbol{j}\in S^3_{N_q}} |\mathbb{W}^3_{\boldsymbol{j}}|$, where $|\mathbb{W}^3_{\boldsymbol{j}}|$ denotes the cardinality of $\mathbb{W}^3_{\boldsymbol{j}}$ appearing in (3-22). For a fixing m, by Lemma 3.6 of [7], there exist constants c_1, c_2 such that for all $q \in \mathbb{Z}_\tau$,

$$\sum_{\boldsymbol{j}\in S^3_{N_q}} |\mathbb{W}^3_{\boldsymbol{j}}| \le c_1 \sum_{\boldsymbol{j}\in S^3_{N_q}} 2^{|\boldsymbol{j}|} \le c_2 N_q^2 2^{N_q}. \tag{3-29}$$

By using inequality (3-29), the number of multiplications used in Step 1 is bounded by $\mathcal{O}(N_q^2 2^{N_q})$. Moreover, the number of multiplications used in Step 2 is not greater than the amount of $\tilde{\eta}_{\boldsymbol{j},\boldsymbol{r}}(\mathscr{E}_q(\Lambda f))$. Thus, for each $q \in \mathbb{Z}_\tau$, the number of multiplications used in Step 2 is bounded by $\mathcal{O}(N_q^2 2^{N_q})$. By Lemma 3.6 of [7], for each $q \in \mathbb{Z}_\tau$, the number of multiplications used in Step 3 is bounded by

$$\mathcal{O}\Bigl(\sum_{u\in\mathbb{Z}_{N_q}} u2^u\Bigr) \le \mathcal{O}(N_q 2^{N_q}).$$

The number of multiplications used in Step 4 is bounded by $2mnN_q$ for each $q \in \mathbb{Z}_\tau$. Therefore, we obtain the desired upper bound on the total number of multiplications used in Algorithm 1. □

The Proposition 3.2 allows us to choose specific τ and $\boldsymbol{N}$ so that the total number of multiplications used in Algorithm 1 is linear up to a logarithmic factor. Specifically, in Algorithm 1 we choose

$$\tau := \lceil m \log n \rceil, \quad \boldsymbol{N} := [N_q := \max\{\lceil \log n - 2q/m \rceil, 1\} : q \in \mathbb{Z}_\tau] \tag{3-30}$$

and define $\mathcal{M}_n := \mathcal{M}_{n,\tau,\boldsymbol{N}}$. In the next theorem, we provide an upper bound on $\mathcal{M}_n$.

Theorem 3.3. Let $m \in \mathbb{N}$ be fixed. If τ and $\boldsymbol{N}$ are chosen as in (3-30), and there exists a positive constant c_1 such that the numbers of multiplications used in computing $f(x)$ and $r(t)$, for each $x \in \Omega$, $t \in \mathbb{R}$ are bounded by c_1, then there exists a positive constant c_2 such that for all $n \in \mathbb{N}$,

$$\mathcal{M}_n \le c_2 n \log^2 n.$$

Proof. This theorem is proved by using Proposition 3.2 with τ and $\boldsymbol{N}$ specified as in (3-30). Let $N'_q := \lceil \log n - q/m \rceil$. We have for all $n \in \mathbb{N}$ that

$$\max\{\lceil \log n - 2q/m \rceil, 1\} \le N'_q,$$

for all $q \in \mathbb{Z}_\tau$. This with Proposition 3.2 implies that there exists a constant c_3 such that for all $n \in \mathbb{N}$,

$$\mathcal{M}_n \le c_3 \Bigl(\sum_{q\in\mathbb{Z}_\tau} (N'^2_q 2^{N'_q}) + n \sum_{q\in\mathbb{Z}_\tau} N'_q \Bigr). \tag{3-31}$$

Lemma 3.7 of [7] ensures that there exists a constant c_4 such that for all $n \in \mathbb{N}$,

$$\sum_{q\in\mathbb{Z}_\tau} (N'^2_q 2^{N'_q}) \le c_4 n \log^2 n. \tag{3-32}$$

Since $\tau = \lceil m \log n \rceil$, there exists a constant c_5 such that for all $n \in \mathbb{N}$,

$$\sum_{q \in \mathbb{Z}_\tau} N'_q \le c_5 \log^2 n. \tag{3-33}$$

Combining (3-31), (3-32) and (3-33) leads to the desired results. □

4. Regularity analysis

For the purpose of estimating the error of the proposed quadrature formula of Algorithm 1, we study the asymptotic behavior of the derivatives of the function Λf and those of its extensions $\mathscr{E}_q(\Lambda f)$, $q \in \mathbb{N}_0$.

Recall that the function Λf may be written as the product of a smooth function

$$(\Xi f)(t, \mu, \lambda) := \lambda r^2(t-\mu) f(\lambda \boldsymbol{r}(t-\mu)) \tag{4-1}$$

and a singular function

$$\Psi(t, \mu, \lambda) := -\tfrac{1}{2\pi} \log|\boldsymbol{r}(t) - \lambda \boldsymbol{r}(t-\mu)|. \tag{4-2}$$

Since the function Ξf is smooth, it suffices to understand the asymptotic behavior of the derivatives of the function Ψ toward its singular points on $T = \{(t, 0, 1) : t \in [0, 2\pi)\}$. In order to consider the function Ψ, we recall that the fundamental solution $\Phi : \mathbb{R}^2 \setminus \{0\} \to \mathbb{R}$ is defined as

$$\Phi(x) = -\tfrac{1}{2\pi} \log|x|,$$

and define a vector-valued function $\boldsymbol{s} : S_0 \setminus T \to \mathbb{R}^2 \setminus \{0\}$ as

$$\boldsymbol{s}(t, \mu, \lambda) := \boldsymbol{r}(t) - \lambda \boldsymbol{r}(t-\mu). \tag{4-3}$$

It is clear that Ψ is the composition of Φ and $\boldsymbol{s}$, that is, $\Psi : S_0 \setminus T \xrightarrow{\boldsymbol{s}} \mathbb{R}^2 \setminus \{0\} \xrightarrow{\Phi} \mathbb{R}$. The singularities of Ψ are caused by the singularity of Φ and the zeros of $\boldsymbol{s}$. We express higher order derivatives of Ψ in terms of higher order derivatives of Φ and $\boldsymbol{s}$, and recall the multivariate Faà di Bruno formula [6; 13] for derivatives of a composition function. To this end, we need multivariate notation,

$$|\boldsymbol{v}| := \sum_{i \in \mathbb{Z}_d} v_i, \quad \boldsymbol{v}! := \prod_{i \in \mathbb{Z}_d} (v_i!), \quad \boldsymbol{j}^{\boldsymbol{v}} := \prod_{i \in \mathbb{Z}_d} j_i^{v_i}, \quad \text{for} \quad \boldsymbol{v} \in \mathbb{N}_0^d,\ \boldsymbol{j} \in \mathbb{R}^d.$$

For $\boldsymbol{m} \in \mathbb{N}_0^3$, there are exactly $|\boldsymbol{m}| + 2$ vectors $\boldsymbol{\gamma}_i \in \mathbb{N}_0^3$ such that $\boldsymbol{0} \prec \boldsymbol{\gamma}_0 \prec \cdots \prec \boldsymbol{\gamma}_{|\boldsymbol{m}|+1} = \boldsymbol{m}$, where the notation $\prec$ denotes the lexicographical order relation: $[a_1, b_1, c_1] \prec [a_2, b_2, c_2]$ if $a_1 < a_2$; or $a_1 = a_2$ and $b_1 < b_2$; or $a_1 = a_2,\ b_1 = b_2$ and $c_1 < c_2$, and we let $\varrho_{\boldsymbol{m}} := \{\boldsymbol{\gamma}_0, \dots, \boldsymbol{\gamma}_{|\boldsymbol{m}|+1}\}$. For $\boldsymbol{m} \in \mathbb{N}_0^3,\ \boldsymbol{z} \in \mathbb{N}_0^2$, we define

$$\sigma_{\boldsymbol{m},\boldsymbol{z}} := \Big\{ [\boldsymbol{k}_0, \dots, \boldsymbol{k}_{|\boldsymbol{m}|+1}] : \boldsymbol{k}_i \in \mathbb{N}_0^2, \sum_{i \in \mathbb{Z}_{|\boldsymbol{m}|+2}} \boldsymbol{k}_i = \boldsymbol{z}, \sum_{i \in \mathbb{Z}_{|\boldsymbol{m}|+2}} |\boldsymbol{k}_i| \boldsymbol{\gamma}_i = \boldsymbol{m}, \boldsymbol{\gamma}_i \in \varrho_{\boldsymbol{m}} \Big\}.$$

The vector-valued function $\boldsymbol{s}$ defined by (4-3) can be expressed as $\boldsymbol{s} = (s_1, s_2)$, where s_1 and s_2 are smooth functions. For $\boldsymbol{\gamma} \in \mathbb{N}_0^3$, let $\boldsymbol{s}^{(\boldsymbol{\gamma})} := (s_1^{(\boldsymbol{\gamma})}, s_2^{(\boldsymbol{\gamma})})$. According to the multivariate Faà di Bruno formula [13], for $\boldsymbol{m} \in \mathbb{N}_0^3$, we have that

(4-4)
$$\Psi^{(\boldsymbol{m})} = (\boldsymbol{m}!) \sum_{|z|=1}^{|\boldsymbol{m}|} \sum_{[\boldsymbol{k}_j] \in \sigma_{\boldsymbol{m},z}} \Phi^{(z)} \circ \boldsymbol{s} \cdot \prod_{j \in \mathbb{Z}_{|\boldsymbol{m}|+2}} \frac{(\boldsymbol{s}^{(\boldsymbol{\gamma}_j)})^{\boldsymbol{k}_j}}{(\boldsymbol{k}_j!)(\boldsymbol{\gamma}_j!)^{|\boldsymbol{k}_j|}},$$

where $\boldsymbol{\gamma}_j \in \varrho_{\boldsymbol{m}}$, $[\boldsymbol{k}_j] := [\boldsymbol{k}_0, \dots, \boldsymbol{k}_{|\boldsymbol{m}|+1}]$.

