# Obstacles to Variational Quantum Optimization from Symmetry Protection
# Supplementary Material

Sergey Bravyi * Alexander Kliesch † Robert Koenig ‡† Eugene Tang §

September 16, 2020

## A QAOA state preparation circuit

In this section we construct a quantum circuit that prepares the level-$p$ QAOA state for any Ising-type Hamiltonian

$$C = \sum_{(j,k)\in E} J_{j,k} Z_j Z_k$$

defined on a graph $G = (V, E)$ with $n$ vertices and maximum vertex degree $D$. This includes the MaxCut Hamiltonian as a special case. Let

$$U = \prod_{a=1}^{p} e^{i\beta_a B} e^{i\gamma_a C}$$

be the requisite circuit. For simplicity, we ignore the initial layer of Hadamard gates that prepares the $|+^n\rangle$ state.

**Lemma A.1.** *The unitary $U$ can be realized by a circuit of depth $d \le p(D+2)$ composed of 1-qubit and 2-qubit gates. If the graph $G$ is $D$-regular and bipartite then $d \le p(D+1)$.*

*Proof.* By Vizing's theorem [24] there is an edge coloring of $G$ with at most $D + 1$ colors. Let $E = E_1 \cup \cdots \cup E_{D+1}$ be such a coloring. For each color $c \in \{1, \cdots, D+1\}$ define a unitary

$$V_c = \prod_{(j,k)\in E_c} e^{i\gamma J_{j,k} Z_j Z_k}$$

Note that $V_c$ is a depth-1 circuit since all edges in $E_c$ are disjoint. Then each entangling layer $e^{i\gamma_a C}$ can be realized by a depth $D+1$ circuit $V_1 V_2 \cdots V_{D+1}$. Each layer $e^{i\beta_a B}$ is a product of single-qubit gates, which has depth one. Thus $U$ has depth at most $p(D + 2)$.

If $G$ is $D$-regular and bipartite, we may reduce the number of edge colors from $D + 1$ to $D$ since all bipartite graphs are $D$-edge-colorable by Kőnig's line coloring theorem. We illustrate the construction of the circuit on Figure 1 for the case $D = 3$ and $p = 1$. □

*IBM T.J. Watson Research Center, Yorktown Heights, NY 10598
†Zentrum Mathematik, Technical University of Munich, 85748 Garching, Germany
‡Institute for Advanced Study, Technical University of Munich, 85748 Garching, Germany
§Institute for Quantum Information and Matter, Caltech, Pasadena, CA 91125

