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2012-12-10

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Abstract— This paper focuses on the problem of optimizing the trajectories of multiple searchers attempting to detect a non-evading moving target whose motion is conditionally deterministic. This problem is a parameter-distributed optimal control problem, as it involves an integration over a space of stochastic parameters as well as an integration over the time domain. In this paper, we consider a wide range of discretization schemes to approximate the integral in the parameter space by a finite summation, which results in a standard control-constrained optimal control problem that can be solved using existing techniques in optimal control theory. We prove that when the sequence of solutions to the discretized problem has an accumulation point, it is guaranteed to be an optimal solution of the original search problem. We also provide a necessary condition that accumulation points of this sequence must satisfy.

I. INTRODUCTION

In this paper we consider a search optimization problem where multiple searchers seek multiple non-evading, moving targets. The problem takes the form of a parameter-distributed optimal control problem, where the searcher dynamics are given by ordinary differential equations (ODEs). The targets follow conditionally deterministic trajectories in the sense that the targets trajectories depend on unknown parameters treated as random variables. The optimal control problem is to determine searcher trajectories that maximize the probability of detecting the targets.

While earlier studies have considered similar problems with assumptions such as simple searcher dynamics and a single target [1]–[3], special target movement in a channel [4], and exponential detection model [5], we consider a broader class of problems with general nonlinear dynamics and detection models. In the literature, previous theoretical works on such search problems have focused on the development of necessary optimality conditions in the tradition of Pontryagin, e.g. [2], [6]–[8], and sufficient conditions for optimality in the tradition of Hamilton-Jacobi-Bellman equation (see, for example, [9], [10]). Numerical algorithms corresponding to those theoretical results for solving optimal search problems were developed in [3], [10]–[14].

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In this paper, we conduct theoretical analysis for the solution of parameter-distributed optimal control problems by employing a direct method. The parameter space is directly discretized, which results in a family of standard optimal control problems that can be solved using existing approaches. The considered direct method is similar to those of [4], [5]. We go beyond [4], which focuses on model formulation and computations exclusively, and show that the discretization scheme is consistent in the sense that globally optimal solutions of the standard optimal control problems converge to a globally optimal solution of the parameter-distributed optimal control problem. In addition, we provide a necessary condition that an accumulation point of a sequence of optimal solutions of the standard optimal control problems must satisfy, when such an accumulation point exists. While [5] also provides consistency results and optimality conditions, they are limited to the case with a two-dimensional parameter space and integration using Simpson's rule. We allow for essentially any numerical integration scheme in an arbitrary finite dimension under mild smoothness assumptions. We also consider a Pontryagin-type necessary condition in contrast to that in [5], which follows an approach by Polak [15], Chapter 4.

The paper is organized as follows: Section II introduces the optimal control model associated with the search problem and its spatial discretization, Section III shows the consistency of the family of approximate standard optimal control problems, Section IV shows the consistency of the dual variables and provides a necessary condition for the accumulation point, and a numerical example is demonstrated in Section V.

II. PROBLEM FORMULATION

Motivated by the problem of optimal search for a target with conditionally deterministic motion, we now introduce the parameter-distributed optimal control problem, which we refer to as Problem B. The focus of this paper will be the consistent approximation of this problem, as well as the formulation of a necessary condition for a solution obtained with this approximation.

Problem B**.** Determine the function pair $(x(t), u(t))$ with $x \in W_{1,\infty}([0,1]; \mathbb{R}^{n_x})$, $u \in L_\infty([0,1]; \mathbb{R}^{n_u})$ that minimizes the cost functional

$$J = \int_A F\left(\int_0^1 r(x(t), y(t, \alpha), u(t)) dt\right) \phi(\alpha) d\alpha \tag{1}$$

subject to initial condition $x(0) = x_0$, the dynamics

$$\dot{x} = f(x(t), u(t))$$

and the control constraint $g(u(t)) \leq 0$ for all $t \in [0, 1]$.

Here $W_{1,\infty}([0,1];\mathbb{R}^{n_x})$ is the space of all essentially bounded functions with essentially bounded distributional derivatives which map the interval $[0,1]$ into the space $\mathbb{R}^{n_x}$ and $L_\infty([0,1];\mathbb{R}^{n_u})$ is the set of all essentially bounded functions mapping the interval $[0,1]$ into the space $\mathbb{R}^{n_u}$. The function $y(t,\alpha):[0,1]\times A \mapsto \mathbb{R}^{n_y}$ represents the trajectory of the target, given that the unknown parameter takes the value α. $r:\mathbb{R}^{n_x}\times\mathbb{R}^{n_y}\times\mathbb{R}^{n_u}\mapsto\mathbb{R}^K$ and $F:\mathbb{R}^K\mapsto\mathbb{R}$ are functions which determine the probability of detecting the target, and are determined by the sensor model. Note that because we allow n_x, n_u, and n_y to be any integer, this formulation can include the case of multiple searchers and multiple targets. The following regularity conditions are assumed:

Assumption 1. *The function g is continuous and the set $U = \{\nu \in \mathbb{R}^{n_u} | g(\nu) \le 0\}$ is compact.*

This is a reasonable assumption to make, as in a real world scenario the set of allowable controls will be bounded and therefore U, being a closed and bounded set, will be compact.

Assumption 2. *There exists a compact set $X \subset \mathbb{R}^{n_x}$ such that for each $u \in L_\infty([0,1],\mathbb{R}^{n_u})$ with $u(t) \in U$ for all $t \in [0,1]$, we have $x(t) \in X$ for all $t \in [0,1]$, where $x(t)$ is the solution to the dynamical system $\dot{x} = f(x(t), u(t))$, with initial condition $x(0) = x_0$. Furthermore, there exists a compact set Y such that $y(t,\alpha) \in Y$ for all $\alpha \in A, t \in [0,1]$.*

We recognize that this assumption will not be true for all nonlinear dynamical systems with bounded controls. However, the assumption will be satisfied for dynamical systems in which f is globally Lipschitz with respect to x. In particular the example examined in this work, a Dubin's vehicle, has globally Lipschitz dynamics. It should be noted that this assumption will also be satisfied for systems which are input-to-state stable and systems with linearly bounded dynamics, which includes a wide class of dynamical systems often used in control problems.

