1990-03-15

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Americal Physical Society

Physical Review A, v. 41, no. 6, March 15, 1990, pp. 3144-3155
http://hdl.handle.net/10945/47579

Dispersion relations for gravity waves in a deep fluid: Second sound in a stormy sea

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(Received 15 September 1989)

We consider the nonlinear processes of interaction between a random field of short deep gravity waves and a deep long gravity wave, as well as the effects of nonlinear interactions among the driven short waves. The interactions are characterized by two types of invariants. First, the short waves possess an adiabatic invariant as regards their interaction with the long waves. Second, a collision between short waves conserves frequency and wave vector. As a consequence of these invariants, extra degrees of freedom appear. This results in a two-fluid description at the surface of the liquid for the mean flow of the long wave plus a distribution of waves. The two-fluid theory implies a spectrum of oscillations of the surface where, besides the usual gravity waves, there exists a surface mode with longitudinal oscillations. The regimes of validity of this hydrodynamic description are discussed, and the solutions for the dispersion relation are also presented. The power spectrum of a stormy sea is derived within the framework of nonlinear wave interactions. Finally, we comment on the relevance of the results to phase-velocity measurements in a wind-wave laboratory experiment, as well as possible experiments to observe the theoretically predicted new mode.

I. INTRODUCTION

The ocean provides one of nature's best laboratories for the study of nonlinear waves. Typical storms can lead to the appearance of surface waves, with wavelengths from 100 m down to a few centimeters and with Mach numbers of order unity. The Reynolds number for these waves is over one million.

The rich assortment of phenomena displayed by this highly nonlinear and highly dispersive system has over many years motivated a number of theoretical investigations related, for instance, to (a) the transfer of energy from wind to surface waves;[1,2] (b) the power spectrum of the stormy sea;[3–6] (c) the interaction of short gravity waves and a nonuniform environment;[7,8] and (d) the redistribution of energy among a random field of waves on the surface of a fluid.[9]

In this paper we focus our attention on the case of a random field of short-wavelength deep gravity waves in the background of a long-wavelength deep gravity wave. For this system our goal is to present and investigate the self-consistent set of equations which unifies items (b), (c), and (d) above.

We find that the presence of a stochastic distribution of surface waves leads to the possibility of new propagating long-wavelength modes. The restoring force for these modes is the elasticity of the power spectrum of the short waves. The appearance of these modes means that a complete description of the state of the fluid requires additional macroscopic variables. We will see that the new variables are, the integrated energy E_T of the random short waves, and the velocity $\mathbf{v}_T$ with which E_T is convected. The variables E_T and $\mathbf{v}_T$ are independent of the usual variables $\bar{\zeta}$ and $\mathbf{v}_s$ which describe the surface elevation and horizontal surface velocity. In our theory $\bar{\zeta}$, $\mathbf{v}_s$ will refer to the motions at long wavelength.

In parallel with the properties of superfluid ^{4}He, we shall refer to the equations of motion for $\bar{\zeta}$, $\mathbf{v}_s$, E_T, and $\mathbf{v}_T$ as the two-fluid theory for surface waves in a deep fluid, and we shall refer to the new surface mode as second sound. The question now arises as to how a Navier-Stokes fluid could have properties such as second sound which are usually thought of as manifestiations of superfluidity. A first step towards resolving this issue is provided by the classical derivation of the two-fluid equations of ^{4}He from the Euler fluid mechanics.[10] Prior to that analysis, superfluid phenomena were generally interpreted in terms of the quantization of flow.[11] Even if one now accepts that the Euler equations can explain the two-fluid theory, the question still remains as to how these phenomena could exist in a real Navier-Stokes fluid with its intrinsic damping.

This question brings us to the concept of wave turbulence. The presence of this state of motion is essential for the observation of second sound in a classical fluid. By wave turbulence we mean the random redistribution of energy in a sea of waves interacting through nonlinear processes. That is, the bandwidth of frequencies is large and the redistribution of energy is dominated by inertial nonlinearities which are large compared to linear irreversible transport processes. For surface waves, the wave turbulent state can occur only when the amplitudes of motion are sufficiently large that effects due to viscosity are negligible. But this is precisely the limit in which the reversible Euler level equations describe the motion.