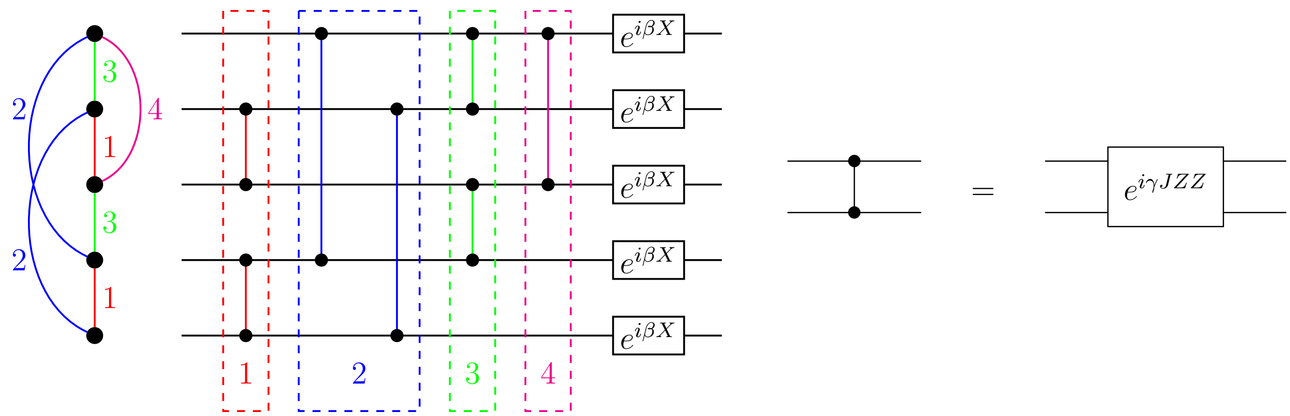


Figure 1: Example for the construction of the circuit given in Lemma A.1: a 4-colorable graph with maximum degree 3 alongside its associated depth-5 quantum circuit for the level-1 QAOA unitary.

# B Optimal variational circuit for the ring of disagrees

In this section we consider the MaxCut Hamiltonian $C$ on the cycle graph $\mathbb{Z}_n$. It is shown that the upper bound

$$\frac{1}{n}\langle +^n|U^\dagger C U|+^n\rangle \leq \frac{2p+1/2}{2p+1}.$$

established in the main text for any $\mathbb{Z}_2$-symmetric range-$p$ unitary $U$ with $p < n/4$ is tight whenever $n$ is an even multiple of $2p+1$. Let

$$|\mathsf{GHZ}_n\rangle = 2^{-1/2}(|0^n\rangle + |1^n\rangle)$$

be the GHZ state of $n$ qubits.

**Lemma B.1.** *Suppose $n = 2p+1$ for some integer $p$. There exists a $\mathbb{Z}_2$-symmetric range-$p$ quantum circuit $V$ such that*

$$|\mathsf{GHZ}_n\rangle = V|+^n\rangle. \tag{1}$$

*Proof.* We shall write $\mathsf{CX}_{c,t}$ for the CNOT gate with a control qubit $c$ and a target qubit $t$. Let $H_j$ be the Hadamard gate acting on the $j$-th qubit and $c = p+1$ be the central qubit. One can easily check that

$$|\mathsf{GHZ}_n\rangle = \left(\prod_{j=1}^{p} \mathsf{CX}_{c,c-j}\mathsf{CX}_{c,c+j}\right) H_c|0^n\rangle.$$

All $\mathsf{CX}$ gates in the product pairwise commute, so the order does not matter. Inserting a pair of Hadamards on every qubit $j \in [n] \setminus \{c\}$ before and after the respective $\mathsf{CX}$ gate and using the identity $(I\otimes H)\mathsf{CX}(I\otimes H) = \mathsf{CZ}$ one gets

$$|\mathsf{GHZ}_n\rangle = \left(\prod_{j\in[n]\setminus\{c\}} H_j\right)\left(\prod_{j=1}^{p} \mathsf{CZ}_{c,c-j}\mathsf{CZ}_{c,c+j}\right)|+^n\rangle. \tag{2}$$

Let $S = \exp\left[i(\pi/4)Z\right]$ be the phase-shift gate. Define the two-qubit Clifford gate

$$\mathsf{RZ} = (S\otimes S)^{-1}\mathsf{CZ} = \exp(-i\pi/4)\exp\left[-i(\pi/4)(Z\otimes Z)\right].$$

Expressing $\mathsf{CZ}$ in terms of $\mathsf{RZ}$ and $S$ in Eq. (2) one gets

$$|\mathsf{GHZ}_n\rangle = S_c^{2p}\left(\prod_{j\in[n]\setminus\{c\}} H_j S_j\right)\left(\prod_{j=1}^{p} \mathsf{RZ}_{c,c-j}\mathsf{RZ}_{c,c+j}\right)|+^n\rangle. \tag{3}$$

Multiply both sides of Eq. (3) on the left by a product of $S$ gates over qubits $j\in[n]\setminus\{c\}$. Noting that

$$SHS = i\exp\left[-i(\pi/4)X\right]$$

one gets (ignoring an overall phase factor)

$$\prod_{j\in[n]\setminus\{c\}} S_j|\mathsf{GHZ}_n\rangle = S_c^{2p}\left(\prod_{j\in[n]\setminus\{c\}} \exp\left[-i(\pi/4)X_j\right]\right)\left(\prod_{j=1}^{p} \mathsf{RZ}_{c,c-j}\mathsf{RZ}_{c,c+j}\right)|+^n\rangle. \tag{4}$$