Assumption 3. *The functions f, r and F are C^1 with respect to their argument. Moreover, the derivative of r is Lipschitz on the set $X \times Y \times U$. Also, $y(t,\cdot):A \to Y$ is continuous for all $t \in [0,1]$.*

Complications arise when attempting to apply standard non-linear optimal control methods to the parameter-distributed optimal control problem because of the integral in the objective functional. Indirect methods require a necessary condition for the problem be known, but the standard Pontryagin Minimum Principle does not directly apply to problems of this type. A necessary condition for optimality is presented in [2], but only for the simplified problem in which the dynamics of the searcher are given by a single-integrator with box constraints. In this paper we present a direct method as a means to solve Problem B, by using a numerical scheme to approximate the integral in the objective functional, and show that this approximation is consistent.

III. CONSISTENT APPROXIMATION OF THE OPTIMAL CONTROL PROBLEM

In this section we introduce a numerical integration scheme to approximate the integral in the objective functional for Problem B, creating a sequence of standard optimal control problems, Problem B^M, which can be solved numerically using existing techniques in the field of computational optimal control theory. We show that the family of standard optimal control problems, Problem B^M, have the property that an accumulation point of the sequence of optimal solutions to Problem B^M will be an optimal solution to Problem B. We show that this property holds for a wide variety of numerical schemes, and furthermore show that this sequence will indeed have an accumulation under certain assumptions on the class of optimal controls to Problem B^M.

A. Numerical Schemes and Problem B^M

In this section we introduce the numerical scheme which will be used in the approximation of Problem B, and the approximated Problem B^M. Because the integral over the parameter space in (1) has an integrad which is continuous as a function of α, we require only that the numerical scheme converge over the class of continuous functions, as stated in the following assumption:

Assumption 4. *For each $M \in \mathbb{N}$, there is a set of nodes $\{\alpha_i^M\}_{i=1}^M \subset A$ and an associated set of weights $\{w_i^M\}_{i=1}^M \subset \mathbb{R}$, such that for any continuous function $h:A \to \mathbb{R}$,*

$$\int_A h(\alpha)d\alpha = \lim_{M\to\infty}\sum_{i=1}^M h(\alpha_i^M)\omega_i^M \qquad (2)$$

Remark 1. *Note that if $h_M:A\to\mathbb{R}$ is continuous for all $M \in \mathbb{N}$ and $\{h_M\}$ converges uniformly to h, then*

$$\lim_{M\to\infty}\sum_{i=1}^M h_M(\alpha_i^M)\omega_i^M = \int_A h(\alpha)d\alpha \qquad (3)$$

Because the function h to be integrated is continuous, numerical quadrature and Simpson's rule will satisfy Assumption 4 in the case where the set A is compact; and are applicable in this scenario to determine the nodes $\{\alpha_i^M\}$ and weights $\{\omega_i^M\}$. Throughout the paper, M will be used to denote the number of nodes used in this approximation. Research into application of other algorithms for a wider class of functions h and spaces A, such as Monte Carlo simulation methods, is ongoing. Once the numerical scheme is chosen, we can then define a new approximated objective function J^M for each $M \in \mathbb{N}$ by

$$J^M = \sum_{i=1}^M F\left(\int_0^1 r(x(t),y(t,\alpha_i^M),u(t))dt\right)\phi(\alpha_i^M)\omega_i^M$$

We introduce new states governed by the equations $\dot{z}_{M,i}(t) = r(x(t),y(t,\alpha_i^M),u(t))$ with $z_{M,i}(0)=0$, so that

$$z_{M,i}(1) = \int_0^1 r(x(s),y(s,\alpha_i^M),u(s))ds \qquad (4)$$

The auxiliary states, $\{z_{M,i}(t)\}_{i=1}^M$, can be used to eliminate the integration over the time domain in J^M. The state $z_{M,i}$ contains all information about the probability of detecting the target conditioned on $\alpha = \alpha_i^M$ and in fact

when detection is given by a Poisson process, $z_{M,i}(t)$ is the expected number of detections up to time t. With the integral over the parameter space approximated by a sum, and the integral over time replaced by the augmented state z, we are now able to define the approximate optimal control problem:

Problem B^M. Determine the state-control function triplet $(x(t), z(t), u(t))$, where $x \in W_{1,\infty}([0,1];\mathbb{R}^{n_x})$, $z \in W_{1,\infty}([0,1];\mathbb{R}^{KM})$, and $u \in L_\infty([0,1];\mathbb{R}^{n_u})$, that minimizes the cost functional

$$J^M = \sum_{i=1}^M F(z_{M,i}(1))\phi(\alpha_i^M)\omega_i^M \qquad (5)$$

subject to the dynamics

$$\dot{x}(t) = f(x(t), u(t)) \qquad (6)$$
$$\dot{z}_{M,i}(t) = r(x(t), y(t, \alpha_i^M), u(t)) \qquad i = 1, \ldots, M, \quad (7)$$

the initial conditions

$$x(0) = x_0, \qquad z_{M,i}(0) = 0 \quad i = 1, \ldots, M, \qquad (8)$$

and the control constraint $g(u(t)) \leq 0$ for all $t \in [0,1]$.

Remark 2. *The spatial discretization in Problem B^M occurs in the parameter space A, not in the spaces $\mathbb{R}^{n_x}$ or $\mathbb{R}^{n_u}$ or the time interval $[0,1]$. Therefore, for any feasible triplet (x, z, u) for Problem B^M, the pair (x, u) is a feasible solution for Problem B. Similarly, for any feasible pair (x, u) for Problem B there exists a feasible triplet (x, z, u) for Problem B^M, with z defined by (7) and (8). Therefore when we refer to a pair (x, u) as being feasible, it will satisfy the feasibility condition for both Problem B and Problem B^M.*

Problem B^M is a standard control-constrained optimal control problem; and can be solved by a variety of computational optimal control methods, such as Runge-Kutta [16] or pseudospectral [17], [18] methods.