Estimation of $\Psi^{(\boldsymbol{m})}$ requires bounding $\Phi^{(z)} \circ \boldsymbol{s}$. To this end, we let $\Upsilon(t, \mu, \lambda) := |\boldsymbol{s}(t, \mu, \lambda)|$ and observe that

(4-5)
$$\Phi^{(z)} \circ \boldsymbol{s} \le c \Upsilon^{-|z|}$$

for some positive constant c. We need to estimate Υ. For this purpose, we define

(4-6)
$$\Theta(t, \mu, \lambda) := \Big(r^2(t-\mu)\,\mu^2 + (r(t-\mu)(1-\lambda) + \frac{\mathrm{d}r}{\mathrm{d}t}(t-\mu)\,\mu)^2\Big)^{1/2}$$

and show that there exist positive constants c_1, c_2, c_3, c_4 such that for all $(t, \mu, \lambda) \in S_0$,

(4-7)
$$c_1 R_T(t, \mu, \lambda) \le c_2 \Theta(t, \mu, \lambda) \le \Upsilon(t, \mu, \lambda) \le c_3 \Theta(t, \mu, \lambda) \le c_4 R_T(t, \mu, \lambda),$$

where
$$R_T(t, \mu, \lambda) := (\mu^2 + (1-\lambda)^2)^{\frac{1}{2}}, \quad (t, \mu, \lambda) \in S_0.$$

To establish (4-7), we need to introduce an additional hypothesis.

Hypothesis 4.1. The boundary $\partial\Omega$ of Ω is of radiation type and the radial distance function r of parametrization $\boldsymbol{r}$ defined in (2-4) satisfies that $r \in C_{2\pi}^{\infty}(\mathbb{R})$ and $r > 0$.

We estimate Θ below.

Lemma 4.2. If $\partial\Omega$ satisfies Hypothesis 4.1, then there exist constants $c_1, c_2 > 0$ such that for all $(t, \mu, \lambda) \in S_0$,

(4-8)
$$c_1 R_T(t, \mu, \lambda) \le \Theta(t, \mu, \lambda) \le c_2 R_T(t, \mu, \lambda).$$

Proof. We first define the function

$$\widetilde{\Theta}(t, \mu, \lambda) := \big(\mu^2 + ((1-\lambda) + p(t, \mu)\mu)^2\big)^{\frac{1}{2}}, \quad \text{where} \quad p(t, \mu) := \frac{\mathrm{d}r}{\mathrm{d}t}(t-\mu)/r(t-\mu),\ (t, \mu, \lambda) \in S_0,$$

and establish there exist two positive constants $r_{\min}, r_{\max}$ such that for all $(t, \mu, \lambda) \in S_0$,

(4-9)
$$r_{min} \widetilde{\Theta}(t, \mu, \lambda) \le \Theta(t, \mu, \lambda) \le r_{\max} \widetilde{\Theta}(t, \mu, \lambda).$$

In fact, since $r \in C_{2\pi}^{\infty}(\mathbb{R})$ and $r(t) > 0$ for all $t \in \mathbb{R}$, there exist constants $r_{\min}, r_{\max}$ such that $0 < r_{\min} \le r(t) \le r_{\max}$ for all $t \in \mathbb{R}$. Thus, it follows from the definition (4-6) of Θ that for all $(t, \mu, \lambda) \in S_0$, inequality (4-9) holds.

It can be verified that

(4-10)
$$a_1(t, \mu)\, R_T(t, \mu, \lambda) \le \widetilde{\Theta}(t, \mu, \lambda) \le a_2(t, \mu)\, R_T(t, \mu, \lambda),$$

where

$$a_1(t,\mu) := \frac{1}{\sqrt{2}}\Big(p^2(t,\mu)+2-\sqrt{p^4(t,\mu)+4p^2(t,\mu)}\Big)^{1/2}$$

and

$$a_2(t,\mu) := \frac{1}{\sqrt{2}}\Big(p^2(t,\mu)+2+\sqrt{p^4(t,\mu)+4p^2(t,\mu)}\Big)^{1/2}.$$

Since $r \in C^\infty_{2\pi}(\mathbb{R})$, there exists a positive constant $p_{\max}$ such that

$$\left|\frac{\mathrm{d}r}{\mathrm{d}t}(t)/r(t)\right| \le p_{\max}, \quad t \in \mathbb{R}. \tag{4-11}$$

By using (4-11), we obtain that for all $(t,\mu) \in [0,2\pi)\times[-\pi,\pi)$,

$$a_2(t,\mu) \le \frac{1}{\sqrt{2}}\Big(p^2_{\max}+2+\sqrt{p^4_{\max}+4p^2_{\max}}\Big)^{1/2} \tag{4-12}$$

and

$$a_1(t,\mu) = a_2^{-1}(t,\mu) \ge \frac{1}{\sqrt{2}}\Big(p^2_{\max}+2-\sqrt{p^4_{\max}+4p^2_{\max}}\Big)^{1/2}. \tag{4-13}$$

Thus, combining (4-9), (4-10), (4-12) and (4-13) leads to (4-8). □

We next estimate Υ with the help of Lemma 4.2.

Lemma 4.3. If $\partial\Omega$ satisfies Hypothesis 4.1, then there exist constants $c_1, c_2 > 0$ such that for all $(t,\mu,\lambda) \in S_0$,

$$c_1 R_T(t,\mu,\lambda) \le \Upsilon(t,\mu,\lambda) \le c_2 R_T(t,\mu,\lambda). \tag{4-14}$$

Proof. This lemma is proved by showing that there exist constants $c', c'' > 0$ such that for all $(t,\mu,\lambda) \in S_0$,

$$c'\Theta(t,\mu,\lambda) \le \Upsilon(t,\mu,\lambda) \le c''\Theta(t,\mu,\lambda). \tag{4-15}$$

The desired result of this lemma is then obtained by combining Lemma 4.2 with (4-15).

It remains to establish (4-15). We define the difference between Υ^2 and Θ^2 by

$$D_\Upsilon(t,\mu,\lambda) := \Upsilon^2(t,\mu,\lambda) - \Theta^2(t,\mu,\lambda), \quad (t,\mu,\lambda) \in S_0.$$

Since $r \in C^\infty_\pi(\mathbb{R})$, by using the Taylor theorem, there exist constants $\delta_1, c_5 > 0$ such that for all $(t,\mu,\lambda) \in S_0$ with $\mu^2 + (1-\lambda)^2 < \delta_1$,

$$|D_\Upsilon(t,\mu,\lambda)| \le c_5(|\mu|^3 + |\mu^2(1-\lambda)|). \tag{4-16}$$

By the definition (4-6) of Θ, there holds that

$$\Theta^2(t,\mu,\lambda) \ge r^2_{\min}\,\mu^2, \tag{4-17}$$

where $r_{\min} > 0$ satisfies that $r(t) \geq r_{\min}$ for all $t \in \mathbb{R}$. Thus, by inequalities (4-16) and (4-17), we have that for all $(t, \mu, \lambda) \in S_0 \setminus T$ with $\mu^2 + (1-\lambda)^2 < \delta_1$, there holds that

$$\left| \frac{D_\Upsilon(t, \mu, \lambda)}{\Theta^2(t, \mu, \lambda)} \right| \leq \frac{c_5(|\mu|^3 + |\mu^2(1-\lambda)|)}{r_{\min}^2 \mu^2} \leq \frac{c_5}{r_{\min}^2}(|\mu| + |1-\lambda|). \tag{4-18}$$

We then consider the inequality (4-15) in two cases: $|\mu| + |1-\lambda| < \delta_2$ and $|\mu| + |1-\lambda| \geq \delta_2$, where $\delta_2 := \min\{\delta_1, r_{\min}^2/(2c_5)\}$. By noting that

$$\Upsilon^2(t, \mu, \lambda) = D_\Upsilon(t, \mu, \lambda) + \Theta^2(t, \mu, \lambda)$$

from (4-18), we have that for all $(t, \mu, \lambda) \in S_0$ with $|\mu| + |1-\lambda| < \delta_2$,

$$\left(1 - \frac{c_5}{r_{\min}^2}(|\mu| + |1-\lambda|)\right)^{1/2} \Theta(t, \mu, \lambda) \leq \Upsilon(t, \mu, \lambda) \leq \left(1 + \frac{c_5}{r_{\min}^2}(|\mu| + |1-\lambda|)\right)^{1/2} \Theta(t, \mu, \lambda). \tag{4-19}$$

By noting that Υ, Θ are continuous on the set S_0 and $\Theta(t, \mu, \lambda) = \Upsilon(t, \mu, \lambda) = 0$ if and only if $(t, \mu, \lambda) \in T$, we know that there exist constants $c_6, c_7 > 0$ such that for all $(t, \mu, \lambda) \in S_0$ with $|\mu| + |1-\lambda| \geq \delta_2$,

$$c_6 \leq \Theta(t, \mu, \lambda) \leq c_7 \quad \text{and} \quad c_6 \leq \Upsilon(t, \mu, \lambda) \leq c_7.$$

Thus we have that for all $(t, \mu, \lambda) \in S_0$ with $|\mu| + |1-\lambda| \geq \delta_2$,

$$\frac{c_6}{c_7}\Theta(t, \mu, \lambda) \leq \Upsilon(t, \mu, \lambda) \leq \frac{c_7}{c_6}\Theta(t, \mu, \lambda). \tag{4-20}$$

Combining (4-19) and (4-20), we conclude that there exist constants $c', c'' > 0$ such that for all $(t, \mu, \lambda) \in S_0$, (4-15) holds. □

We now estimate the upper bound of $|\Psi^{(\boldsymbol{m})}(t, \mu, \lambda)|$.