Using the identity

$$\prod_{j\in[n]\setminus\{c\}} S_j|\mathsf{GHZ}_n\rangle = S_c^{2p}|\mathsf{GHZ}_n\rangle.$$

one can cancel $S_c^{2p}$ that appears in both sides of Eq. (4). We arrive at Eq. (1) with

$$V = \left(\prod_{j\in[n]\setminus\{c\}} \exp\left[-i(\pi/4)X_j\right]\right)\left(\prod_{j=1}^{p} \mathsf{RZ}_{c,c-j}\mathsf{RZ}_{c,c+j}\right)$$

The circuit diagram of $V$ in the case $n=7$ is shown in Figure 2. Obviously, $V$ is $\mathbb{Z}_2$-symmetric since any individual gate commutes with $X^{\otimes n}$. Let us check that $V$ has range-$p$. Consider any single-qubit observable $O_q$ acting on the $q$-th qubit. Consider three cases. *Case 1:* $q=c$. Then $V^\dagger O_q V$ may be supported on all $n$ qubits. However, $[c-p,c+p]=[1,n]$, so the $p$-range condition is satisfied trivially. *Case 2:* $1\le q<c$. Then all gates $\mathsf{RZ}_{c,c+j}$ in $V$ cancel the corresponding gates in $V^\dagger$, so that $V^\dagger O_q V$ has support in the interval $[1,c]\subseteq[q-p,q+p]$. Thus the $p$-range condition is satisfied. *Case 3:* $c<q\le n$. This case is equivalent to Case 2 by symmetry. □

Recall that the ring of disagrees Hamiltonian has the form

$$C = \frac{1}{2}\sum_{j\in\mathbb{Z}_n}(I - Z_j Z_{j+1}).$$

**Lemma B.2.** *Consider any integers $n,p$ such that $n$ is even and $n$ is a multiple of $2p+1$. Then there exists a $\mathbb{Z}_2$-symmetric range-$p$ circuit $U$ such that*

$$\frac{1}{n}\langle +^n|U^\dagger C U|+^n\rangle = \frac{2p+1/2}{2p+1}.$$

*Proof.* Let $W$ be the $\mathbb{Z}_2$-symmetric range-$p$ unitary operator preparing the GHZ state on $2p+1$ qubits starting from $|+^{2p+1}\rangle$, see Lemma B.1. Suppose $n=m(2p+1)$ for some even integer $m$. Define