B. Consistency of Problem B^M

The following Theorem 1 shows that if a sequence of optimal solutions to Problem B^M converges as $M \to \infty$, then the limit will be an optimal solution to Problem B. Such a consistency property guarantees that Problem B^M indeed is a valid approximation of the original non-standard optimal control Problem B.

Remark 3. *Before stating the Theorem, we first make a note on the notation to be used. We define the set $\mathcal{N}_\infty^\# = \{V \subset \mathbb{N} | V \text{ infinite}\}$. That is, $\mathcal{N}_\infty^\#$ is the set of all subsequences of $\mathbb{N}$, which are designated by the index set $V \subset \mathbb{N}$. When $M \to \infty$ as usual in $\mathbb{N}$, we write $\lim_{M \to \infty}$. However, in the case of convergence with respect to a subsequence designated by an index set V, we write $\lim_{M \in V}$. For sequences of feasible pairs $\{x_M, u_M\}$, the notation $\lim_{M \to \infty}\{x_M, u_M\} = \{x, u\}$ will mean that x_M converges pointwise to x and u_M converges pointwise to u. Similarly $\lim_{M \in V}\{x_M, u_M\} = \{x, u\}$ will refer to pointwise convergence of the state-control pair along the subsequence indexed by V.*

We also need the following lemma:

Lemma 1. *Let $\mathcal{A}$ be the set of feasible pairs to Problem B, that is the set of all pairs $(x, u) \in W_{1,\infty}([0,1];\mathbb{R}^{n_x} \times L_\infty([0,1];\mathbb{R}^{n_u})$ such that $u(t) \in U$ and $x(t) = x_0 + \int_0^t f(x(s), u(s))ds$ for all $t \in [0,1]$. Then the set $\mathcal{A}$ is closed in the topology of pointwise convergence.*

Proof: Suppose $\{x_M, u_M\} \in \mathcal{A}$ and $\lim_{M \to \infty}\{x_M, u_M\} = \{x, u\}$. By the continuity of g, $g(u(t)) \leq 0$ for all $t \in [0,1]$. Now consider

$$\left\| x(t) - x_0 - \int_0^t f(x(s), u(s))ds \right\|$$

$$= \left\| x(t) - x_M(t) + \int_0^t f(x_M(s), u_M(s))ds - \int_0^t f(x(s), u(s))ds \right\|$$

$$\leq \|x(t) - x_M(t)\| + \left\| \int_0^t [f(x_M(s), u_M(s)) - f(x(s), u(s))]ds \right\|$$

$$\leq \|x(t) - x_M(t)\| + \int_0^t L[\|x(s) - x_M(s)\| + \|u(s) - u_M(s)\|]ds$$

Where we have used fact that f is C^1 therefore Lipschitz on the compact set $X \times U$. Because $x(s), x_M(s) \in X$ and $u(s), u_M(s) \in U$ where X and U are compact, $\|x(s) - x_M(s)\|$ and $\|u(s) - u_M(s)\|$ are bounded for all $s \in [0,1]$ $M \in \mathbb{N}$. Therefore by the dominated convergence theorem we have, for all $t \in [0,1], \epsilon > 0$,

$$\left\| x(t) - x_0 - \int_0^t f(x(s), u(s))ds \right\| < \epsilon$$

Therefore $\{x, u\} \in \mathcal{A}$. $\qquad\square$

Lemma 1 shows that if $\{x_M, u_M\}$ is a sequence of feasible pairs, then any accumulation point of this sequence will also be a feasible pair. We use this fact to prove the that an accumulation point of a sequence of optimal pairs to Problem B^M will be an optimal pair to Problem B, which is the result in Theorem 1.

Theorem 1. *Suppose Assumptions 1-4 hold, and in addition there exists $V \in \mathcal{N}_\infty^\#$ and a set of optimal pairs $\{x_M^*, u_M^*\}_{M \in V}$ for Problem B^M such that $\lim_{M \in V}\{x_M^*, u_M^*\} = \{x^\infty, u^\infty\}$. Then $\{x^\infty, u^\infty\}$ is an optimal solution to Problem B.*

Proof: Let $V \in \mathcal{N}_\infty^\#$ and $\{x_M^*, u_M^*\}_{M \in V}$ be a set of optimal pairs to Problem B^M such that $\lim_{M \in V}\{x_M^*, u_M^*\} = \{x^\infty, u^\infty\}$. By Lemma 1, $\{x^\infty, u^\infty\}$ is a feasible solution to Problem B. Next, we prove the optimality of $\{x^\infty, u^\infty\}$.

From Assumption 3, r is bounded and Lipschitz on $X \times Y \times U$ and F is uniformly continuous on $r(X, Y, U)$. From the Lipschitz continuity of r, we have, for all $\alpha \in A$

$$\int_0^1 \|r(x_M^*(t), y(t, \alpha), u_M^*(t)) - r(x^\infty(t), y(t, \alpha), u^\infty(t)\|dt$$

$$\leq \int_0^1 L(\|x_M^*(t) - x^\infty(t)\| + \|u_M^*(t) - u^\infty(t)\|)dt$$

By the dominated convergence theorem,

$$\lim_{M \in V} \int_0^1 L(\|x_M^*(t) - x^\infty(t)\| + \|u_M^*(t) - u^\infty(t)\|)dt = 0$$

and this convergence must be uniform in α. Then by the uniform continuity of F there must exist, for each $\epsilon > 0$,

$N \in \mathbb{N}$ such that for each $M \in V$ with $M > N$ and $\alpha \in A$

$$\left\| F\left(\int_0^1 r(x_M^*(t), y(t,\alpha), u_M^*(t))dt\right) - F\left(\int_0^1 r(x^\infty(t), y(t,\alpha), u^\infty(t))dt\right)\right\| < \epsilon \quad (9)$$

