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Review article on the role of visualisation in mathematics conceptualisation and learning

Miguel de Guzmán, 'El papel de la visualización [The rôle of visualisation]', chapter 0 of **El rincón de la pizarra** [The corner of the Blackboard] (Pirámide, Madrid, 1996)

Visualisation is a natural element at the root of mathematical thought, in the discovery of relations between mathematical objects, and in the transmission and communication of mathematics. This review focuses on the introductory chapter of one of the most famous books on this topic, by a widely published writer on pedagogy of mathematics, discussing at the same time implications for undergraduate teaching in the subject.

Miguel de Guzmán is Professor of Mathematics and director of the Mathematical Education programme at the Universidad Complutense de Madrid. The book's title comes from an anecdote about Norbert Wiener teaching at MIT. Though the story is doubtless apocryphal (and is in circulation with both mathematicians and physicists – including Feynman – in the protagonist's rôle) it serves very well to justify the importance of visualisation in mathematics as a worthy topic for the book, and we relate it again here:

> *Wiener (or whoever) had nearly filled the blackboard with a detailed rigorous proof of a complex result, when he stopped staring at the line he had just written, seemingly at a dead end. But Wiener moved to the corner of the blackboard and sketched a few pictures keeping his back to the audience, and suddenly his face lit up. He silently erased his diagrams, and resumed his symbolic proof from where he had left off with no further difficulties.*

The great challenge proposed and thoroughly examined in this introductory chapter is to resist the temptation to erase the diagrams, and instead to explain and inculcate into our students the why and how of this traditionally deprecated mathematical language. It is of great practical interest to me, given the lack of 'formal' or 'traditional' reasoning skills in many of my students, to investigate this approach.

The term visualisation here is not that of affective subconscious restructuring, as introduced by transactional analyst in the 1950s, Guzmán notes, but rather a cognitive representation of the rich visual component of mathematical ideas, concepts and methods. As Guzmán defines it:

- **Visualisation** is the explicit attention a mathematician pays to possible concrete representations of abstract relations. Using these one naturally forms versatile 'networks' of meaning, facilitating the choice of the most efficient approaches to the problem at hand.

- **Mathematisation** is the process, starting from perception of similarities of form, of abstraction of common features, and of rational, symbolic elaboration of these abstractions allowing straightforward manipulation of the structure underlying the initial observations. Observation of *multiplicity* is thus abstracted to arithmetic, which is in turn itself abstracted to *algebra* (the structures underlying the arithmetic operations between quantities) which is in turn…

Human vision is not a photographic process, even from the purely neurological point of view. When considering mathematical visualisation, it becomes in fact a complex commulative system of coding and *decoding*. Guzmán himself emphasises that the teaching and learning of visualisation is not easy; it must be an interactive process of immersion and culturing in the sociohistorical fabric of mathematics. He gives the “beautiful” geometric proof of Pythagorus’ theorem as an example of how misleadingly easy such proofs appear to those already in the know.

Four important qualitative classes of visulation are identified by Guzmán, termed the *isomorphic*, *homeomorphic*, *analogous* and *diagrammatic* forms. Isomorphic visualisation involves a one-to-one matching of the visual and mathematical elements, such as the *Argand diagram* without which, he convincingly argues, mathematicians would never have adopted complex numbers as a ‘serious’ notion. In homeomorphic visualisation, as exemplified by the Birkhoff-MacLane proof of the Schroeder-Bernstein Theorem, there are visual elements without any exact abstract counterpart but which hint at the method of proof. A superb example of analogous visualisation worthy of Archimedes is given: when asked to find which quadrilateral of given side lengths has largest area, a group of students imagine it as a wire-frame model filled with soap film which will naturally adopt the largest-area configuration to minimise surface tension. From here they note that since the forces, acting at the midpoint and perpendicular to each side, must be concurrent, the quadrilateral must be inscribed in a circle with the sides as chords. Diagrammatic visualisation, finally, is more of a mnemonic device and is generally more symbolic and personal than the others. As a pedagogical point, however, care must be taken even here not to try to hide such devices from students, to allow them to participate not just in the results of the theory but in the processes of reaching them.

An extremely valuable insight into the current state of visualisation in mathematics is afforded by the historical background Guzmán presents. The Greeks, to whom ‘theorem’ and ‘contemplate’ were almost the same word, visualised quite sophisticated algebraic calculations with small stones (*‘calculi’* as the Romans would say), I wonder if my students, or the readers of this review, would visualise the algebraic formulae

$$1+3+5+\dots+(2n-1)=n^2$$
$$2(1+2+3+4+\dots+n)=n(n+1)$$

with such deceptively ‘obvious’ pictures as those found naturally by the Greeks. Euclid and Archimedes are excellent sources of visual reasoning, and in Plato one finds explicitly the idea that the picture of a circle is not the circle itself - an idea which now sounds like Magritte!

In the twentieth century, however, visualisation was relegated to the second division. The ideas of non-Euclidean geometry, the justification of infinitesimal calculus by the Weierstraß arithmetisation of analysis, the easy-but-false visual proofs of the Jordan curve theorem and the four-colour theorem – all of these contributed, according to Guzmán, to the stigmatisation of visual arguments, and to the introduction of purely symbolic ‘modern mathematics’ into our secondary and even primary schools.

As Miguel de Guzmán points out, a picture is worth a thousand words, but only if you know how to decode it. It is our responsibility to show our students the other side of the mathematical formalism that has become the syllabus, in the hope that they may develop their own ways of learning, drawing their own maps. The notion of what constitutes a proof is by no means easy for them to grasp; it is to be hoped that with the ability to translate visually the language in which proofs may be expressed, the fundamental concept may rise to the surface.

Andrew Tonks
Department of Computing, Communications Technology and Mathematics
London Metropolitan University