$$U = \overline{X}W^{\otimes m},$$

where

$$\overline{X} = (X\otimes I)^{\otimes n/2}.$$

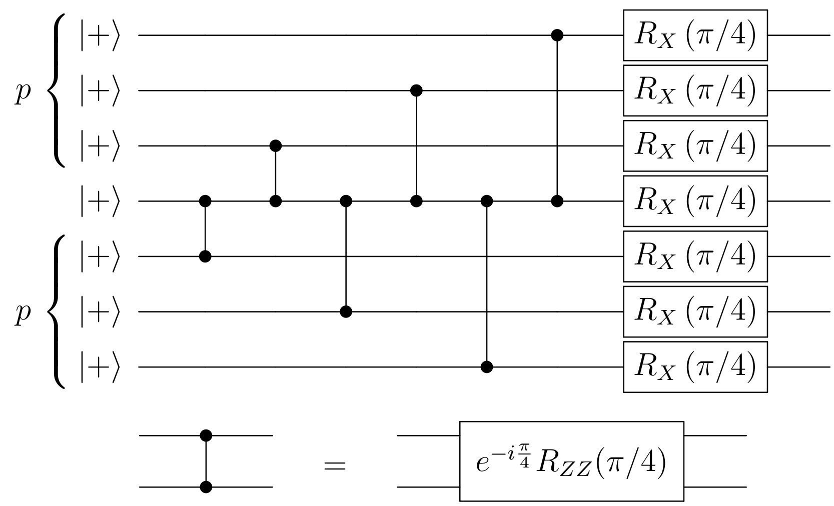


Figure 2: The $\mathbb{Z}_2$-symmetric range-3 quantum circuit to prepare the GHZ state $|\mathsf{GHZ}_{2p+1}\rangle$ of $2p+1 = 7$ qubits ($p = 3$). Here, $R_O(\theta) = \exp(-i\theta O)$.

Since each copy of $W$ acts on a consecutive interval of qubits and has range $p$, one infers that $U$ has range $p$. We have

$$\overline{X}^\dagger C\overline{X} = \sum_{k\in\mathbb{Z}_n} G_k, \quad \text{where} \quad G_k = \frac{1}{2}(I + Z_k Z_{k+1}).$$

The state $W^{\otimes m}|+^n\rangle$ is a tensor product of GHZ states supported on consecutive tuples of $2p + 1$ qubits. The expected value of $G_k$ on the state $W^{\otimes m}|+^n\rangle$ equals 1 if $G_k$ is supported on one of the GHZ states. Otherwise, if $G_k$ crosses the boundary between two GHZ states, the expected value of $G_k$ on the state $W^{\otimes m}|+^n\rangle$ equals 1/2. Thus

$$\langle +^n|U^\dagger CU|+^n\rangle = \sum_{k\in\mathbb{Z}_n} \langle +^n|(W^{\otimes m})^\dagger G_k W^{\otimes m}|+^n\rangle = m(2p + 1/2) = n\left(\frac{2p + 1/2}{2p + 1}\right).$$

□

# C Numerical simulation of level-1 QAOA and RQAOA

In this section we provide details of the simulation reported on Figure 1 in the main text. Let $J$ be a real symmetric matrix of size $n$. Consider an Ising-type Hamiltonian

$$C = \sum_{1\le j<k\le n} J_{j,k} Z_j Z_k \ .$$

Here $J_{j,k}$ are arbitrary real coefficients. Below we show how to compute the mean value of a Pauli operator $Z_j Z_k$ on the level-1 QAOA state

$$|\psi(\beta,\gamma)\rangle = e^{i\beta B}e^{i\gamma C}|+^n\rangle$$

in time $O(n)$ using an an explicit analytic formula. Such a formula was derived for the MaxCut cost function by Wang et al. [27, Theorem 1]. Here we provide a generalization to arbitrary Ising

Hamiltonians. Since the total number of terms in the cost function is $O(n^2)$, simulating each step of RQAOA takes time at most $O(n^3)$. Assuming that $n_c = O(1)$, the number of steps is roughly $n$ so that the full simulation cost is $O(n^4)$. Crucially, the simulation cost of this method does not depend on the depth of the variational circuit. This is important because RQAOA may potentially increase the depth from $O(1)$ to $O(n)$ since it adds many new terms to the cost function.

**Lemma C.1.** *Fix a pair of qubits* $1 \le j < k \le n$. *Let* $c = \cos(2\beta)$ *and* $s = \sin(2\beta)$. *Then*

$$\langle\psi(\beta,1)|Z_jZ_k|\psi(\beta,1)\rangle = (s^2/2)\prod_{p\neq j,k}\cos\left[2J_{j,p}-2J_{k,p}\right] - (s^2/2)\prod_{p\neq j,k}\cos\left[2J_{j,p}+2J_{k,p}\right]$$
$$+ cs\cdot\sin(2J_{j,k})\left[\prod_{p\neq j,k}\cos(2J_{j,p}) + \prod_{p\neq j,k}\cos(2J_{k,p})\right]. \qquad (5)$$

Here we only consider the case $\gamma = 1$ since $\gamma$ can be absorbed into the definition of $J$.