This implies, by the statement in Remark 1,

$$\lim_{M \in V} J^M(x_M^*, u_M^*) = J(x^\infty, u^\infty).$$

Suppose $\{x, u\}$ is a feasible pair for Problem B. Then, based on the optimality of $\{x_M^*, u_M^*\}$ and Remark 2, $J^M(x_M^*, u_M^*) \leq J^M(x, u)$ for all $M \in V$. Thus

$$\begin{aligned} J(x^\infty, u^\infty) &= \lim_{M \in V} J^M(x_M^*, u_M^*) \\ &\leq \lim_{M \in V} J^M(x, u) = J(x, u). \end{aligned}$$

Therefore $\{x^\infty, u^\infty\}$ is an optimal pair for Problem B, since it produces the minimum cost among all feasible solutions. $\qquad\square$

Theorem 1 shows that if a subsequence of optimal solutions to Problem B^M converges, this limit point will be an optimal solution to Problem B. This shows that the Problem B^M is indeed a good approximation to Problem B in the case where the solutions to Problem B^M converge in the pointwise sense. However, without additional restrictions on the class of controls to be allowed, it does not guarantee the existence of a convergent subsequence. Using the following generalization of Helly's Selection Theorem, we show the existence of a convergent subsequence for a certain class of controls.

Theorem 2. *[19] Let (X, d) be a complete metric space and $\{h_n\}_{n \in \mathbb{N}}$ a sequence of functions from $[a, b]$ into X such that*
  1) *For each $t \in [a, b]$, the set $\{h_n(t)\}_{n \in \mathbb{N}}$ has compact closure.*
  2) *the functions $\{h_n\}_{n \in \mathbb{N}}$ have uniformly bounded varrations.*

Then there exists a subsequence of the sequence $\{h_n\}_{n \in \mathbb{N}}$ converging pointwise in X to a function $h : [a, b] \mapsto X$ of bounded variation.

This theorem allows us to prove the following corollary, which guarantees the existence of an optimal solution to Problem B when the optimal controls to problem B^M are known to be of a certain class.

Corollary 1. *Suppose Assumptions 1-4 hold, and in addition there exists $V \in \mathcal{N}_\infty^\#$ and a set of optimal solutions $\{x_M^*, u_M^*\}_{M \in V}$ to Problem B^M, such that $\{u_M^*\}_{M \in V}$ have uniformly bounded variation. Then there exists $V' \subseteq V$ such that $\lim_{M \in V'}\{x_M^*, u_M^*\} = \{x^\infty, u^\infty\}$ for some $\{x^\infty, u^\infty\} \in \mathcal{A}$.*

Proof: Because $\dot{x} = f(x, u)$ and f is bounded on $X \times U$, $\{x_M^*\}$ is of uniformly bounded variation on X. $\{u_M^*\}$ is of uniformly bounded variation on U by the hypothesis. Therefore $\{x_M^*, u_M^*\}$ is of uniformly bounded variation on $X \times U$. Furthermore, $\{(x_M^*(t), u_M^*(t))\}_{M \in V}$ is relatively compact, as it is a subset of a compact space.

Therefore by Theorem 2, there exists a $V' \subset V$ such that $\lim_{M \in V'}\{x_M^*, u_M^*\} = \{x^\infty, u^\infty\}$. $\qquad\square$

It is known that for non-linear optimal control problems, the optimal control often belongs to the class of bang-bang controllers. As long as the constraint function g is well-behaved and the number of jump discontinuities in the control is bounded, these controls will satisfy the conditions in Corllary 1, therefore the Corollary guarantees the existence of an accumulation point of optimal pairs to Problem B^M. From Theorem 1, it is known that this accumulation point will be a optimal pair to Problem B, which shows the existence of an optimal solution to this problem in the case where the controls are known to be of the bang-bang type.

IV. ADJOINT EQUATION AND HAMILTONIAN FOR PROBLEM B

In this section we introduce the adjoint equations and Hamiltonian for Problem B^M. By examining the limiting behavior as $M \to \infty$, we are able to introduce adjoint equations and a Hamiltonian for Problem B. This allows us to establish a necessary condition which must be satisfied by accumulation points of sequences of optimal pairs to Problem B^M.

A. Necessary Condition for Problem B^M

First, we note that Problem B^M admits a minimum principle in the fashion of Pontryagin [20]. To formulate the necessary conditions, we introduce the adjoint variables (costates) λ corresponding to (6), and the adjoint variables η_M corresponding to (7). It should be noted that the dimension of the vectors $z_M(t)$ and $\eta_M(t)$ depends on the number of nodes M used in the discretization process. The Hamiltonian for the Problem B^M is

$$\mathbf{H}^M(x, \lambda, z_M, \eta_M, u) = \dot{x}^T \lambda + [\dot{z}_M]^T \eta_M, \quad (10)$$

With this Hamiltonian in place, we are ready to state the necessary condition for Problem B^M:

Necessary Condition of Problem $\mathbf{B^M}$: Let $\{x_M^*, u_M^*\}$ be an optimal pair for the Problem B^M, then there must exist absolutely continuous costate trajectories λ_M^* and η_M^* such that the following conditions hold for almost every $t \in [0, 1]$:

$$u^*(t) = \arg\min_{u \in U} \mathbf{H}^M\left(x_M^*(t), \lambda_M^*(t), z_M^*(t), \eta_M^*(t), u(t)\right)$$

$$\dot{x}_M^* = \frac{\partial \mathbf{H}^M}{\partial \lambda_M^*} \qquad \dot{\lambda}_M^* = -\frac{\partial \mathbf{H}^M}{\partial x_M^*} \quad (11)$$

$$\dot{z}_M^* = \frac{\partial \mathbf{H}^M}{\partial \eta_M^*} \qquad \dot{\eta}_M^* = -\frac{\partial \mathbf{H}^M}{\partial z_M^*} \quad (12)$$

Moreover, λ_M^* and η_M^* satisfy the transversality conditions

$$\frac{\partial J^M}{\partial x}|_{x_M^*(1)} - \lambda_M^*(1) = 0 \quad (13)$$

$$\frac{\partial J^M}{\partial z_M}|_{z_M^*(1)} - \eta_M^*(1) = 0 \quad (14)$$

We wish to use this necessary condition for Problem B^M to derive an equivalent necessary condition for problem B. However, this is problematic for two reasons. First, the necessary condition for Problem B^M depends explicitly on the variables z_M, but there are no equivalent variables present in the formulation of Problem B. Second, the dimension of z_M and η_M are dependent on the number of nodes M used in the discretization, and the dimension of these variables goes to infinity as we increase M. Therefore, even though the limits of x_M^*, u_M^*, and λ_M^* as $M \to \infty$ may exist, we cannot discuss the limit of $\mathbf{H}^M$ in any meaningful sense. In order to circumvent these difficulties, we define a reduced Hamiltonian $\mathbf{H}_{red}^M$ for Problem B^M which depends only on the variables $x(t), u(t)$, and $\lambda(t)$. We can then discuss the convergence of $\mathbf{H}_{red}^M$, as the dimensions of $x(t), u(t)$, and $\lambda(t)$ are fixed.