To the extent that the wave motion in a medium is driven far off equilibrium, a real classical fluid will mimic superfluidity and the elasticity of the wave turbulence will yield two branches in the dispersion relation.[12]

As regards an application of our results to surface waves in the ocean we will have in mind that E_T is the integrated intensity of surface wave motion that is maintained by a prevailing wind. For the purpose of analysis we will assume in this paper that the spectral distribution of short waves is spatially isotropic. When we refer to the Mach number of a surface wave we mean the velocity amplitude divided by the phase velocity at that wavelength. Similarly the Reynolds number R of a surface wave equals the product of velocity amplitude with wavelength divided by kinematic viscosity. The limit of very large R leads to the possibility of second sound as developed in this work.

Although we have succeeded in developing a systematic two-fluid picture from nonlinear wave interactions, it is perhaps convenient at this point to give some of the physical arguments that can lead to it. The developments on the problem of nonlinear waves can be summarized by the study of three main mechanisms.

The first, often referred to as the WKB approximation, considers a nonuniform background medium in which waves propagate. Here it is assumed that the properties of the medium vary only slightly over the dominant length and time scales for the fluctuations, which in turn are slowly varying wave trains locally sinusoidal but with wave number and frequency changing with position and time. For example, short gravity waves superimposed on a much longer wave of the same type will have wavelengths that expand in proportion to the stretching of the surface by the long waves, while their energy will change as a result of the rate of working by the radiation stress against the mean strain.[7] Consequently the short-wave energy is not conserved, but the wave action is.

Similarly, the high-frequency waves cause a reaction in the flow of the background fluid, that is, the waves can do work on the surrounding medium. Unlike the effect mentioned above, the short waves in this case do not influence the mean motion to leading order. Yet changes in the mean energy are comparable to the changes in wave energy. This second nonlinear mechanism is not fortuitous, and the combined effect with the first one is, in general, unavoidable, because a wave train moving in a velocity gradient will incur changes in its energy as momentum is transferred by the waves down the gradient and energy is exchanged with the mean flow. In an extension of some results derived by Longuet-Higgins and Stewart,[8] we show in Sec. II B that the collective motion of short random waves (for which the statistical properties vary only slowly with position) induces a mass flux into the mean motion of a long wave. Variations in the mass flux produce convergence at the surface; pushing water up producing variations in the surface elevation, and pushing water down resulting in an induced flow in the deeper region.

The combined effect of these two nonlinear mechanisms leads to short-wave energy and mass flux excesses at the crests of the long wave. Both the stiffness and inertia of a long wave in the field of random gravity waves will increase and so cause a change in the phase velocity of the long wave. This effect is, however, not quantitatively accessible from the theoretical formulation as it stands, since a spectral wave action density for the random field of waves cannot be determined unambiguously from these two mechanisms (Sec. III B).

The third mechanism describes interactions among the high-frequency random waves and thus provides a means of calculating the spectral density. Then, due to collisions between the random waves, energy is continuously redistributed among them in a way prescribed by a collision integral.[9] The collision integral determines the stationary distributions of noise and therefore also the spectrum of noise under appropriate physical conditions (Sec. II D). This collision term in general destroys the adiabatic invariance for the wave action. But perhaps the most important consequence is that in a collision of waves the wave vector **k** is an additive conserved property independent of the total fluid momentum. The hydrodynamic description for the long modulations of the distribution of short waves then admits the presence of an additional vector degree of freedom which will characterize the distribution of noise.[10,12] This extra degree of freedom represents a broken symmetry of the original set of hydrodynamic variables, and for the case of deep gravity waves it corresponds to the difference $\mathbf{v}_s - \mathbf{v}_T$ between the long-wave horizontal orbital particle velocity at the surface and the macroscopic Stokes drift of the high-frequency random wave field.