Proposition 4.4. Let $\boldsymbol{m} := [m_0, m_1, m_2] \in \mathbb{N}_0^3$ be fixed and $|\boldsymbol{m}|_\infty > 0$. If $\partial\Omega$ satisfies Hypothesis 4.1, then there exists a constant $c_{\boldsymbol{m}} > 0$ such that for all $(t, \mu, \lambda) \in S_0 \setminus T$,

$$|\Psi^{(\boldsymbol{m})}(t, \mu, \lambda)| \leq c_{\boldsymbol{m}} R_T^{-(m_1+m_2)}(t, \mu, \lambda). \tag{4-21}$$

Proof. According to formula (4-4), it suffices to consider the upper bound of

$$Z_{[\boldsymbol{k}_j]}(t, \mu, \lambda) := \Phi^{(z)} \circ \boldsymbol{s}(t, \mu, \lambda) \cdot \prod_{j \in \mathbb{Z}_{|\boldsymbol{m}|+2}} \frac{(\boldsymbol{s}^{(\boldsymbol{\gamma}_j)}(t, \mu, \lambda))^{\boldsymbol{k}_j}}{(\boldsymbol{k}_j!)(\boldsymbol{\gamma}_j!)^{|\boldsymbol{k}_j|}}, \tag{4-22}$$

for all $(t, \mu, \lambda) \in S_0 \setminus T$ with given $[\boldsymbol{k}_j] \in \sigma_{\boldsymbol{m},z}$. To this end, we introduce an index set

$$\kappa_{\boldsymbol{m}} := \{j : j \in \mathbb{Z}_{|\boldsymbol{m}|+2}, \boldsymbol{\gamma}_j \in \varrho_{\boldsymbol{m}}, (\boldsymbol{\gamma}_j)_0 \neq 0, (\boldsymbol{\gamma}_j)_1 = (\boldsymbol{\gamma}_j)_2 = 0\},$$

and define two functions

$$Z_{1,[\boldsymbol{k}_j]}(t, \mu, \lambda) := \prod_{j \in \kappa_{\boldsymbol{m}}} \frac{(\boldsymbol{s}^{(\boldsymbol{\gamma}_j)}(t, \mu, \lambda))^{\boldsymbol{k}_j}}{(\boldsymbol{k}_j!)(\boldsymbol{\gamma}_j!)^{|\boldsymbol{k}_j|}}, \tag{4-23}$$

$$Z_{2,[\boldsymbol{k}_j]}(t, \mu, \lambda) := \prod_{j \in \mathbb{Z}_{|\boldsymbol{m}|+2} \setminus \kappa_{\boldsymbol{m}}} \frac{(\boldsymbol{s}^{(\boldsymbol{\gamma}_j)}(t, \mu, \lambda))^{\boldsymbol{k}_j}}{(\boldsymbol{k}_j!)(\boldsymbol{\gamma}_j!)^{|\boldsymbol{k}_j|}}. \tag{4-24}$$

We then rewrite (4-22) as

$$Z(t,\mu,\lambda) = \Phi^{(z)} \circ \boldsymbol{s}(t,\mu,\lambda) \cdot Z_{1,[\boldsymbol{k}_j]}(t,\mu,\lambda) \cdot Z_{2,[\boldsymbol{k}_j]}(t,\mu,\lambda) \tag{4-25}$$

and estimate $\Phi^{(z)} \circ \boldsymbol{s}$, $Z_{1,[\boldsymbol{k}_j]}$ and $Z_{2,[\boldsymbol{k}_j]}$, separately.

We first provide an upper bound for $|\Phi^{(z)} \circ \boldsymbol{s}|$. It follows from (4-5) and (4-14) in Lemma 4.3 that there exists a positive constant c_1 such that for all $(t,\mu,\lambda) \in S_0 \setminus T$,

$$|(\Phi^{(z)} \circ \boldsymbol{s})(t,\mu,\lambda)| \le c_1 R_T^{-|z|}(t,\mu,\lambda). \tag{4-26}$$

We next bound the function $|Z_{1,[\boldsymbol{k}_j]}(t,\mu,\lambda)|$. The vector-valued function $\boldsymbol{s}(t,\mu,\lambda)$ in $Z_{1,[\boldsymbol{k}_j]}$ can be expressed as

$$\boldsymbol{s}(t,\mu,\lambda) = (s_1(t,\mu,\lambda), s_2(t,\mu,\lambda)),$$

where

$$s_1(t,\mu,\lambda) = r_1(t) - \lambda r_1(t-\mu), \quad s_2(t,\mu,\lambda) = r_2(t) - \lambda r_2(t-\mu),$$

and $r_1, r_2 \in C_{2\pi}^{\infty}(\mathbb{R})$. We first consider upper bounds of $|s_i^{(\boldsymbol{\gamma}_j)}(t,\mu,\lambda)|$, where $j \in \kappa_{\boldsymbol{m}}$ and $\boldsymbol{\gamma}_j \in \varrho_{\boldsymbol{m}}$. By using the Taylor theorem, there exists a constant $c_{\boldsymbol{\gamma}_j} > 0$ such that for all $(t,\mu,\lambda) \in S_0$,

$$|s_i^{(\boldsymbol{\gamma}_j)}(t,\mu,\lambda)| \le c_{\boldsymbol{\gamma}_j}(|\mu| + |1-\lambda|), \quad i = 0, 1. \tag{4-27}$$

Applying the Cauchy–Schwartz inequality to (4-27) yields

$$|s_i^{(\boldsymbol{\gamma}_j)}(t,\mu,\lambda)| \le c_{\boldsymbol{\gamma}_j}\sqrt{2}\, R_T(t,\mu,\lambda), \quad i = 0, 1. \tag{4-28}$$

Thus, for $[\boldsymbol{k}_j] \in \sigma_{\boldsymbol{m},z}$, substituting (4-28) into (4-23) yields a positive constant c_2 such that for all $(t,\mu,\lambda) \in S_0 \setminus T$,

$$|Z_{1,[\boldsymbol{k}_j]}(t,\mu,\lambda)| \le c_2 R_T^{\vartheta}(t,\mu,\lambda), \tag{4-29}$$

where $\vartheta := \sum_{j \in \kappa_{\boldsymbol{m}}} |\boldsymbol{k}_j|$. By the smoothness of $\boldsymbol{s}$, we know that there exists a positive constant c_3 such that for all $(t,\mu,\lambda) \in S_0 \setminus T$,

$$|Z_{2,[\boldsymbol{k}_j]}(t,\mu,\lambda)| \le c_3. \tag{4-30}$$

Combining the estimates (4-26), (4-29) and (4-30) yields a positive constant c_4 such that for all $(t,\mu,\lambda) \in S_0 \setminus T$,

$$|Z(t,\mu,\lambda)| \le c_4 R_T^{-|z|+\vartheta}(t,\mu,\lambda). \tag{4-31}$$

Finally, we establish that $|z| - \vartheta \le m_1 + m_2$. Since $\sum_{j \in \mathbb{Z}_{|\boldsymbol{m}|+2}} \boldsymbol{k}_j = z$, we have that

$$|z| - \vartheta = |z| - \sum_{j \in \kappa_{\boldsymbol{m}}} |\boldsymbol{k}_j| = \sum_{j \in \mathbb{Z}_{|\boldsymbol{m}|+2} \setminus \kappa_{\boldsymbol{m}}} |\boldsymbol{k}_j|. \tag{4-32}$$

By the definition of $\kappa_{\boldsymbol{m}}$, $\sum_{j \in \mathbb{Z}_{|\boldsymbol{m}|+2}} |\boldsymbol{k}_j| \gamma_j = \boldsymbol{m}$ and (4-32), we obtain that

$$|z| - \vartheta \le m_1 + m_2. \tag{4-33}$$

By substituting (4-33) into (4-31), we obtain the desired inequality (4-21). □

In the remainder of this section, we investigate the smoothness of Λf and its extension $\mathscr{E}_q(\Lambda f)$ on the domain S_q. We need the bounded mixed derivatives space defined by

$$X^m(\mathbb{E}) := \{\omega : \mathbb{E} \to \mathbb{R} : \omega^{(\boldsymbol{\alpha})} \in C(\mathbb{E}),\ |\boldsymbol{\alpha}|_\infty \le m\}$$

with the norm and the seminorm, respectively, by

$$\|\omega\|_{X^m(\mathbb{E})} := \max\{\|\omega^{(\boldsymbol{\alpha})}\|_\infty : \boldsymbol{\alpha} \in \mathbb{N}_0^3,\ |\boldsymbol{\alpha}|_\infty \le m\}, \quad |\omega|_{X^m(\mathbb{E})} := \max\{\|\omega^{(\boldsymbol{\alpha})}\|_\infty : \boldsymbol{\alpha} \in \mathbb{N}_0^3,\ |\boldsymbol{\alpha}|_\infty = m\},$$

where $\mathbb{E} \subset \mathbb{R}^3$.