B. The Reduced Hamiltonian for Problem B^M

Here we show that the necessary condition from Section IV-A admits a form which does not explicity depend on the variables z_M and η_M. The Hamiltonian (10) can be written as

$$\mathbf{H}^M(x, \lambda, z_M, \eta_M, u) = [f(x, u)]^T \lambda + \sum_{i=1}^M [r(x, y(\alpha_i^M), u)]^T \eta_{M,i} \tag{15}$$

For an optimal solution, this leads to

$$\dot{\eta}_M^* = -\frac{\partial \mathbf{H}^M}{\partial z_M^*} = 0.$$

Therefore η_M^* is constant and given by $\eta_M^*(t) = \eta_M^*(1)$. From the transversality condition (14) and the objective function (5) we have

$$\eta_{M,i}^*(t) = \eta_{M,i}^*(1) = \frac{\partial J^M}{\partial z_{M,i}^*}\Big|_{z_{M,i}^*(1)}$$
$$= F_z(z_{M,i}^*(1))\omega_i^M \phi(\alpha_i^M) \quad \text{for } i = 1, \ldots M. \tag{16}$$

Here F_z is the gradient of F with respect to the augmented state z. Now the Hamiltonian, evaluated at $(x_M^*, \lambda_M^*, z_M^*, \eta_M^*, u)$ can be rewritten from (4), (15), and (16) to give

$$\mathbf{H}^M(x_M^*(t), \lambda_M^*(t), z_M^*(t), \eta_M^*(t), u(t))$$
$$= [f(x_M^*(t), u(t))]^T \lambda(t) + \sum_{i=1}^M [r(x_M^*(t), y(t, \alpha_i^M), u(t))]^T \eta_{M,i}^*(t)$$
$$= [f(x_M^*(t), u(t))]^T \lambda_M^*(t)$$
$$+ \sum_{i=1}^M [r(x_M^*(t), y(t, \alpha_i^M), u(t))]^T F_z(z_{M,i}^*(1))\omega_i^M \phi(\alpha_i^M). \tag{17}$$

From this form of the Hamiltonian and the dynamics (11) we get the following adjoint equation for λ_M^*:

$$\dot{\lambda}_M^*(t) = -[f_x(x_M^*(t), u_M^*(t))]^T \lambda_M^*(t) -$$
$$\sum_{i=1}^M [r_x(x_M^*(t), y(t, \alpha_i^M), u_M^*(t))]^T F_z(z_{M,i}^*(1))\omega_i^M \phi(\alpha_i^M), \tag{18}$$

with the end condition given by the transversality condition (13):

$$\lambda_M^*(1) = \frac{\partial J^M}{\partial x}\Big|_{x^*(1)} = 0.$$

Remark 4. *The value $z_{M,i}(1)$ is given by the integral $\int_0^1 r(x(s), y(s, \alpha_i^M), u(s))ds$, hence we can eliminate the explicit dependence of both the Hamiltonian (17) and the adjoint equation (20) on the variables $\{z_{M,i}\}_{i=1}^M$. Therefore the necessary condition for Problem B^M can be stated only in terms of the original state variable x, adjoint variable λ, and the control u.*

By making the substition suggesting in Remark 4 in equations (17) and (18), we can define the reduced necessary condition for Problem B^M. First we define the reduced Hamiltonian of the problem and the corresponding adjoint variables.

$$\mathbf{H}_{red}^M(x(t), \lambda(t), u(t))$$
$$= [f(x(t), u(t))]^T \lambda(t) + \sum_{i=1}^M [r(x(t), y(t, \alpha_i^M), u(t))]^T$$
$$\times F_z\left(\int_0^1 r(x(s), y(s, \alpha_i^M), u(s))ds\right) \omega_i^M \phi(\alpha_i^M) \tag{19}$$
$$\dot{\lambda}_M^* = -[f_x(x_M^*(t), u_M^*(t))]^T \lambda_M^*(t) -$$
$$\sum_{i=1}^M [r_x(x_M^*(t), y(t, \alpha_i^M), u_M^*(t))]^T$$
$$\times F_z\left(\int_0^1 r(x_M^*(s), y(s, \alpha_i^M), u_M^*(s))ds\right) \omega_i^M \phi(\alpha_i^M) \tag{20}$$
$$\lambda_M^*(1) = 0 \tag{21}$$

Necessary Condition of Problem $\mathbf{B^M}$: Let $\{x_M^*, u_M^*\}$ be an optimal pair for the Problem B^M, then it must satisfy

$$u_M^*(t) = \arg\min_{u \in U} \mathbf{H}_{red}^M(x_M^*(t), \lambda_M^*(t), u(t))$$

for almost every $t \in [0, 1]$, where $\mathbf{H}_{red}^M$ is given by (19), λ_M^* is given by (20) and (21).

Finally, to use this new necessary condition for Problem B^M to define a necessary condition for Problem B, we must show that the adjoint states and resulting reduced Hamiltonian converge.