The combined effects of these three nonlinear mechanisms in general, and the difference $\mathbf{v}_s - \mathbf{v}_T$ in particular, bring about an extra phasing in the motion: the short-wave energy can be increased at the troughs and decreased at the crests of the long-wave motion. This extra surface mode indeed corresponds to longitudinal compressions with almost no motion normal to the surface.

These results are presented in Secs. II and III of this paper where the dynamics of the motion are characterized. In Sec. III B the limits of applicability of the results are discussed. In Sec. III C a comparison of some of the results with phase-velocity measurements is made and possible experiments are discussed in the light of the theoretical predictions of the dynamics of the motion of this two-fluid picture. Finally, in Sec. IV we comment on some possible consequences of the theory to the long-standing problem of energy transfer from the wind to long waves.

II. FORMULATION OF THE PROBLEM

A. Energy of a system of gravity waves

We shall be concerned mainly with the oscillatory part of the motion of an incompressible inviscid fluid in a homogeneous gravitational field. The depth of the fluid is infinite and the undisturbed fluid surface coincides with the xy plane at $z=0$, where z labels the vertical coordinate away from the surface. Potential flow is assumed and effects due to surface tension are neglected. The potential in the whole volume satisfies Laplace's equation, as well as the condition $\partial\phi/\partial z \rightarrow 0$ as $z \rightarrow -\infty$.

The shape of the surface $z=\zeta(\mathbf{r},t)$ and the velocity potential at the free surface $\psi=\phi(\mathbf{r},z,t)|_{z=\zeta}$ are canonical variables[13] in that

$$\frac{\partial\zeta}{\partial t}=\frac{\delta H}{\delta\psi}, \tag{2.1a}$$

$$\frac{\partial\psi}{\partial t}=-\frac{\delta H}{\delta\zeta}, \tag{2.1b}$$

where

$$H=\tfrac{1}{2}\int d\mathbf{r}\int_{-\infty}^{\zeta}\left[(\nabla\phi)^2+\left[\frac{\partial\phi}{\partial z}\right]^2\right]dz+\tfrac{1}{2}g\int\zeta^2 d\mathbf{r} \tag{2.2}$$

is the total energy per unit mass of the fluid. Here $\mathbf{r}$ labels the coordinates of a point at the surface and ∇ is the two-dimensional gradient in the xy plane.

Equation (2.1a) is the kinematic boundary condition for Laplace's equation, while (2.1b) is Bernoulli's integral at the free surface. Specifying ζ and ψ fully defines the fluid flow since the boundary value for Laplace's equation has a unique solution.

At the risk of reproducing known results, but with the intention of presenting the complete theory as well as attempting to correct for abundant misprints in Zakharov's paper, we first consider the problem of nonlinear gravity waves. For this purpose we write the solution to Laplace's equation satisfying the condition at infinity, in terms of its Fourier components

$$\phi(\mathbf{r},z,t)=\frac{1}{2\pi}\int\hat{\phi}_k e^{kz}e^{i\mathbf{k}\cdot\mathbf{r}}d\mathbf{k} \tag{2.3a}$$

and

$$\zeta(\mathbf{r},t)=\frac{1}{2\pi}\int\hat{\zeta}_k e^{i\mathbf{k}\cdot\mathbf{r}}d\mathbf{k}, \tag{2.3b}$$

where k is the magnitude of the two-dimensional wave vector $\mathbf{k}$.

In terms of the Fourier components $\hat{\phi}_k$, $\hat{\zeta}_k$, the energy of the system of gravity waves becomes

$$\begin{aligned}H=&\tfrac{1}{2}\int(k\hat{\phi}_k\hat{\phi}_{-k}+g\hat{\zeta}_k\hat{\zeta}_{-k})d\mathbf{k}\\&+\frac{1}{4\pi}\int(kk_1-\mathbf{k}_1\cdot\mathbf{k})\hat{\phi}_0\hat{\phi}_1\hat{\zeta}_2\delta_{0+1+2}d012\\&+\frac{1}{8\pi^2}\int(kk_1-\mathbf{k}\cdot\mathbf{k}_1)k_1\hat{\phi}_0\hat{\phi}_1\hat{\zeta}_2\hat{\zeta}_3\delta_{0+1+2+3}d0123,\end{aligned} \tag{2.4}$$

