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Bootstrap Percolation in the Random Geometric Graph

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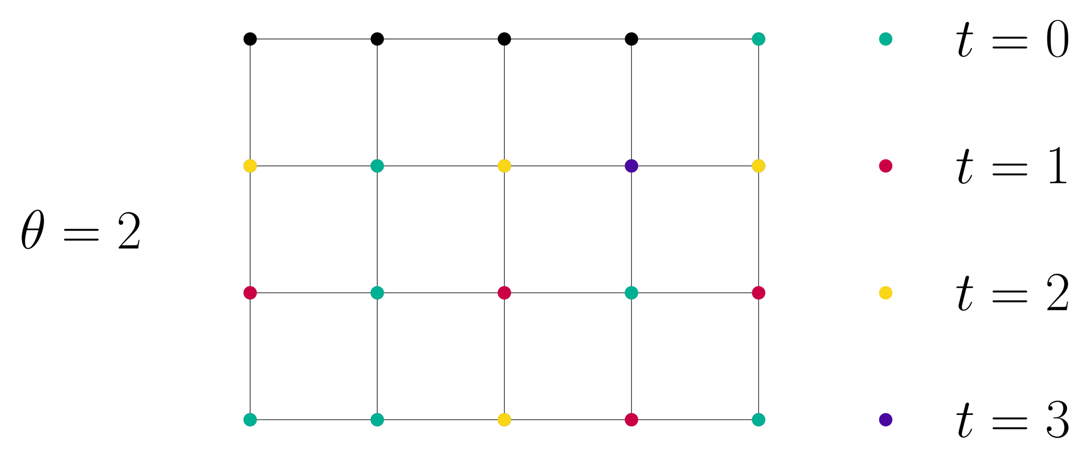
Bootstrap Percolation

Bootstrap Percolation, sometimes used to model the spread of a disease, is a dynamic process on a graph in which a vertex becomes infected if it has too many edges to infected vertices. This can be stated precisely as follows.

The Process:

1. Start with a set of "infected" vertices, I_0, on a graph and select $\theta > 0$.
2. For each t, define I_t to be I_{t-1} along with vertices in $V \setminus I_{t-1}$ which have at least θ edges into I_{t-1}.
3. We say the system *percolates* if all vertices eventually become infected and does not percolate otherwise



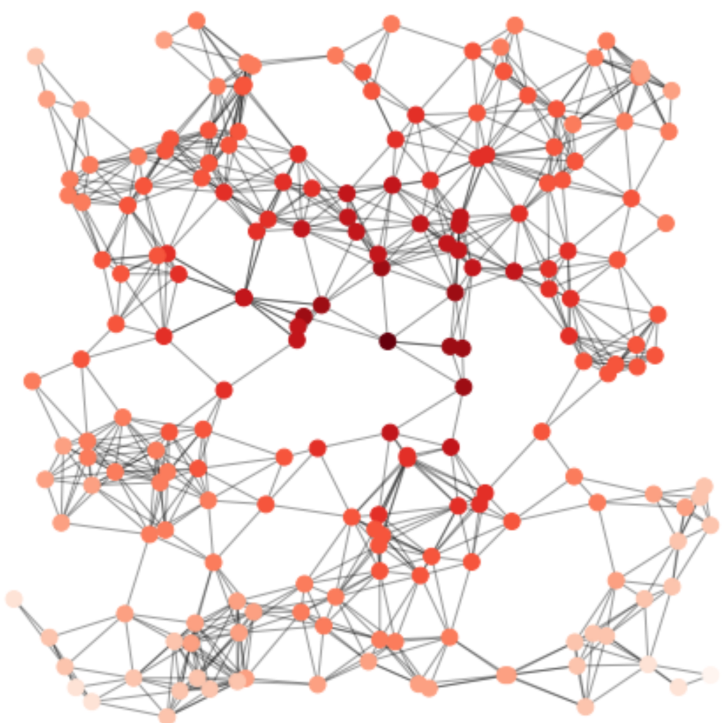
In the above example, since $\theta = 2$, the infection will continue to spread until all vertices are infected, which means the system percolates.

The Random Geometric Graph

The **Random Geometric Graph** is sometimes used to model random networks in which it is likely that vertices "cluster". The following is a rigorous definition of this random graph model.

The Process to Generate $G_{n,r}$:

1. Select an $n \in \mathbb{N}$ and an r satisfying $0 < r < 1$.
2. Choose n points uniformly at random from $[0,1]^2$.
3. Join points by an edge if their distance is less than r.