C. Convergence of Adjoint States and Reduced Hamiltonians

We now demonstrate the convergence of the adjoint states λ_M^* and reduced Hamiltonians $\mathbf{H}_{red}^M$, which allows us to determine a Hamiltonian $\mathbf{H}$ for Problem B and leads to a corresponding minimum principle. It should be noted that the number of nodes M, and the corresponding discretization scheme given by $\{\alpha_i^M\}_{i=1}^M, \{\omega_i^M\}_{i=1}^M$ enter into the reduced Hamiltonian and adjoint equations only through the sums in (19) and (20). Therefore to show the convergence of these functions, we show that these sums converge.

Lemma 2. *Suppose Assumptions 1-4 hold, and in addition there exists $V \in \mathcal{N}_\infty^\#$ with $\{x_M^*, u_M^*\}_{M \in V}$ a sequence of optimal pairs for the Problem B^M such that $\lim_{M \in V}\{x_M^*, u_M^*\} = \{x^\infty, u^\infty\}$. Then the following limits*

hold:

1.
$$\lim_{M \in V} \sum_{i=1}^{M} \left[r(x_M^*(t), y(t, \alpha_i^M), u_M^*(t)) \right]^T$$
$$\times F_z \left(\int_0^1 r(x_M^*(s), y(s, \alpha_i^M), u_M^*(s))ds \right) \omega_i^M \phi(\alpha_i^M)$$
$$= \int_A \left[r(x^\infty(t), y(t, \alpha), u^\infty(t)) \right]^T$$
$$\times F_z \left(\int_0^1 r(x^\infty(s), y(s, \alpha), u^\infty(s))ds \right) \phi(\alpha)d\alpha \quad (22)$$

2.
$$\lim_{M \in V} \sum_{i=1}^{M} \left[r_x(x_M^*(t), y(t, \alpha_i^M), u_M^*(t)) \right]^T$$
$$\times F_z \left(\int_0^1 r(x_M^*(s), y(s, \alpha_i^M), u_M^*(s))ds \right) \omega_i^M \phi(\alpha_i^M)$$
$$= \int_A \left[r_x(x^\infty(t), y(t, \alpha), u^\infty(t)) \right]^T$$
$$F_z \left(\int_0^1 r(x^\infty(s), y(s, \alpha), u^\infty(s))ds \right) \phi(\alpha)d\alpha \quad (23)$$

For a proof of Lemma 2, see [21].

Based on Lemma 2, the consistent approximation of the adjoint equations can be established. For this purpose, let λ^∞ be the solution of the initial value problem

$$\dot{\lambda}^\infty(t) = -\left[f_x(x^\infty(t), u^\infty(t)) \right]^T \lambda^\infty$$
$$- \int_A \left[r_x(x^\infty(t), y(t, \alpha), u^\infty(t)) \right]^T$$
$$\times F_z \left(\int_0^1 r(x^\infty(s), y(s, \alpha), u^\infty(s))ds \right) \phi(\alpha)d\alpha, \quad (24)$$

with final condition $\lambda^\infty(1) = 0$; and the Hamiltonian of Problem B as

$$\mathbf{H}(x(t), \lambda(t), u(t))$$
$$= [f(x(t), u(t))]^T \lambda + \int_A [r(x(t), y(t, \alpha), u(t))]^T$$
$$\times F_z \left(\int_0^1 r(x(s), y(s, \alpha), u(s))ds \right) \phi(\alpha)d\alpha \quad (25)$$

Lemma 3. *Suppose Assumptions 1-4 hold. Let $V \in \mathcal{N}_\infty^\#$ and let $\{x_M^*, u_M^*\}_{M \in V}$ be a set of optimal solutions to Problem B^M such that $\lim_{M \in V}\{x_M^*, u_M^*\} = \{x^\infty, u^\infty\}$. Let λ_M^* be the corresponding solutions to (20), λ^∞ be the solution to (24), $\mathbf{H}_{red}^M$ be given by (19), and $\mathbf{H}$ be given by (25). Then*

1. $\lim_{M \in V} \lambda_M^*(t) = \lambda^\infty(t)$

2. $\lim_{M \in V} \mathbf{H}_{red}^M(x_M^*(t), \lambda_M^*(t), u_M^*(t)) = \mathbf{H}(x^\infty(t), \lambda^\infty(t), u^\infty(t))$

For a proof of Lemma 3, see [21].

Given the convergence of the adjoint variables and Hamiltonians, we can now show that if the solutions to Problem B^M have an accumulation point, this accumulation point must minimize the Hamiltonian for Problem B.

Theorem 3. *Suppose Assumptions 1-4 hold. Let $V \in \mathcal{N}_\infty^\#$ and let $\{x_M^*, u_M^*\}$ be a sequence of optimal pairs to Problem B^M such that $\lim_{M \in V}\{x_M^*, u_M^*\} = \{x^\infty, u^\infty\}$. Then there exists an absolutely continuous costate trajectory λ^∞ satisfying (24) such that the following holds for almost every $t \in [0, 1]$*

$$u^\infty(t) = \arg\min_U \mathbf{H}(x^\infty(t), \lambda^\infty(t), u)$$

where $\mathbf{H}$ is given by (25).

Proof: Let $V \in \mathcal{N}_\infty^\#$ and let $\{x_M^*, u_M^*\}$ be a sequence of optimal pairs to Problem B^M such that $\lim_{M \in V}\{x_M^*, u_M^*\} = \{x^\infty, u^\infty\}$. Due to the result of Lemma 3, $\lim_{M \in V} \lambda_M^* = \lambda^\infty$ and $\lim_{M \in V} \mathbf{H}_{red}^M(x_M^*, \lambda_M^*, u_M^*) = \mathbf{H}(x^\infty, \lambda^\infty, u^\infty)$. Then for any admissible $u \in U$

$$\mathbf{H}(x^\infty, \lambda^\infty, u^\infty) = \lim_{M \in V} \mathbf{H}_{red}^M(x_M^*, \lambda_M^*, u_M^*)$$
$$\leq \lim_{M \in V} \mathbf{H}_{red}^M(x_M^*, \lambda_M^*, u) = \mathbf{H}(x^\infty, \lambda^\infty, u). \quad \square$$

While there may exist optimal pairs which are not accumulation points of the set of optimal pairs to Problem B^M, the result is still useful for checking the optimality of a solution computed numerically by discretizing the integral in the objective of Problem B and then solving the resulting Problem B^M, as a solution obtained in this way will necessarily be a accumulation point of the approximated problem. The result in Theorem 3 is a more general result than that in [2], since it provides a minimum principle for the problem with multiple searchers, multiple targets, and nonlinear dynamics with a control constraint. We show the existence of a solution satisfying the condition of Theorem 3 under certain assumptions in the following Corollary.