where we are using the shorthand notation $\hat{\phi}_i=\hat{\phi}(\mathbf{k}_i)$, $d12\ldots=d\mathbf{k}_1 d\mathbf{k}_2\ldots$, and $\delta_{0+1+\ldots}=\delta(\mathbf{k}+\mathbf{k}_1+\cdots)$, and "0" corresponds to $\mathbf{k}$. Expression (2.4) for the energy is valid up to fourth order in the amplitudes of the waves. We now need to express (2.4) in terms of surface quantities $\hat{\psi}_k$, $\hat{\zeta}_k$. We note that the Fourier components $\hat{\phi}_k$ can be determined in terms of the Fourier components $\hat{\psi}_k$ and $\hat{\zeta}_k$ by using (2.3a) evaluated at the free surface $z=\zeta(\mathbf{r},t)$. This yields

$$\begin{aligned}\hat{\phi}_k=&\hat{\psi}_k-\frac{1}{2\pi}\int k_1\hat{\psi}_1\hat{\zeta}_2\delta_{0-1-2}d12\\&-\frac{1}{8\pi^2}\int k_1(k_1-|\mathbf{k}-\mathbf{k}_2|-|\mathbf{k}-\mathbf{k}_3|)\\&\times\hat{\psi}_1\hat{\zeta}_2\hat{\zeta}_3\delta_{0-1-2-3}d123.\end{aligned} \tag{2.5}$$

Substituting this into (2.4) and keeping terms up to fourth order results in the expression for the energy as a function of only surface quantities

$$\begin{aligned}H=&\tfrac{1}{2}\int(k\hat{\psi}_k\hat{\psi}_{-k}+g\hat{\zeta}_k\hat{\zeta}_{-k})d\mathbf{k}\\&-\frac{1}{4\pi}\int\tilde{X}(2|01)\hat{\psi}_0\hat{\psi}_1\hat{\zeta}_2\delta_{0+1+2}d012\\&-\frac{1}{32\pi^2}\int\tilde{Y}(01|23)\hat{\psi}_0\hat{\psi}_1\hat{\zeta}_2\hat{\zeta}_3\delta_{0+1+2+3}d0123,\end{aligned} \tag{2.6}$$

where

$$\begin{aligned}&\tilde{X}(2|01)=kk_1+\mathbf{k}\cdot\mathbf{k}_1,\\&\tilde{Y}(01|23)=kk_1(2k+2k_1-|\mathbf{k}+\mathbf{k}_2|-|\mathbf{k}+\mathbf{k}_3|\\&\qquad-|\mathbf{k}_1+\mathbf{k}_2|-|\mathbf{k}_1+\mathbf{k}_3|),\end{aligned} \tag{2.7}$$

are symmetric functions under interchange of their paired arguments. In obtaining $\tilde{Y}(01|23)$ the δ restriction in the integrand has been used in order to obtain a symmetric form. We now consider the canonical transformation to the variables ia_k^* and a_k (Ref. 13):

$$\begin{aligned}\hat{\psi}_k&=-i\left[\frac{\omega_k}{2k}\right]^{1/2}(a_k-a^*_{-k}),\\\hat{\zeta}_k&=\left[\frac{k}{2\omega_k}\right]^{1/2}(a_k+a^*_{-k}),\end{aligned} \tag{2.8}$$

where $\omega_k^2=gk$ is the dispersion law for infinitesimal amplitude gravity waves in deep water. In terms of the variables a_k and a_k^*, Hamilton's equations become