Ex: $G_{200,0.05}$

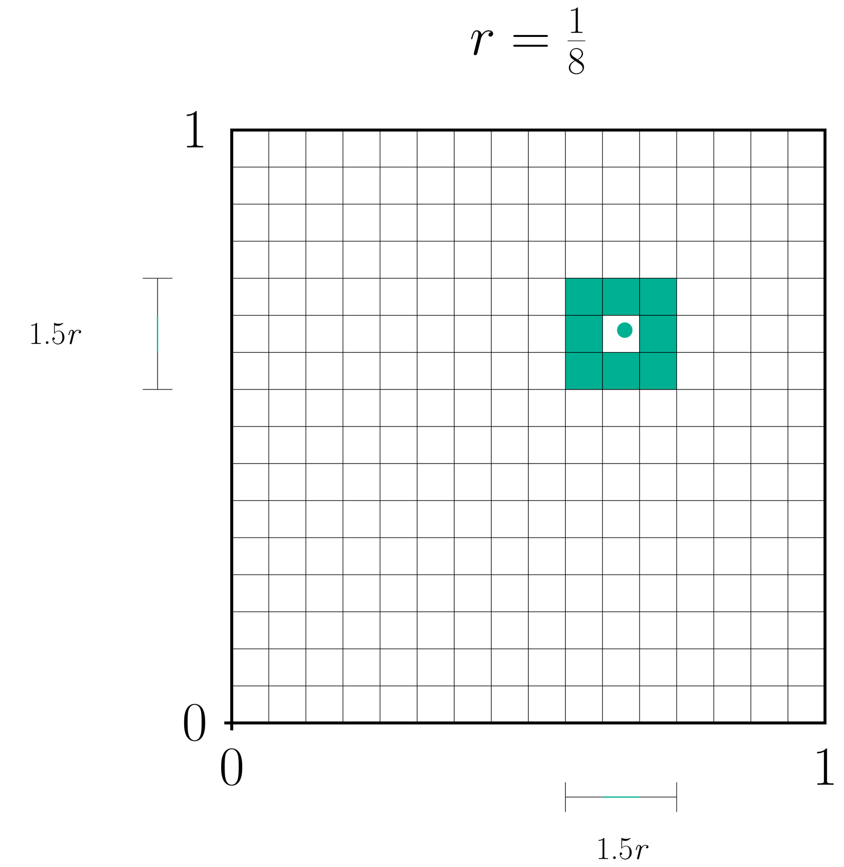
First Result

Theorem: Suppose θ is constant and $r \gg n^{-0.25}$. If $|I_0| \gg (\frac{1}{r^2})^{\theta - 1/\theta}$, then $G_{n,r}$ percolates whp.

Note: "whp" means with probability tending to 1 as $n \to \infty$.

Sketch of Proof: For this sketch, fix $\theta = 2$ and $r = \frac{1}{k}$ with $k \in \mathbb{N}$.

1. Create a grid of $0.5r \times 0.5r$ boxes in $[0,1]^2$.
2. Claim: Every box contains at least $0.125nr^2$ vertices whp.
3. Claim: If *any* box contains at least θ infected vertices, then the system percolates.
4. Claim: If $|I_0| \gg k$, then *some* grid box contains at least θ vertices from I_0 whp.



Together these claims imply that the system percolates. Claim 4 means that some box (i.e. the one with the green dot) will have enough initially infected vertices to become totally infected. Because of this, Claim 3 gives the result.

Misc Background Information

Connectivity: A graph is called *connected* if there is a path along edges between any two vertices.

Theorem: If $r \gg \sqrt{\ln(n)/n}$, then $G_{n,r}$ is connected whp.

Local Resilience of Graph Properties: This should be thought of as a measure of how robustly a graph satisfies a property. Formally, if G satisfies $\mathcal{P}$, we say that G has *local resilience* α with respect to $\mathcal{P}$ if deleting an α-fraction of the edges at each vertex results in a graph which does not satisfy $\mathcal{P}$.

Connectivity Threshold

Theorem: The local resilience for connectivity in $G_{n,r}$ is 0.5.

Note: this matches the value for other random graph models although the proof here is slightly different than in those models.

Sketch of Proof:

1. Deleting $0.5d(v)$ edges at each vertex is enough:
   1. Delete every edge crossing the line $x = 0.3$.
      - Notice that this disconnects the graph.
   2. Claim: Any vertex (x, y) whose x-coordinate satisfies $x < 0.3$ or $x > 0.3$ loses at most $0.5d(v)$ edges.
      - This requires a standard concentration bound on how edges look at a vertex.
2. Deleting $0.5d(v)$ edges at each vertex is required:
   1. 'Grid' $[0,1]^2$ with $\frac{1}{\ell}r \times \frac{1}{\ell}r$ squares (ℓ TBD).
      - Vertices are adjacent to $(2\ell - 1)^2$ boxes each of area $(\frac{r}{\ell})^2$
      - Pairs of vertices in side-adjacent boxes share $(2\ell - 1)(2\ell - 2)$ boxes
      - So # of common neighbors for vertices in the same box or in side-adjacent boxes is at least:
        $$\frac{(2\ell - 1)(2\ell - 2)}{\ell^2} r^2 n > \left(4 - \frac{6}{\ell}\right) r^2 n$$
      - **Key Point:** To destroy connectivity, at least one edge must be deleted per common neighbor.
   2. Fix $\varepsilon > 0$ and delete $(0.5 - \varepsilon)4r^2 n$ edges per vertex.
      - $2(0.5 - \varepsilon)4r^2 n = (4 - 8\varepsilon)r^2 n \Rightarrow$ if $\ell > \frac{6}{8\varepsilon}$, we cannot eliminate all common neighbors.

Future Work

1. Our work extends previous work on this problem. In particular, our results work with a larger r value than previous work. We are, however, limited in what we can show. So, we hope to extend the range of values of r and θ for which we know whether $G_{n,r}$ is likely to percolate or not.
2. To our knowledge, local resilience problems on $G_{n,r}$ have not been studied. Thus, we hope to examine such problems for other graph properties such as, "has a perfect matching" or "is Hamiltonian".

Acknowledgements

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