Corollary 2. *Suppose Assumptions 1-4 hold, and in addition there exists $V \in \mathcal{N}_\infty^\#$, a set of optimal solutions $\{x_M^*, u_M^*\}_{M \in V}$ to Problem B^M, such that $\{u_M^*\}_{M \in V}$ have uniformly bounded variation. Then there exists a solution to Problem B satisfying the necessary condition of Theorem 3.*

Proof: This corollary follows directly from Theorem 3 and Corollary 1. $\square$

The existence of a necessary condition such as that found in Theorem 3 allows us to assess the optimality of a numerically computed control.

V. APPLICATION ON OPTIMAL SEARCH FOR A TARGET WITH CONDITIONALLY DETERMINISTIC MOTION

In this section, we apply the results of the previous sections to a problem in optimal search theory. In this example, taken from [5] and [22], we consider an optimal search problem inspired by a real-world scenario.

A hostile target, whose location is unknown, is travelling towards a friendly ship, called the "high value unit" or "HVU." The starting location of this target is unknown, but the trajectory of the target, conditioned on the starting location, is known for all possible starting locations. The objective of the problem is to find a search path for a single searcher with given intitial position which will maximize the chance of detecting the target, before the target can reach the "HVU." The searcher is assumed to be a Dubin's vehicle with known constant velocity v and turning rate bounded by $K \in \mathbb{R}^+$. The dynamics are then given by

$$\dot{x}_1 = v\cos(x_3) \quad \dot{x}_2 = v\sin(x_3) \quad \dot{x}_3 = u \quad |u| \leq K$$

For the simulation we use the values $v = 120$, and $K = 50$.

For the detection probability we adopt the model in which r is explicitly the detection rate, that is the probability of the searcher detecting the target in the time interval $[t, t + \Delta t]$ is

given by $r(x(t), y(t, \alpha))\Delta t + o(\Delta t)$, when $y(t, \alpha)$ is the true position of the target. Note that in this scenario the detection rate function is independent of the control $u(t)$. When r satisfies this property, F is given by $F(z) = \exp(-z)$. The specific form of the detection rate function is given by the Poisson scan model:

$$r(x(t), y(t, \alpha)) = \beta \, \Phi \left(\frac{F - D(||x(t) - y(t, \alpha)||^2 - b)}{\sigma} \right),$$

where $\Phi(\cdot)$ is the standard normal cumulative distribution function, β is the scan opportunity rate, F^k is the so-called "figure of merit" (a sonar characteristic), and σ reflects the variability in the "signal excess". In the simulation we use the values

$$\beta = 1.1 \quad F = 90 \quad b = 20 \quad D = 0.3 \quad \sigma = 100$$

The optimal control Problem B is then to find a trajectory $x : [0, 1] \mapsto \mathbb{R}^3$ and control $u : [0, 1] \mapsto \mathbb{R}$, which minimize the objective function

$$\int_0^{70} \exp \left(- \int_0^1 \beta \, \Phi \left(\frac{F - D(||x(t) - y(t, \alpha)||^2 - b)}{\sigma} \right) dt \right) \phi(\alpha) d\alpha$$

where $\phi(\alpha)$ is the beta probability distribution functioned scaled to the interval $[0, 70]$.

We obtain an approximate solution to the optimal control problem by first discretizing the integral over the parameter α in the objective functional, and solving the resulting Problem B^M using a standard computational optimal control technique. In this section we consider an Euler discretization in the parameter space, and will solve the resulting standard optimal control problem using a direct method based on an Euler discretization in the time domain. After applying both discretizations, the resulting constrained optimization problem is solved using the NLP package SNOPT [23]. A discretization using 50 nodes in the parameter space and 120 nodes in the time domain is used with an Euler scheme, and the calculated optimal trajectories are shown in Figures 1 and 2. In Figure 1 the initial starting location of the target is distributed according to a $Beta(7, 2)$ distribution, whereas in Figure 2 the targets are distributed according to a $Beta(2, 7)$ distribution.

We use the necessary condition to assess the optimality of this trajectory by examining the adjoint variables, which can be computed by directly integrating the adjoint equations. We first construct the Hamiltonian according to (25) as

$$\mathbf{H}(x, \lambda, u) = v\big(\cos(x_3(t))\lambda_1(t) + \sin(x_3(t))\lambda_2(t)\big) + \lambda_3 u$$

$$- \int_0^{70} \beta \, \Phi \left(\frac{F - D(||x(t) - y(t, \alpha)||^2 - b)}{\sigma} \right) \times$$

$$\exp \left(- \int_0^1 \beta \, \Phi \left(\frac{F - D(||x(s) - y(s, \alpha)||^2 - b)}{\sigma} \right) ds \right) \phi(\alpha) d\alpha$$