In the rest of this paper, we assume that the composition of function f and the radial parametrization $\tilde{\boldsymbol{r}}(t, \mu, \lambda) := \lambda \boldsymbol{r}(t - \mu)$ has mth-order bounded mixed derivatives. We define the space

$$Y^m(\Omega) := \{\omega : \Omega \to \mathbb{R} : \omega \circ \tilde{\boldsymbol{r}} \in X^m(S_0)\}$$

with the norm

$$\|\omega\|_{Y^m(\Omega)} := \|\omega \circ \tilde{\boldsymbol{r}}\|_{X^m(S_0)}, \quad \text{where} \quad \tilde{\boldsymbol{r}}(t, \mu, \lambda) := \lambda \boldsymbol{r}(t - \mu).$$

Clearly, $f \circ \tilde{\boldsymbol{r}} \in X^m(S_0)$ when $f \in Y^m(\Omega)$. We show in the following proposition that for all $f \in Y^m(\Omega)$, $q \in \mathbb{N}_0$, $(\Lambda f)|_{\overline{D}_q} \in X^m(\overline{D}_q)$ and estimate the derivatives of Λf on D_q.

Proposition 4.5. Let $m \in \mathbb{N}$ be fixed. If $\partial\Omega$ satisfies Hypothesis 4.1, then for all $f \in Y^m(\Omega)$ and $q \in \mathbb{N}_0$, there holds that $(\Lambda f)|_{\overline{D}_q} \in X^m(\overline{D}_q)$. Moreover, there exists a positive constant c such that for all $q \in \mathbb{N}_0$, $f \in Y^m(\Omega)$, $\alpha \in \mathbb{N}_0^3$ with $|\boldsymbol{\alpha}|_\infty = m$,

$$\|(\Lambda f)^{(\boldsymbol{\alpha})}|_{\overline{D}_q}\|_\infty \le c 2^{q(\alpha_1+\alpha_2)} \|f\|_{Y^m(\Omega)}. \tag{4-34}$$

Proof. We first prove that $(\Lambda f)|_{\overline{D}_q} \in X^m(\overline{D}_q)$. Since the mth-order bounded mixed derivatives function space have the property that $\omega_1 \cdot \omega_2 \in X^m(\overline{D}_q)$, when $\omega_1, \omega_2 \in X^m(\overline{D}_q)$, the proof of $(\Lambda f)|_{\overline{D}_q} \in X^m(\overline{D}_q)$ is done by proving that both $\Psi|_{\overline{D}_q}$ and $(\Xi f)|_{\overline{D}_q}$ are in $X^m(\overline{D}_q)$. By the definition of Ψ, we can easily obtain that $\Psi|_{\overline{D}_q} \in X^m(\overline{D}_q)$. Since $f \in Y^m(\Omega)$, by the definition of Ξf, we have that $\Xi f \in X^m(S_0)$. Thus, we obtain that $(\Lambda f)|_{\overline{D}_q} \in X^m(\overline{D}_q)$.

We next prove inequality (4-34). By applying the Leibniz rule to $\Lambda f = (\Xi f)\Psi$, we know that there exists a positive constant c_1 such that for all $q \in \mathbb{N}$, $f \in Y^m(\Omega)$, $\boldsymbol{\alpha} \in \mathbb{N}_0^3$ with $|\boldsymbol{\alpha}|_\infty = m$,

$$\|(\Lambda f)^{(\boldsymbol{\alpha})}|_{\overline{D}_q}\|_\infty \le c_1 \sum_{\boldsymbol{\beta} \le \boldsymbol{\alpha}} \|\Psi^{(\boldsymbol{\beta})}|_{\overline{D}_q}\|_\infty \, \|(\Xi f)^{(\boldsymbol{\alpha}-\boldsymbol{\beta})}|_{\overline{D}_q}\|_\infty, \tag{4-35}$$

where we say $\boldsymbol{\beta} \le \boldsymbol{\alpha}$ for any $\boldsymbol{\alpha} := [\alpha_0, \alpha_1, \alpha_2]$, $\boldsymbol{\beta} := [\beta_0, \beta_1, \beta_2] \in \mathbb{N}_0^3$, if $\beta_0 \le \alpha_0$, $\beta_1 \le \alpha_1$ and $\beta_2 \le \alpha_2$. We bound $\|\Psi^{(\boldsymbol{\beta})}|_{\overline{D}_q}\|_\infty$ and $\|(\Xi f)^{(\boldsymbol{\alpha}-\boldsymbol{\beta})}|_{\overline{D}_q}\|_\infty$, separately. When $q \in \mathbb{N}_0$, $(t, \mu, \lambda) \in D_q$, substituting the inequality $R_T(t, \mu, \lambda) \ge 2^{-q-1}\pi$ into the right-hand side of (4-21) in Proposition 4.4 yields that there exists a positive constant c_2 such that for all $q \in \mathbb{N}_0$, $\boldsymbol{\alpha} \in \mathbb{N}_0^3$ with $|\boldsymbol{\alpha}|_\infty = m$ and $\boldsymbol{\beta} \in \mathbb{N}_0^3$ with $\boldsymbol{\beta} \le \boldsymbol{\alpha}$,

$$\|\Psi^{(\boldsymbol{\beta})}|_{\overline{D}_q}\|_\infty \le c_2 2^{q(\alpha_1+\alpha_2)}. \tag{4-36}$$

By applying the Leibniz rule to Ξf defined by (4-1), we find that there exists a constant c_3 such that for all $q \in \mathbb{N}$, $f \in Y^m(\Omega)$, $\boldsymbol{\alpha} \in \mathbb{N}_0^3$ with $|\boldsymbol{\alpha}|_\infty = m$ and $\boldsymbol{\beta} \in \mathbb{N}_0^3$ with $\boldsymbol{\beta} \le \boldsymbol{\alpha}$,

$$\|(\Xi f)^{(\boldsymbol{\alpha}-\boldsymbol{\beta})}|_{\overline{D}_q}\|_\infty \le c_3 \|f\|_{Y^m(\Omega)}. \tag{4-37}$$

Substituting (4-36) and (4-37) into the right hand side of (4-35), we obtain estimate (4-34). □

In order to ensure $\mathscr{E}_q(\Lambda f) \in X^m(\overline{D}_q)$, we next establish that $\mathscr{E}_q \omega$ is in the space $X^m(S_q)$, when the function ω is in $X^m(\overline{D}_q)$.

Proposition 4.6. Let $m \in \mathbb{N}$ and $q \in \mathbb{N}_0$ be fixed. If $\omega \in X^m(\overline{D}_q)$, then $\mathscr{E}_q \omega \in X^m(S_q)$. Moreover, there exists a constant c such that for all $q \in \mathbb{N}_0$, $\omega \in X^m(\overline{D}_q)$ and $\boldsymbol{\alpha} := [\alpha_0, \alpha_1, \alpha_2] \in \mathbb{N}_0^3$ with $|\boldsymbol{\alpha}|_\infty \le m$,

$$\|(\mathscr{E}_q \omega)^{(\boldsymbol{\alpha})}\|_\infty \le c \|\omega^{(\boldsymbol{\alpha})}\|_\infty. \tag{4-38}$$

Proof. The statement $\mathscr{E}_q \omega \in X^m(S_q)$ follows from Lemma 3.1 immediately.