*Proof.* Given a 2-qubit observable $O$ define the mean value

$$\mu(O) = \langle\psi(\beta,1)|O_{j,k}|\psi(\beta,1)\rangle.$$

We are interested in the observable $O = ZZ \equiv Z\otimes Z$.

We note that all terms in $B$ and $C$ that act trivially on $\{j,k\}$ do not contribute to $\mu(O)$. Such terms can be set to zero. Given a 2-qubit observable $O$, define a mean value

$$\mu'(O) = \langle +^n|e^{iC'}O_{j,k}e^{-iC'}|+^n\rangle, \quad \text{where} \quad C' = \sum_{p\neq j,k}(J_{j,p}Z_j + J_{k,p}Z_k)Z_p. \qquad (6)$$

Using the identities

$$\begin{aligned}
e^{i\beta(X_j+X_k)}Z_jZ_ke^{-i\beta(X_j+X_k)} &= c^2Z_jZ_k + s^2Y_jY_k + cs(Z_jY_k+Y_jZ_k),\\
e^{iJ_{j,k}Z_jZ_k}Z_jZ_ke^{-iJ_{j,k}Z_jZ_k} &= Z_jZ_k,\\
e^{iJ_{j,k}Z_jZ_k}Y_jY_ke^{-iJ_{j,k}Z_jZ_k} &= Y_jY_k\\
e^{iJ_{j,k}Z_jZ_k}Z_jY_ke^{-iJ_{j,k}Z_jZ_k} &= \cos(2J_{j,k})Z_jY_k + \sin(2J_{j,k})X_k,\\
e^{iJ_{j,k}Z_jZ_k}Y_jZ_ke^{-iJ_{j,k}Z_jZ_k} &= \cos(2J_{j,k})Y_jZ_k + \sin(2J_{j,k})X_j,
\end{aligned}$$

and noting that $\mu'(ZZ) = 0$ one easily gets

$$\mu(ZZ) = s^2\cdot\mu'(YY) + cs\cdot\cos(2J_{j,k})\left[\mu'(ZY)+\mu'(YZ)\right] + cs\cdot\sin(2J_{j,k})\left[\mu'(XI)+\mu'(IX)\right]. \quad (8)$$

Using the explicit form of $C'$ one gets

$$e^{-iC'}|+^n\rangle = \frac{1}{2}\sum_{a,b=0,1}|a,b\rangle_{j,k}\otimes|\Phi(a,b)\rangle_{\mathsf{else}}, \qquad (9)$$

where $|\Phi(a,b)\rangle$ is a tensor product state of $n-2$ qubits defined by

$$|\Phi(a,b)\rangle = \bigotimes_{p\neq j,k}|J_{j,p}(-1)^a + J_{k,p}(-1)^b\rangle_p \quad \text{where} \quad |\theta\rangle \equiv e^{-i\theta Z}|+\rangle.$$

Combining Eqs. (6),(9) one gets

$$\mu'(O) = (1/4)\sum_{a,b,a',b'=0,1}\langle a',b'|O|a,b\rangle\cdot\langle\Phi(a',b')|\Phi(a,b)\rangle. \qquad (10)$$

Using the tensor product form of the states $|\Phi(a,b)\rangle$ and the identity $\langle\theta'|\theta\rangle = \cos(\theta-\theta')$ gives

$$\langle\Phi(a',b')|\Phi(a,b)\rangle = \prod_{p\neq j,k} \cos\left[J_{j,p}(-1)^a - J_{j,p}(-1)^{a'} + J_{k,p}(-1)^b - J_{k,p}(-1)^{b'}\right]. \tag{11}$$

From Eqs. (10),(11) one can easily compute the mean value $\mu'(O)$ for any 2-qubit observable.