$$\frac{\partial a_k}{\partial t}=-i\frac{\delta H}{\delta a_k^*}, \tag{2.9}$$

where

$$\begin{aligned}H=&\int\omega_k a_k a_k^* d\mathbf{k}+\int d012[V^{(1)}_{0|12}(a_0^*a_1a_2+a_0a_1^*a_2^*)\delta_{0-1-2}+\tfrac{1}{3}V^{(2)}_{012}(a_0a_1a_2+a_0^*a_1^*a_2^*)\delta_{0+1+2}]\\&+\int d0123[\tfrac{1}{2}W^{(1)}_{01|23}\delta_{0+1-2-3}a_0^*a_1^*a_2a_3+\tfrac{1}{3}W^{(2)}_{0|123}(a_0a_1^*a_2^*a_3^*+a_0^*a_1a_2a_3)\delta_{0-1-2-3}],\end{aligned} \tag{2.10}$$

with

$$
\begin{aligned}
V^{(1)}_{0|12} &= X(-0|12) - X(2|1-0) - X(1|-02)\ ,\\
V^{(2)}_{012} &= X(0|12) + X(2|10) + X(1|02)\ ,\\
W^{(1)}_{01|23} &= Y(-0-1|23) - Y(-02|3-1)\\
&\quad - Y(-03|2-1) - Y(-12|3-0)\\
&\quad - Y(-13|2-0) + Y(23|-0-1)\ ,\\
W^{(2)}_{0|123} &= Y(32|1-0) + Y(31|2-0) + Y(21|3-0)\\
&\quad - Y(-01|23) - Y(-02|13) - Y(-03|12)\ ,
\end{aligned}
\tag{2.11}
$$

and

$$
\begin{aligned}
X(2|01) &= \frac{1}{8\pi\sqrt{2}}\left[\frac{\omega\omega_1\omega_2}{kk_1k_2}\right]^{1/2}\frac{k_2}{\omega_2}\tilde{X}(2|01)\ ,\\
Y(01|23) &= \frac{1}{(2\pi)^2 16}\left[\frac{kk_1k_2k_3}{\omega\omega_1\omega_2\omega_3}\right]^{1/2}\frac{\omega\omega_1}{kk_1}\tilde{Y}(01|23)\ .
\end{aligned}
\tag{2.12}
$$

The symmetries for V, W have been made explicit through the grouping of the terms and a bar.

B. Reaction of high-frequency waves on the flow of a long coherent wave

Longuet-Higgins and Stewart[8] considered the problem of the work done by gravity waves of nonuniform amplitude on the surrounding medium. The extension of their results that we have in mind here is that of a sea of random waves riding on the field of a long coherent wave of small amplitude. By a coherent wave, we mean that the average over the ensemble of phases

$$
\langle a_q \rangle = A_q \neq 0\ , \tag{2.13}
$$

where $\mathbf{q}$ is the wave vector of the coherent wave with $q/k \ll 1$, k being the wave number of a high-frequency component of the noise. To leading order ϵ in the measure of the slope of a typical high-frequency component ($\epsilon = k\hat{\zeta}_k d\mathbf{k}$, where $d\mathbf{k}$ is some bandwidth such as the mode spacing) the short waves do not influence the field of the coherent wave, which in turn obeys the linearized equations of motion. The theory that we develop in Sec. II C, however, treats wave momentum and wave energy at $O(\epsilon^2)$ and we thus should carry the long-wave motion theory to this same order. The approach that we will be using is due to Kontorovich *et al.*,[14] however, we will depart from their conclusions.

From Hamilton's equation (2.9) we readily obtain the evolution equation for the coherent wave up to $O(\epsilon^2)$ by taking an ensemble average

$$
\frac{\partial A_q}{\partial t} + i\omega_q A_q = -2i\int V^{(1)}_{k|q,k-q}\langle a_k a^*_{k-q}\rangle d\mathbf{k}\ . \tag{2.14}
$$

Kontorovich *et al.*[14] include the contribution of terms that they call anomalous correlators ($\langle a_k a_{k-q}\rangle$, $\langle a^*_k a^*_{k-q}\rangle$). However, it is shown in the Appendix, that not only is their contribution of higher order in ϵ, but also that over the slow time and length scales of variation of the spectral noise intensity ($q/k \equiv \gamma \ll 1$) these terms and cubic terms in the equation for A_q would cancel each other out up to terms of $O(\gamma^2)$.