It remains to prove inequality (4-38). By the definition (3-8) of $\mathscr{E}_q \omega$, we only need to estimate $\|\omega_{q,1}^{(\boldsymbol{\alpha})}|_{S_{q+1}}\|_\infty$, $\|\omega_{q,2}^{(\boldsymbol{\alpha})}|_{S_{q+1}}\|_\infty$ and $\|\omega_{q,\text{corner}}^{(\boldsymbol{\alpha})}|_{S_{q+1}}\|_\infty$. To estimate $\|\omega_{q,1}^{(\boldsymbol{\alpha})}|_{S_{q+1}}\|_\infty$, for all $(t, \mu, \lambda) \in S_{q+1}$, we compute

$$\omega_{q,1}^{(\boldsymbol{\alpha})}(t,\mu,\lambda) = \sum_{j=\alpha_2}^{m} \omega^{([\alpha_0,\alpha_1,j])}(t,\mu,\lambda_{q+1}) \frac{1}{(j-\alpha_2)!} (\lambda - \lambda_{q+1})^{j-\alpha_2}$$

and observe that $|\lambda - \lambda_q| \le 1$. Hence, there exists a positive constant c_1 such that for all $\omega \in X^m(\overline{D}_q)$, $q \in \mathbb{N}_0$ and $\boldsymbol{\alpha} \in \mathbb{N}_0^3$ with $|\boldsymbol{\alpha}|_\infty \le m$,

$$\|\omega_{q,1}^{(\boldsymbol{\alpha})}|_{S_{q+1}}\|_\infty \le c_1 \|\omega^{(\boldsymbol{\alpha})}\|_\infty. \tag{4-39}$$

To estimate $\|\omega_{q,2}^{(\boldsymbol{\alpha})}|_{S_{q+1}}\|_\infty$, for all $(t, \mu, \lambda) \in S_{q+1}$, we have that

$$\omega_{q,2}^{(\boldsymbol{\alpha})}(t,\mu,\lambda) = \sum_{j=0}^{m} (2\mu_{q+1})^{j-\alpha_1} \Bigg[\omega^{([\alpha_0,j,\alpha_2])}(t,-\mu_{q+1},\lambda)\, q_j^{(\alpha_1)}\left(\frac{\mu+\mu_{q+1}}{2\mu_{q+1}}\right) + (-1)^{j-\alpha_1} \omega^{([\alpha_0,j,\alpha_2])}(t,\mu_{q+1},\lambda)\, q_j^{(\alpha_1)}\left(\frac{\mu_{q+1}-\mu}{2\mu_{q+1}}\right)\Bigg],$$

where q_j are defined in (3-7). Thus, there exists a positive constant c_2 such that for all $\omega \in X^m(\overline{D}_q)$, $q \in \mathbb{N}_0$ and $\boldsymbol{\alpha} \in \mathbb{N}_0^3$ with $|\boldsymbol{\alpha}|_\infty \le m$,

$$\|\omega_{q,2}^{(\boldsymbol{\alpha})}|_{S_{q+1}}\|_\infty \le c_2 \|\omega^{(\boldsymbol{\alpha})}\|_\infty. \tag{4-40}$$

Finally, we estimate $\|\omega_{q,\text{corner}}^{(\boldsymbol{\alpha})}|_{S_{q+1}}\|_\infty$. Since for all $(t, \mu, \lambda) \in S_{q+1}$,

$$\omega_{q,\text{corner}}^{(\boldsymbol{\alpha})}(t,\mu,\lambda) = \sum_{j=0}^{m}\sum_{k=0}^{m} \frac{(2\mu_{q+1})^j}{(k-\alpha_2)!}(\lambda-\lambda_{q+1})^{k-\alpha_2} \Bigg[\omega^{([\alpha_0,j,k])}(t,-\mu_{q+1},\lambda_{q+1})\, q_{j-\alpha_1}^{(\alpha_1)}\left(\frac{\mu+\mu_{q+1}}{2\mu_{q+1}}\right) + (-1)^{j-\alpha_1} \omega^{([\alpha_0,j,k])}(t,\mu_{q+1},\lambda_{q+1})\, q_j^{(\alpha_1)}\left(\frac{\mu_{q+1}-\mu}{2\mu_{q+1}}\right)\Bigg].$$

Hence, there exists a positive constant c_3 such that for all $\omega \in X^m(\overline{D}_q)$, $q \in \mathbb{N}_0$ and $\boldsymbol{\alpha} \in \mathbb{N}_0^3$ with $|\boldsymbol{\alpha}|_\infty \le m$,

$$\|\omega^{(\boldsymbol{\alpha})}_{q,\mathrm{corner}}|_{S_{q+1}}\|_\infty \le c_3 \|\omega^{(\boldsymbol{\alpha})}\|_\infty. \tag{4-41}$$

Note the definition (3-8) of $\mathscr{E}_q\omega$. Combining inequalities (4-39), (4-40) and (4-41) yields the desired estimate (4-38). □

By Propositions 4.5 and 4.6, we obtain the next proposition.

Proposition 4.7. Let $m \in \mathbb{N}$ and $q \in \mathbb{N}_0$ be fixed. If $\partial\Omega$ satisfies Hypothesis 4.1, then for all, $f \in Y^m(\Omega)$, there holds $\mathscr{E}_q(\Lambda f) \in X^m(S_q)$, and there exists a positive constant c such that for all $q \in \mathbb{N}_0$, $f \in Y^m(\Omega)$, $\alpha \in \mathbb{N}_0^3$ with $|\boldsymbol{\alpha}|_\infty = m$,

$$\|(\mathscr{E}_q(\Lambda f))^{(\boldsymbol{\alpha})}\|_\infty \le c 2^{q(\alpha_1+\alpha_2)}\|f\|_{Y^m(\Omega)}. \tag{4-42}$$

Proof. First, we prove that $\mathscr{E}_q(\Lambda f) \in X^m(\Omega)$. Since $\partial\Omega$ satisfies Hypothesis 4.1 and $f \in Y^m(\Omega)$, by Proposition 4.5, we have that $(\Lambda f)|_{\overline{D}_q} \in X^m(\overline{D}_q)$. According to Proposition 4.5, we obtain that $\mathscr{E}_q(\Lambda f) \in Y^m(\Omega)$.

It remains to establish estimate (4-42). Since $\partial\Omega$ satisfies Hypothesis 4.1, by Proposition 4.5, there exists a positive constant c_1 such that for all $q \in \mathbb{N}_0$, $f \in Y^m(\Omega)$, $\alpha \in \mathbb{N}_0^3$ with $|\boldsymbol{\alpha}|_\infty = m$,

$$\|(\Lambda f)^{(\boldsymbol{\alpha})}|_{\overline{D}_q}\|_\infty \le c_1 2^{q(\alpha_1+\alpha_2)}\|f\|_{Y^m(\Omega)}. \tag{4-43}$$

By identifying ω with Λf in estimate (4-38) and combining (4-43), we obtain the desired estimate (4-42). □

5. Error estimates

We estimate the accuracy of the proposed quadrature formula (3-21).

We first estimate the difference between $\mathscr{E}_q(\Lambda f)$ and $\widetilde{\mathscr{S}}_{N_q}(\mathscr{E}_q(\Lambda f))$. To this end, we recall an estimate of the difference between $\omega \in X^m(I^3)$ and its sparse approximation $\mathscr{S}_N\omega$. The following error estimate of the sparse grid approximation was obtained in Theorem 2.12 of [17]: For fixed $m \in \mathbb{N}$, there exits a positive constant c such that for all $N \in \mathbb{N}$ and $\omega \in X^m(I^3)$,

$$\|\omega - \mathscr{S}_N\omega\|_\infty \le cN^2 2^{-mN}|\omega|_{X^m(I^3)}. \tag{5-1}$$

Application of estimate (5-1) yields the next lemma.

Lemma 5.1. Let $q \in \mathbb{N}_0$, $m \in \mathbb{N}$ be fixed. If the boundary $\partial\Omega$ satisfies the Hypothesis 4.1, then there exists a positive constant c such that for all $N \in \mathbb{N}$ and $f \in Y^m(\Omega)$,

$$\|\mathscr{E}_q(\Lambda f) - \widetilde{S}_N(\mathscr{E}_q(\Lambda f))\|_\infty \le cN^2 2^{-mN}\|f\|_{Y^m(\Omega)}. \tag{5-2}$$

Proof. From the definition of $\widetilde{S}_N$, we have that

$$\|\mathscr{E}_q(\Lambda f) - \widetilde{S}_N(\mathscr{E}_q(\Lambda f))\|_\infty = \|\mathscr{E}_q(\Lambda f)\circ\boldsymbol{\phi}_q^{-1} - S_N(\mathscr{E}_q(\Lambda f)\circ\boldsymbol{\phi}_q^{-1})\|_\infty. \tag{5-3}$$

It suffices to bound the right hand side of (5-3). Since $f \in Y^m(\Omega)$, Proposition 4.7 ensures that $\mathscr{E}_q(\Lambda f) \in X^m(\overline{D}_q)$. By estimate (5-1), there exists a constant c_1 such that for all $q \in \mathbb{N}$ and $f \in Y^m(\Omega)$,

$$\|\mathscr{E}_q(\Lambda f)\circ \boldsymbol{\phi}_q^{-1} - \mathscr{S}_N(\mathscr{E}_q(\Lambda f)\circ\boldsymbol{\phi}_q^{-1})\|_\infty \le c_1 N^2 2^{-mN} |\mathscr{E}_q(\Lambda f)\circ\boldsymbol{\phi}_q^{-1}|_{X^m(I^3)}. \tag{5-4}$$

It remains to establish that there exists a constant c_2 such that for all $q \in \mathbb{N}$ and $f \in Y^m(\Omega)$,

$$|\mathscr{E}_q(\Lambda f)\circ\boldsymbol{\phi}_q^{-1}|_{X^m(I^3)} \le c_2 \|f\|_{Y^m(\Omega)}. \tag{5-5}$$

We now prove estimate (5-5). By the chain rule of derivatives, we have that

$$\|(\mathscr{E}_q(\Lambda f)\circ\boldsymbol{\phi}_q^{-1})^{(\boldsymbol{\alpha})}\|_\infty = \pi^{\alpha_0+\alpha_1} 2^{-(q-1)(\alpha_1+\alpha_2)} \|(\mathscr{E}_q(\Lambda f))^{(\boldsymbol{\alpha})}\|_\infty. \tag{5-6}$$

Substituting (4-42) into (5-6) yields a constant c_3 such that for all $q \in \mathbb{N}$, $f \in Y^m(\Omega)$ and $\boldsymbol{\alpha} \in \mathbb{N}_0^3$ with $|\boldsymbol{\alpha}|_\infty = m$,

$$\|(\mathscr{E}_q(\Lambda f)\circ\boldsymbol{\phi}_q^{-1})^{(\boldsymbol{\alpha})}\|_\infty \le c_3 \|f\|_{Y^m(\Omega)}.$$

Thus, from the definition of $|\cdot|_{X^m(I^3)}$, we obtain estimate (5-5). By combining (5-3), (5-4) and (5-5), we complete the proof of this lemma. □

We are now ready to estimate the error of the quadrature formula (3-21).