Consider first the case $O = YY$. Then the only terms contributing to Eq. (10) are those with $a' = a\oplus 1$ and $b' = b\oplus 1$. The identity $\langle a\oplus 1|Y|a\rangle = -i(-1)^a$ gives

$$\mu'(YY) = -(1/4)\sum_{a,b=0,1}(-1)^{a+b}\prod_{p\neq j,k}\cos\left[2J_{j,p}(-1)^a + 2J_{k,p}(-1)^b\right],$$

that is,

$$\mu'(YY) = (1/2)\prod_{p\neq j,k}\cos\left[2J_{j,p} - 2J_{k,p}\right] - (1/2)\prod_{p\neq j,k}\cos\left[2J_{j,p} + 2J_{k,p}\right]. \tag{12}$$

Next consider the case $O = YZ$. Note that the matrix elements $\langle a',b'|O|a,b\rangle$ have zero real part. From Eqs. (10),(11) one infers that $\mu'(YZ)$ has zero real part. This implies

$$\mu'(YZ) = \mu'(ZY) = 0. \tag{13}$$

Finally, consider the case $O = XI$. Then the only terms that contribute to Eq. (10) are those with $a' = a\oplus 1$ and $b' = b$. We get

$$\mu'(XI) = \prod_{p\neq j,k}\cos\left(2J_{j,p}\right). \tag{14}$$

Here we noted that the inner product Eq. (11) with $a' = a\oplus 1$ and $b' = b$ does not depend on $a,b$. By the same argument,

$$\mu'(IX) = \prod_{p\neq j,k}\cos\left(2J_{k,p}\right). \tag{15}$$

Combining Eq. (8) and Eqs. (12),(13),(14),(15) one arrives at Eq. (5). □

Clearly, the ability to simulate level-1 RQAOA with Ising-type cost functions on a classical computer in polynomial time precludes exponential quantum speedups. However, as far as we know, higher-level RQAOA with $p\geq 2$ lacks efficient classical simulation leaving room for a quantum advantage.

# D RQAOA optimally solves the ring of disagrees

In this section we prove that the level-1 RQAOA optimally solves the ring of disagrees model. This is in sharp contrast to the standard QAOA which achieves approximation ratio at most $(2p{+}1)/(2p{+}2)$ for any level $p$, as was shown in Ref. [22]. More generally, we show that the level-1 RQAOA optimally solves any $1D$ Ising model where the coupling coefficients are either $+1$ or $-1$.

**Lemma D.1.** *Consider a cost function*

$$C(x) = \sum_{k\in\mathbb{Z}_n} J_k(-1)^{x_k+x_{k+1}}$$

*with $n$ variables $x\in\{0,1\}^n$ located at vertices of the cycle graph $\mathbb{Z}_n$. Assume that $J_k\in\{1,-1\}$ for all $k\in\mathbb{Z}_n$. Then the level-1 RQAOA outputs $x^*\in\{0,1\}^n$ such that $C(x^*) = \max_x C(x)$.*

*Proof.* Let

$$C = \sum_{k \in \mathbb{Z}_n} J_k Z_k Z_{k+1} \tag{16}$$

be the corresponding Hamiltonian. First, we observe that $\langle \psi(\beta,\gamma)|Z_i Z_j|\psi(\beta,\gamma)\rangle = 0$ if $\mathrm{dist}(i,j) > 2$ since in this case the operators $U^{-1}Z_iU$ and $U^{-1}Z_jU$ have disjoint support. Lemma C.1 shows that

$$\langle \psi(\beta,\gamma)|Z_i Z_j|\psi(\beta,\gamma)\rangle = \begin{cases} \frac{1}{2} J_i \sin(4\beta)\sin(4\gamma) & \text{if } j = i+1 \\ \frac{1}{4} J_i J_{i+1} \sin^2(2\beta)\sin^2(4\gamma) & \text{if } j = i+2 \\ 0 & \text{otherwise} \end{cases} \tag{17}$$