By Eqn. (24), adjoint variables are given by $\lambda_1(1) = \lambda_2(1) = \lambda_3(1) = 0$, for $i = 1, 2$,

$$\dot{\lambda}_i(t) = \int_0^{70} \phi(\alpha) \frac{2\beta D}{\sigma\sqrt{\pi}} \big(y_i(t, \alpha) - x_i(t)\big) \times$$

$$\exp \left(- \left(\frac{F - D||x(t) - y(t, \alpha)||^2 - b}{\sigma} \right)^2 \right) \times \quad (26)$$

$$\exp \left(- \int_0^1 \beta \, \Phi \big(\frac{F - D(||x(s) - y(s, \alpha)||^2) - b}{\sigma} \big) ds \right) d\alpha$$

and

$$\dot{\lambda}_3(t) = -\lambda_1(t)v \sin(x_3(t)) + \lambda_2(t)v \cos(x_3(t)) \quad (27)$$
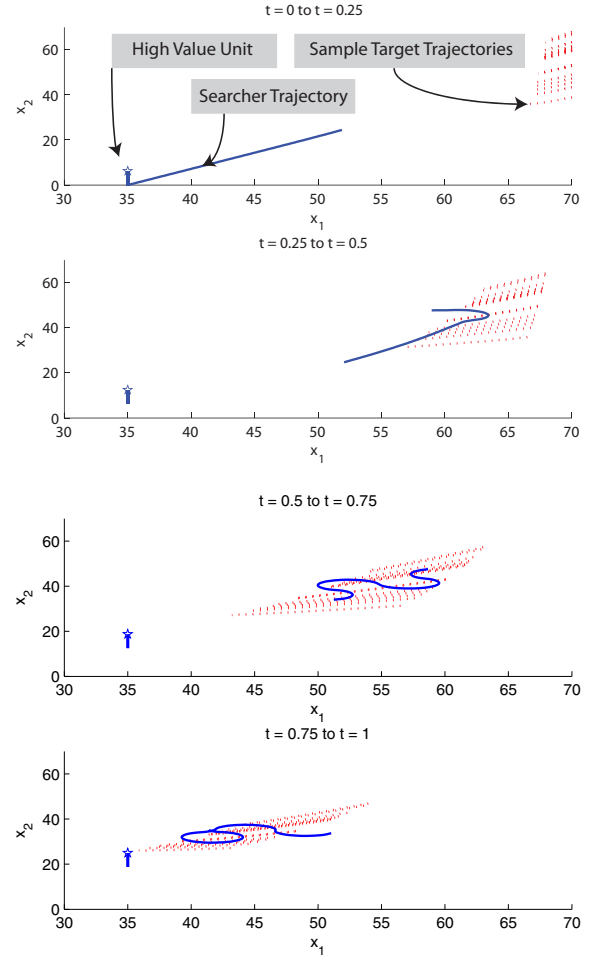

Fig. 1. Calculated optimal trajectory for the searcher and a random sample of target trajectories from a $Beta(7, 2)$ distribution.

This is a system of ordinary differential equations depending on the known trajectory $x(t)$, thus can be calculated by backward propagating the system from $t = 1$.

Observe that the control u enters into the Hamiltonian $\mathbf{H}$ only through the linear term $\lambda_3 u$. Therefore, the optimal control must satsify

$$u^*(t) = \begin{cases} K & \lambda_3(t) < 0 \\ -K & \lambda_3(t) > 0 \end{cases} \quad (28)$$

when $\lambda_3 = 0$, the problem is singular. The adjoint variable λ_3 is determined from the KKT multipliers obtained from SNOPT. Figure 3 shows that the optimal control calculated using the numerical method satisfies the condition in (28).

VI. CONCLUSION

A computational scheme is proposed for the problem of optimizing the trajectories of multiple searchers attempting to detect a moving target. The proposed scheme discretizes the original problem into a sequence of standard optimal control problems. We show that an accumulation point of the solution of the discretized problem is guaranteed to be an optimal solution of the original search problem. We also
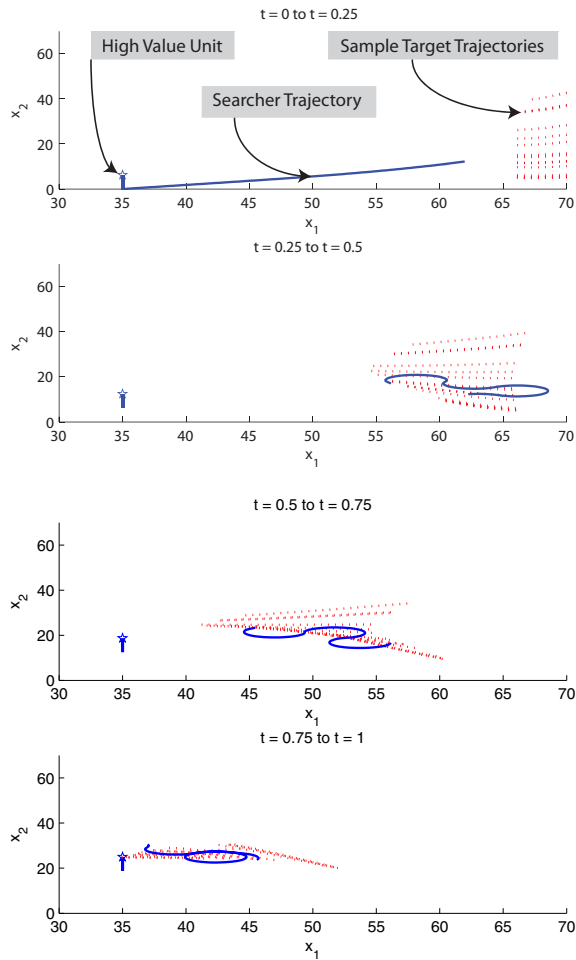
Fig. 2. Calculated optimal trajectory for the searcher and a random sample of target trajectories from a $Beta(2, 7)$ distribution.
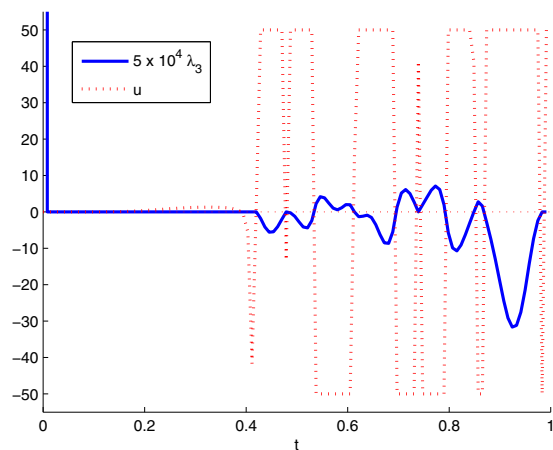

Fig. 3. Adjoint variable λ_3 and optimal control u.

provide a necessary condition that accumulation points must satisfy.

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