Proposition 5.2. Let $m \in \mathbb{N}$ be fixed. If $\partial\Omega$ satisfies Hypothesis 4.1, then there exists a constant c such that for all $\tau \in \mathbb{N}$, $\boldsymbol{N} \in \mathbb{N}^\tau$, $|k| \in \mathbb{N}_0$ and $f \in Y^m(\Omega)$,

$$|\widehat{\mathcal{N}f\circ \boldsymbol{r}}(k) - \mathscr{Q}_{\tau,\boldsymbol{N}} f(k)| \le c\left(\tau 2^{-2\tau} + \sum_{q\in\mathbb{Z}_\tau} N_q^2 2^{-mN_q-2q}\right)\|f\|_{Y^m(\Omega)}. \tag{5-7}$$

Proof. According to the triangle inequality, we have that

$$|\widehat{\mathcal{N}f\circ \boldsymbol{r}}(k) - \mathscr{Q}_{\tau,\boldsymbol{N}} f(k)| \le \zeta(k) + \varphi(k), \tag{5-8}$$

where

$$\zeta(k) := \left|\widehat{\mathcal{N}f\circ \boldsymbol{r}}(k) - \sum_{q\in\mathbb{Z}_\tau} \widehat{(\Lambda f)}_q(k)\right| \quad\text{and}\quad \varphi(k) := \left|\sum_{q\in\mathbb{Z}_\tau} \widehat{(\Lambda f)}_q(k) - \mathscr{Q}_{\tau,\boldsymbol{N}} f(k)\right|.$$

We next estimate $\zeta(k)$ and $\varphi(k)$ separately.

We first estimate $\zeta(k)$. By the definition (3-4) of $\widehat{\mathcal{N}f\circ \boldsymbol{r}}(k)$ for all $\tau \in \mathbb{N}$ and $k \in \mathbb{N}_0$ there holds

$$\zeta(k) \le \iiint_{S_\tau} |(\Lambda f)(t,\mu,\lambda)\, e_{-k}(t)|\, \mathrm{d}t\, \mathrm{d}\mu\, \mathrm{d}\lambda.$$

Note that $r \in C_{2\pi}^\infty(\mathbb{R})$. Since $\Lambda f = \Psi\cdot\Xi f$, $\Psi = -\frac{1}{2\pi}\log\circ\Upsilon$ and there exists a positive constant c_r such that for all $f \in Y^m(\Omega)$,

$$\|\Xi f\|_\infty \le c_r \|f\|_\infty,$$

we conclude that for all $\tau \in \mathbb{N}$, $k \in \mathbb{N}_0$, and $f \in Y^m(\Omega)$,

$$\zeta(k) \le \frac{c_r}{2\pi}\|f\|_\infty \iiint_{S_\tau} |\log\Upsilon(t,\mu,\lambda)|\, \mathrm{d}t\, \mathrm{d}\mu\, \mathrm{d}\lambda. \tag{5-9}$$

Applying the logarithmic function to (4-14), there exists a positive constant c_1 such that for all $(t,\mu,\lambda)\in S_0$,

$$|\log \Upsilon(t,\mu,\lambda)| \le c_1 + |\log R_T(t,\mu,\lambda)|. \tag{5-10}$$

By applying the polar coordinates transform to R_T with respect to (μ,λ), there exists a positive constant c_2 such that for all $\tau\in\mathbb{N}$,

(5-11)
$$\iiint_{S_\tau}(c_1+|\log R_T(t,\mu,\lambda)|)\,\mathrm{d}t\,\mathrm{d}\mu\,\mathrm{d}\lambda \le 4\pi^2 c_1\cdot 2^{-2\tau} + \int_0^{2\pi}\int_0^{\pi}\int_0^{2^{-\tau}\pi} |\log\rho|\,\rho\,\mathrm{d}\rho\,\mathrm{d}\theta\,\mathrm{d}t \le c_2\tau 2^{-2\tau}.$$

Combining (5-9), (5-10) and (5-11) yields a constant c_3 such that for all $\tau\in\mathbb{N}$, $k\in\mathbb{N}_0$ and $f\in Y^m(\Omega)$,

$$\zeta(k)\le c_3\tau 2^{-2\tau}\|f\|_\infty. \tag{5-12}$$

From the definition of $\|\cdot\|_{Y^m(\Omega)}$, we have that for all $f\in Y^m(\Omega)$,

$$\|f\|_\infty \le \|f\|_{Y_m(\Omega)}. \tag{5-13}$$

Combining (5-12) and (5-13) leads to the estimate

$$\zeta(k)\le c_3\tau 2^{-2\tau}\|f\|_{Y_m(\Omega)}. \tag{5-14}$$

We next estimate $\varphi(k)$. According to $\mathscr{E}_q(\Lambda f)(t,\mu,\lambda)=(\Lambda f)(t,\mu,\lambda)$, for all $(t,\mu,\lambda)\in D_q$, the definition (3-5) of $\widehat{(\Lambda f)}_q(k)$ and the definition (3-21) of $Q_{\tau,\boldsymbol{N},m}$, we have that

$$\varphi(k)\le \sum_{q\in\mathbb{Z}_\tau}\iiint_{D_q}|\mathscr{E}_q(\Lambda f)(t,\mu,\lambda)\,e_{-k}(t)-\widetilde{\mathscr{S}}_{N_q}(\mathscr{E}_q(\Lambda f))(t,\mu,\lambda)\,e_{-k}(t)|\,\mathrm{d}t\,\mathrm{d}\mu\,\mathrm{d}\lambda.$$

Thus, there exist a constant c_4 such that for all $\tau\in\mathbb{N}$, $\boldsymbol{N}:=[N_q:q\in\mathbb{Z}_\tau]\in\mathbb{N}^\tau$, $k\in\mathbb{N}_0$ and $f\in Y^m(\Omega)$,

$$\varphi(k)\le c_4\sum_{q\in\mathbb{Z}_\tau}2^{-2q}\|\mathscr{E}_q(\Lambda f)-\widetilde{S}_{N_q}(\mathscr{E}_q(\Lambda f))\|_\infty. \tag{5-15}$$

Thus, applying Lemma 5.1 to the right-hand side of (5-15) yields that there exist a constant c_5 such that for all $\tau\in\mathbb{N}$, $\boldsymbol{N}:=[N_q:q\in\mathbb{Z}_\tau]\in\mathbb{N}^\tau$, $k\in\mathbb{N}_0$ and $f\in Y^m(\Omega)$,

$$\varphi(k)\le c_5\sum_{q\in\mathbb{Z}_\tau}N_q^2 2^{-mN_q-2q}\|f\|_{Y_m(\Omega)}. \tag{5-16}$$

By substituting (5-15) and (5-16) into (5-8), we obtain the desired estimate (5-7). □

We next present the main result of this section.

Theorem 5.3. Let $m\in\mathbb{N}$ be fixed. Suppose that τ and $\boldsymbol{N}$ are chosen as in (3-30) and let $\mathscr{Q}_{m,n}:=\mathscr{Q}_{\tau,\boldsymbol{N}}$. If $\partial\Omega$ satisfies Hypothesis 4.1, then there exists a constant c such that for all $n\in\mathbb{N}$, $|k|\in\mathbb{N}_0$, $f\in Y^m(\Omega)$,

$$|\widehat{\mathscr{N}f\circ\boldsymbol{r}}(k)-\mathscr{Q}_{m,n}(\Lambda f,k)|\le cn^{-m}\log^3 n\|f\|_{Y^m(\Omega)}. \tag{5-17}$$

Proof. We prove this theorem by using Proposition 5.2 with τ and $\boldsymbol{N}$ chosen as in (3-30). Let $N'_q := \lceil \log n - q/m \rceil$. By Proposition 5.2, there exits a positive a constant c_1 such that for all $n \in \mathbb{N}$, $|k| \in \mathbb{N}_0$, $f \in Y^m(\Omega)$,

$$|\widehat{\mathcal{N} f \circ \boldsymbol{r}}(k) - \mathcal{Q}_{m,n}(\Lambda f, k)| \le c_1 \biggl(\lceil m \log n \rceil 2^{-2\lceil m \log n \rceil} + \sum_{q \in \mathbb{Z}_\tau} N'^2_q 2^{-mN'_q} \biggr) \|f\|_{Y^m(\Omega)}. \tag{5-18}$$

It can be verified that there exists a constant c_2 such that for $n \in \mathbb{N}$,

$$\lceil m \log n \rceil 2^{-2\lceil m \log n \rceil} \le c_2 n^{-2m} \log n. \tag{5-19}$$

Also, there exists a constant c_3 such that for $n \in \mathbb{N}$,

$$\sum_{q \in \mathbb{Z}_\tau} N'^2_q 2^{-mN'_q} \le c_3 \, n^{-m} \log^3 n. \tag{5-20}$$

Substituting (5-19) and (5-20) into (5-18) establishes estimate (5-17). □

6. Numerical results

We present two numerical examples to demonstrate the accuracy of the proposed algorithm over its computational costs. In example 1, we calculate the kth Fourier coefficient of the Newton potential $\widehat{\mathcal{N} f \circ \boldsymbol{r}}(k)$ defined in (3-1) by using Algorithm 1 to verify the computational complexity in Theorem 3.3 and the error estimate in Theorem 5.3. In example 2, we solve the BIE of the Dirichlet problem for the Poisson equation defined in (2-5) with applying the Algorithm 1 to compute the Newton potential in the right-hand side. We present the convergence order of the solution of BIE (2-5) and the accuracy of the PDE solution u at some chosen points. We compute these examples by using Algorithm 1 with τ and $\boldsymbol{N}$ chosen as in (3-30). All programs are run on a workstation with 2.10 GHz Intel(R) Xeon(R) E5-2620 v2 CPU and 96 GB memory.