when $J_k \in \{1, -1\}$ for every $k \in \mathbb{Z}_n$. Here we assumed $i < j$. Thus

$$|\langle \psi(\beta,\gamma)|Z_i Z_{i+2}|\psi(\beta,\gamma)\rangle| \le 1/4 \tag{18}$$

for all $\beta, \gamma$. Let $\beta^*, \gamma^*$ be the optimal angles maximizing the variational energy $\langle \psi(\beta,\gamma)|C|\psi(\beta,\gamma)\rangle$. Then we can infer from Eq. (17) that

$$\langle \psi(\beta^*,\gamma^*)|Z_i Z_{i+1}|\psi(\beta^*,\gamma^*)\rangle = J_i/2\ . \tag{19}$$

Combined with Eq. (17) and Eq. (18) we conclude that the maximally correlated pair of variables are nearest neighbors, that is,

$$\arg \max_{(i,j): i<j} |\langle \psi(\beta^*,\gamma^*)|Z_i Z_j|\psi(\beta^*,\gamma^*)\rangle| = (i^*, i^*+1) \tag{20}$$

for some $i^* \in \mathbb{Z}_n$. Without loss of generality, assume that $i^* = n-2$. Then, according to Eq. (20), the RQAOA algorithm eliminates the variable $Z_{n-1}$. By Eq. (19), the corresponding parity constraint is

$$Z_{n-1} = Z_{n-2} J_{n-2}. \tag{21}$$

The resulting reduced graph obtained from $\mathbb{Z}_n$ by contracting the edge $(n-1, n-2)$ is isomorphic to $\mathbb{Z}_{n-1}$. It is easy to check that the new cost function Hamiltonian $C'$ acting on $n-1$ qubits is

$$C' = 1 + \sum_{k \in \mathbb{Z}_{n-1}} J'_k Z_k Z_{k+1} \tag{22}$$

with

$$J'_i = \begin{cases} J_i & \text{if} \quad i \neq n-2 \\ J_{n-2} J_{n-1} & \text{if} \quad i = n-2 \end{cases} \tag{23}$$

We note that the transformation Eq. (23) preserves the parity of the couplings in the sense that

$$\prod_{k \in \mathbb{Z}_n} J_k = \prod_{k \in \mathbb{Z}_{n-1}} J'_k\ . \tag{24}$$

Proceeding inductively, one eliminates variables $Z_{n-1}$, $Z_{n-2}$, $\ldots$, $Z_{n_c}$ while imposing parity constraints (cf. Eq. (21))

$$\begin{aligned} Z_{n-1} &= Z_{n-2} J_{n-2} \\ Z_{n-2} &= Z_{n-3} J'_{n-3} \\ &\vdots \end{aligned}$$

arriving at the cost function Hamiltonian $C''$ for an Ising chain of length $n_c$ having couplings $\pm 1$. Because of Eq. (24) and because the Hamiltonian Eq. (16) is frustrated if and only if $\prod_{k \in \mathbb{Z}_n} J_k = -1$, we conclude that any maximum $x^* \in \{0,1\}^{n_c}$ of $C''(x)$ satisfies

$$C''(x^*) = \begin{cases} n_c & \text{if } \prod_{k \in \mathbb{Z}_{n_c}} J_k = 1 \\ n_c - 2 & \text{if } \prod_{k \in \mathbb{Z}_{n_c}} J_k = -1. \end{cases}$$

Because the cost function acquires a constant energy shift in every variable elimination, see Eq. (22), the final output $x$ of the RQAOA algorithm satisfies

$$C(x) = n - n_c + C''(x^*) = \begin{cases} n & \text{if } \prod_{k \in \mathbb{Z}_n} J_k = 1 \\ n - 2 & \text{if } \prod_{k \in \mathbb{Z}_n} J_k = -1. \end{cases}$$

This implies the claim. □

# References


[1] C.-K. Chiu, J. C. Y. Teo, A. P. Schnyder, and S. Ryu, Rev. Mod. Phys. **88**, 035005 (2016).