For the purpose of evaluating the proposed algorithm, we define the computed error of the algorithm by

$$E_{\mathbb{F}}(N) := \max_{k \in \mathbb{F}} \{ |\widehat{\mathcal{N} f \circ \boldsymbol{r}}(k) - Q_{2^N,m}(\Lambda f, k)| \},$$

where $\mathbb{F} \subset \mathbb{Z}$ is the index set to be specified later, N denotes the level of the sparse approximation and $Q_{2^N,m}(\Lambda f, k)$ is defined in Theorem 5.3. The accuracy order with respect to $E_{\mathbb{F},N}$ is defined by

$$\mathrm{AO}_{\mathbb{F}}(N) := \log_2 \frac{E_{\mathbb{F}}(N)}{E_{\mathbb{F}}(N+1)}.$$

According to Theorem 5.3, we estimate the accuracy order as

$$\mathrm{IAO}_m(N) := \log_2 \frac{N^3 \, 2^{-mN}}{(N+1)^3 \, 2^{-m(N+1)}} = m - 3 \log_2(1 + 1/N).$$

We see that $\mathrm{IAO}_m(N) \approx m$, when N is big enough.

Example 1: We compute the kth Fourier coefficient of the Newton potential $\widehat{\mathcal{N} f \circ \boldsymbol{r}}(k)$ defined in (3-1) with

$$f(x, y) := e^x \cos(y),$$

n	# mesh points	time (s)	$m=1$ $E_{\mathbb{L}^1_{64}}$	$\mathrm{AO}_{\mathbb{L}^1_{64}}$	IAO_1
2^5	1023	0.01	1.64E-00		
2^6	2815	0.02	1.04E-00	0.65	0.21
2^7	7423	0.02	6.70E-01	0.63	0.33
2^8	18943	0.04	4.78E-01	0.48	0.42
2^9	47103	0.07	2.94E-01	0.70	0.49
2^{10}	114687	0.15	1.93E-01	0.60	0.54
2^{11}	274431	0.32	1.10E-01	0.81	0.58
2^{12}	647167	0.71	7.05E-02	0.64	0.62
2^{13}	1507327	1.37	4.16E-02	0.76	0.65
2^{14}	3473407	2.75	2.39E-02	0.79	0.67
2^{15}	7929855	6.24	1.39E-02	0.78	0.70
2^{16}	17956863	13.61	7.75E-03	0.84	0.72
2^{17}	40370175	29.24	4.44E-03	0.80	0.73

Table 1. Results of Algorithm 1 with the piecewise constant interpolation.

n	# mesh points	time (s)	$m=2$ $E_{\mathbb{L}^1_{64}}$	$\mathrm{AO}_{\mathbb{L}^1_{64}}$	IAO_2
2^5	8184	0.03	5.04E-02		
2^6	22520	0.06	2.94E-02	0.71	1.21
2^7	59384	0.12	1.38E-02	1.06	1.33
2^8	151544	0.26	5.60E-03	1.29	1.42
2^9	376824	0.53	2.04E-03	1.44	1.49
2^{10}	917496	1.10	7.01E-04	1.61	1.54
2^{11}	2195448	2.39	2.29E-04	1.61	1.58
2^{12}	5177336	5.24	7.26E-05	1.65	1.62
2^{13}	12058616	11.60	2.23E-05	1.70	1.65
2^{14}	27787256	26.28	6.74E-06	1.72	1.67
2^{15}	63438840	58.35	1.99E-06	1.75	1.70
2^{16}	143654904	121.05	5.84E-07	1.76	1.72
2^{17}	322961400	265.69	1.68E-07	1.79	1.73

Table 2. Results of Algorithm 1 with the piecewise linear interpolation.

for each $k \in \mathbb{F} = \mathbb{L}^1_{64} := \{x \in \mathbb{Z} : |x| \le 63\}$. For this case, the boundary is chosen to the circle with $r(t) = 1$ in (2-4). Numerical results of this example are shown in Tables 1 and 2. We also compare in Figure 5 the performance of the proposed algorithm with that of the fast multipole method (FMM).

We first use Algorithm 1 with the piecewise constant interpolation ($m = 1$) to compute approximate values of the Newton potential $\widehat{\mathcal{N} f \circ \boldsymbol{r}}(k)$. In this case, we have $\tau = 2\lceil \log_2 n \rceil$, $N_q = \max\{\lceil \log_2 n \rceil - q, 1\}$, for all $q \in \mathbb{Z}_\tau$. Numerical results presented in Table 1 confirm the quasilinear accuracy order. We then

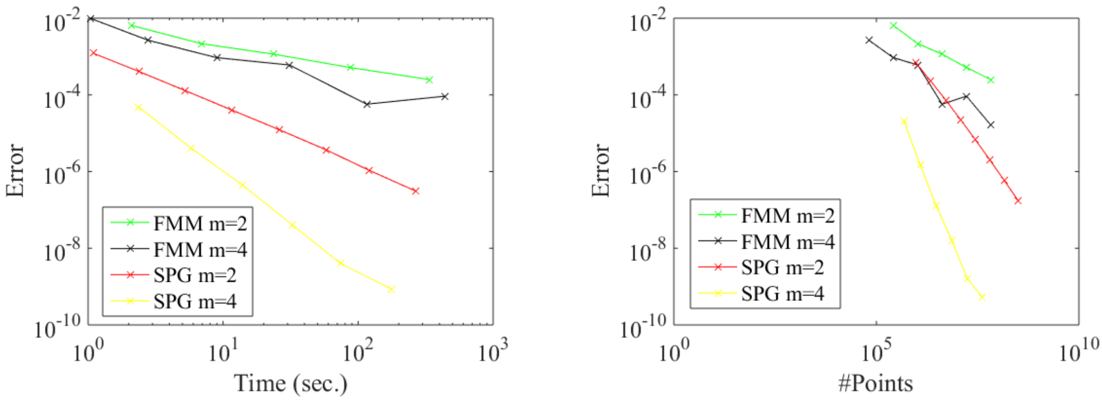

Figure 5. Comparison with the fast multipole method with a uniform grid: errors over the computing time (left) and errors over the number of grid points used (right).

use Algorithm 1 with the piecewise linear interpolation ($m = 2$) to compute approximate values of the Newton potential $\widehat{\mathcal{N} f \circ \boldsymbol{r}}(k)$. In this case, we have $\tau = \lceil \log_2 n \rceil$, $N_q = \max\{\lceil \log_2 n \rceil - 2q, 1\}$, for all $q \in \mathbb{Z}_\tau$. Numerical results presented in Table 2 confirm the quasiquadratic accuracy order. Moreover, both Tables 1 and 2 show that the computing time of Algorithm 1 grows quasilinearly as n increases, as estimated in Theorem 3.3.

The comparison of the proposed method with the FMM are shown in Figure 5. The red curve represents the results of the proposed method with the linear interpolation on sparse grids (SPG $m = 2$) and the yellow curve represents the results of the method with the cubic interpolation (SPG $m = 4$). We embed the domain Ω to a larger square domain and apply the FMM on uniform grids. The green curve indicates the FMM with the piecewise linear approximation. The black curve indicates the FMM with the piecewise cubic approximation. Figure 5 shows that the proposed method outperforms FMM with uniform grids in computing the Newton potential.

Example 2 We solve the Dirichlet problem of the Poisson equation (2-1) with the boundary curve being an ellipse described by

$$r(t) := \frac{0.4}{\sqrt{0.16\cos^2(t) + \sin^2(t)}}$$

using the BIE method reviewed in Section 2 with the corresponding Newton potential computed by using the quadrature method proposed in this paper. For the purpose of comparison, we choose the forcing function as

$$f(x, y) := -e^x y^3 - 6e^x y,$$

and the boundary value

$$u(x, y) := e^x y^3, \quad (x, y) \in \partial\Omega.$$

Specifically, we apply Algorithm 1 to compute the Newton potential in the right-hand side of the related BIE (2-5) and solve this equation by the fast Fourier–Galerkin method described in (2-9). We then present the convergence of the solution of BIE (2-5) and the accuracy of the value of PDE solution u on points specified.

n	N_0	L^2 error	accuracy order
32	8	1.15E-02	
64	9	4.40E-03	1.38
128	10	1.64E-03	1.41
256	11	5.54E-04	1.56
512	12	1.78E-04	1.64

n	solution of the BIE (2-5) L^2 error	convergence order
32	2.08E-01	
64	9.69E-02	1.10
128	6.24E-02	0.63
256	3.95E-02	0.66
512	2.35E-02	0.74

Table 3. Left: accuracy of the computed Newton potential by using Algorithm 1 with $m = 2$. Right: errors and convergence orders of the BIE (2-5).

We first compute the Fourier coefficients of the Newton potential $\widehat{\mathcal{N} f \circ \boldsymbol{r}}(k)$, where $k \in \mathbb{L}_n^1$, by using Algorithm 1 with the linear interpolation. That is, we choose the parameters $m = 2$, $\tau = 2\lceil \log_2 n \rceil$ and $N_q = \max\{\lceil \log_2 n \rceil + 3 - q, 1\}$ for all $q \in \mathbb{Z}_\tau$. In Table 3, left, we present the L^2 errors of the Newton potential on the boundary computed by Algorithm 1.

We then solve the Dirichlet problem (2-1) with the computed values of the Newton potential. The equation is solved is solved by the fast fully discrete Fourier–Galerkin methods (2-9) originally presented in [18] with $m = 2$, $N = \lceil \log_2 n \rceil$. We present the L^2 errors and convergence orders of the approximate solutions of BIE (2-5) in Table 3, right. According to [18], in this case the desired theoretical convergence order for the fully discrete Fourier–Galerkin method is nearly 0.5. We observe from this example that the BIE solutions with the Newton potential computed by using the proposed quadrature method preserve the convergence order guaranteed by the theoretical estimate.

We further compute the approximate solutions of the Dirichlet problem of the Poisson equation by using the fast Fourier–Galerkin method for its BIE with the proposed quadrature method for computing the Newton potential. In Table 4, we present errors of the approximate solutions of the Dirichlet problem at three points

$$P_\alpha := \alpha(\cos(\pi/4), 0.4\sin(\pi/4)), \quad \alpha = 0.25, 0.5, 0.75.$$

These numerical results show that the approximate solutions of the Dirichlet problem obtained by using the BIE with the proposed quadrature method for computing the resulting Newton potential have excellent performance. This further confirms that the proposed quadrature method is accurate enough to serve as an effective method for computing the Newton potential in the context of the fast Fourier–Galerkin method

	α		
n	0.25	0.5	0.75
32	8.57E-05	4.99E-04	2.15E-04
64	1.25E-05	1.51E-04	5.23E-05
128	8.99E-07	4.71E-05	9.84E-05
256	6.88E-06	3.15E-05	1.31E-05
512	8.90E-07	3.89E-06	1.77E-06

Table 4. Results of the Dirichlet problem (2-1).

for solving the BIE for the Poisson equation. This overcomes the obstacle of applying the BIE to the Poisson equation that the resulting Newton potential has to be efficiently and accurately computed. The proposed quadrature method enables us to efficiently solve the Poisson equation by employing the fast Fourier–Garlerkin method developed in [8; 21; 18].

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References

[1] T. Apel, A.-M. Sändig, and J. R. Whiteman, "Graded mesh refinement and error estimates for finite element solutions of elliptic boundary value problems in non-smooth domains", *Math. Methods Appl. Sci.* **19**:1 (1996), 63–85.

[2] K. E. Atkinson, *The numerical solution of integral equations of the second kind*, Cambridge Monographs on Applied and Computational Mathematics **4**, Cambridge University Press, Cambridge, 1997.

[3] I. Babuška, R. B. Kellogg, and J. Pitkäranta, "Direct and inverse error estimates for finite elements with mesh refinements", *Numer. Math.* **33**:4 (1979), 447–471.

[4] S. C. Brenner, J. Cui, T. Gudi, and L.-Y. Sung, "Multigrid algorithms for symmetric discontinuous Galerkin methods on graded meshes", *Numer. Math.* **119**:1 (2011), 21–47.

[5] H. Brunner, "Nonpolynomial spline collocation for Volterra equations with weakly singular kernels", *SIAM J. Numer. Anal.* **20**:6 (1983), 1106–1119.

[6] C. F. F. di Bruno, "Note sur une nouvelle formule de calcul différentiel", *Quarterly J. Pure Appl. Math.* **1** (1857), 359–360.

[7] H.-J. Bungartz and M. Griebel, "Sparse grids", *Acta Numer.* **13** (2004), 147–269.

[8] H. Cai and Y. Xu, "A fast Fourier–Galerkin method for solving singular boundary integral equations", *SIAM J. Numer. Anal.* **46**:4 (2008), 1965–1984.

[9] Y. Cao, Y. Jiang, and Y. Xu, "Orthogonal polynomial expansions on sparse grids", *J. Complexity* **30**:6 (2014), 683–715.

[10] Z. Chen, C. A. Micchelli, and Y. Xu, "A construction of interpolating wavelets on invariant sets", *Math. Comp.* **68**:228 (1999), 1569–1587.

[11] X. Chen, R. Wang, and Y. Xu, "Fast Fourier–Galerkin methods for nonlinear boundary integral equations", *J. Sci. Comput.* **56**:3 (2013), 494–514.

[12] Z. Chen, C. A. Micchelli, and Y. Xu, *Multiscale methods for Fredholm integral equations*, Cambridge Monographs on Applied and Computational Mathematics **28**, Cambridge University Press, Cambridge, 2015.

[13] G. M. Constantine and T. H. Savits, "A multivariate Faà di Bruno formula with applications", *Trans. Amer. Math. Soc.* **348**:2 (1996), 503–520.

[14] F. Ethridge and L. Greengard, "A new fast-multipole accelerated Poisson solver in two dimensions", *SIAM J. Sci. Comput.* **23**:3 (2001), 741–760.

[15] L. C. Evans, *Partial differential equations*, Graduate Studies in Mathematics **19**, American Mathematical Society, Providence, 1998.

[16] L. N. G. Filon, "On a quadrature formula for trigonometric integrals", *Proc. Roy. Soc. Edinburgh* **49** (1930), 38–47.

[17] Y. Jiang and Y. Xu, "Fast discrete algorithms for sparse Fourier expansions of high dimensional functions", *J. Complexity* **26**:1 (2010), 51–81.

[18] Y. Jiang and Y. Xu, "Fast Fourier–Galerkin methods for solving singular boundary integral equations: numerical integration and precondition", *J. Comput. Appl. Math.* **234**:9 (2010), 2792–2807.

[19] Y. Jiang and Y. Xu, "B-spline quasi-interpolation on sparse grids", *J. Complexity* **27**:5 (2011), 466–488.

[20] Y. Jiang and Y. Xu, "Fast computation of the multidimensional discrete Fourier transform and discrete backward Fourier transform on sparse grids", *Math. Comp.* **83**:289 (2014), 2347–2384.

[21] Y. Jiang, B. Wang, and Y. Xu, "A fast Fourier–Galerkin method solving a boundary integral equation for the biharmonic equation", *SIAM J. Numer. Anal.* **52**:5 (2014), 2530–2554.

[22] H. Kaneko and Y. Xu, "Gauss-type quadratures for weakly singular integrals and their application to Fredholm integral equations of the second kind", *Math. Comp.* **62**:206 (1994), 739–753.

[23] R. Kress, *Linear integral equations*, Applied Mathematical Sciences **82**, Springer, Berlin, 1989.

[24] Y. Ma and Y. Xu, "Computing highly oscillatory integrals", *Math. Comp.* **87**:309 (2018), 309–345.

[25] W. McLean, "A spectral Galerkin method for a boundary integral equation", *Math. Comp.* **47**:176 (1986), 597–607.

[26] W. McLean, S. Prössdorf, and W. L. Wendland, "A fully-discrete trigonometric collocation method", *J. Integral Equations Appl.* **5**:1 (1993), 103–129.

[27] S. G. Mikhlin, *Mathematical physics, an advanced course*, North-Holland Series in Applied Mathematics and Mechanics **11**, North-Holland Publishing, Amsterdam-London, 1970.

[28] G. Of, O. Steinbach, and P. Urthaler, "Fast evaluation of volume potentials in boundary element methods", *SIAM J. Sci. Comput.* **32**:2 (2010), 585–602.

[29] F.-J. Sayas, "Fully discrete Galerkin methods for systems of boundary integral equations", *J. Comput. Appl. Math.* **81**:2 (1997), 311–331.

[30] B. Wang, R. Wang, and Y. Xu, "Fast Fourier–Galerkin methods for first-kind logarithmic-kernel integral equations on open arcs", *Sci. China Math.* **53**:1 (2010), 1–22.

[31] Y. Xu and Y. Zhao, "Quadratures for improper integrals and their applications in integral equations", pp. 409–413 in *Proc. Sympos. Appl. Math.*, vol. 48, edited by W. Gautschi, Amer. Math. Soc., Providence, RI, 1994.

[32] Y. Xu and Y. Zhao, "An extrapolation method for a class of boundary integral equations", *Math. Comp.* **65**:214 (1996), 587–610.

[33] Y. Xu and Y. Zhao, "Quadratures for boundary integral equations of the first kind with logarithmic kernels", *J. Integral Equations Appl.* **8**:2 (1996), 239–268.

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