[2] X. Chen, Z.-C. Gu, and X.-G. Wen, Phys. Rev. B **82**, 155138 (2010).

[3] A. Y. Kitaev, Annals of Physics **303**, 2 (2003).

[4] S. Bravyi, M. B. Hastings, and F. Verstraete, Phys. Rev. Lett. **97**, 050401 (2006).

[5] D. Aharonov and Y. Touati, preprint (2018), arXiv:1810.03912.

[6] M. H. Freedman and M. B. Hastings, Quantum Information and Computation **14** (2014).

[7] M. B. Hastings, Quantum Information and Computation **13**, 393 (2013).

[8] L. Eldar and A. W. Harrow, Local Hamiltonians whose ground states are hard to approximate, in *2017 IEEE 58th Annual Symposium on Foundations of Computer Science (FOCS)*, pp. 427–438, 2017.

[9] D. Aharonov, I. Arad, and T. Vidick, SIGACT News **44**, 47 (2013).

[10] C. Nirkhe, U. Vazirani, and H. Yuen, Approximate Low-Weight Check Codes and Circuit Lower Bounds for Noisy Ground States, in *Proceedings of ICALP 2018*, edited by I. C. et al., , Leibniz International Proceedings in Informatics (LIPIcs) Vol. 107, pp. 91:1–91:11, Dagstuhl, Germany, 2018, Dagstuhl–Leibniz-Zentrum fuer Informatik.

[11] Z.-C. Gu and X.-G. Wen, Phys. Rev. B **80**, 155131 (2009).

[12] F. D. M. Haldane, Phys. Rev. Lett. **50**, 1153 (1983).

[13] I. Affleck, T. Kennedy, E. H. Lieb, and H. Tasaki, Phys. Rev. Lett. **59**, 799 (1987).

[14] N. Schuch, D. Pérez-García, and I. Cirac, Phys. Rev. B **84**, 165139 (2011).

[15] F. Pollmann, A. M. Turner, E. Berg, and M. Oshikawa, Phys. Rev. B **81**, 064439 (2010).

[16] T.-C. Wei, I. Affleck, and R. Raussendorf, Phys. Rev. A **86**, 032328 (2012).

[17] J. Miller and A. Miyake, Phys. Rev. Lett. **114**, 120506 (2015).

[18] M. Morgenstern, Journal of Combinatorial Theory, Series B **62**, 44 (1994).

[19] E. Farhi, J. Goldstone, and S. Gutmann, preprint (2014), arXiv:1411.4028.

[20] M. B. Hastings, preprint (2019), arXiv:1905.07047.

[21] M. X. Goemans and D. P. Williamson, J. ACM **42**, 1115 (1995).

[22] G. Mbeng, R. Fazio, and G. Santoro, preprint (2019), arXiv:1906.08948.

[23] P. Raghavan and C. D. Tompson, Combinatorica 7, 365, (1987).

[24] V. G. Vizing, Diskret. Analiz **3**, 25 (1964).

[25] A. W. Marcus, D. A. Spielman, and N. Srivastava, Interlacing families iv: Bipartite Ramanujan graphs of all sizes, in *2015 IEEE 56th Annual Symposium on Foundations of Computer Science*, pp. 1358–1377, 2015.

[26] A. W. Marcus, D. A. Spielman, and N. Srivastava, Annals of Mathematics **182**, 307 (2015).

[27] Z. Wang, S. Hadfield, Z. Jiang, and E. G. Rieffel, Phys. Rev. A **97**, 022304 (2018).

[28] M. Van Den Nest, Quantum Info. Comput. **11**, 784 (2011).

[29] S. Bravyi *et al.*, Quantum **3**, 181 (2019).

[30] E. Farhi and A. W. Harrow, preprint (2016), arXiv:1602.07674.