Using (2.8), one finds

$$
\begin{aligned}
\frac{\partial\hat{\zeta}_q}{\partial q} - q\hat{\psi}_q &= -2i\left[\frac{q}{2\omega_q}\right]^{1/2}\\
&\quad\times\int\big(V^{(1)}_{k|q,k-q}\langle a_k a^*_{k-q}\rangle\\
&\qquad - V^{(1)}_{k|-q,k+q}\langle a^*_k a_{k+q}\rangle\big)d\mathbf{k}\ ,\\
\frac{\partial\hat{\psi}_q}{\partial t} + g\hat{\zeta}_q &= -2\left[\frac{\omega_q}{2q}\right]^{1/2}\\
&\quad\times\int\big(V^{(1)}_{k|q,k-q}\langle a_k a^*_{k-q}\rangle\\
&\qquad + V^{(1)}_{k|-q,k+q}\langle a^*_k a_{k+q}\rangle\big)d\mathbf{k}\ .
\end{aligned}
\tag{2.15}
$$

If we now assume that the modulation of the high-frequency noise components is $O(\gamma)$, we can then employ a double Fourier representation of slow (coherent) and fast (random) modes, in the same spirit of a multiple scale perturbation theory. Here q/k is the (small) length scale ratio that determines the perturbation expansion. Thus with the approximation $V^{(1)}_{k|q,k-q} \simeq V^{(1)}_{k|qk}$, etc., Eq. (2.15) implies the following boundary value problem for $\bar{\phi}$, the long-wave velocity potential:

$$
\begin{aligned}
&\nabla^2\bar{\phi} + \frac{\partial^2\bar{\phi}}{\partial z^2} = 0, \quad -\infty < z < 0\\
&\bar{\phi}\to 0 \quad \text{as } z\to-\infty\ ,
\end{aligned}
\tag{2.16a}
$$

while at $z=0$

$$
\frac{\partial\bar{\zeta}}{\partial t} - \frac{\partial\bar{\phi}}{\partial z} = -\nabla\cdot\int \mathbf{k}n_k(\mathbf{r},t)d\mathbf{k}\ , \tag{2.16b}
$$

and

$$
\frac{\partial\bar{\phi}}{\partial t} + g\bar{\zeta} = 0\ , \tag{2.16c}
$$

with $\bar{\phi}(\mathbf{r},z=0,t) = \bar{\psi}(r,t)$ and

$$
n_k(\mathbf{r},t) = \frac{1}{8\pi^2}\int\big(\langle a_k a^*_{k-q}\rangle + \langle a^*_k a_{k+q}\rangle\big)e^{i\mathbf{q}\cdot\mathbf{r}}d\mathbf{q}\ , \tag{2.17}
$$

the spatially inhomogeneous wave action per unit mass density.

A physical interpretation of (2.16b) can be obtained by noting that the surface momentum density carried by the short waves is

$$
\mathbf{J} = \rho\int \mathbf{k}n_k(\mathbf{r},t)d\mathbf{k}\ . \tag{2.18}
$$

Variations in $\mathbf{J}$ over the long wavelength q^{-1} can therefore lead to variations in height $\bar{\zeta}$ not accounted for by the long-wave velocity potential $\bar{\phi}$. Since the surface mass is zero $\mathbf{J}$ is the total mass current. Alternatively the motion $\bar{\phi}$ generated by a "virtual pressure" p_s applied at the upper surface of the fluid[8] leads to the same results as (2.16) provided

$$
p_s = g\int dt\,\nabla\cdot\mathbf{J}\ . \tag{2.19}
$$

As the small expansion parameters ϵ,γ for length scale and amplitude are independent, modification of (2.16) to include the nonlinear long-wavelength interactions is justified. In this case, we find

$$\nabla^2\bar{\phi}+\frac{\partial^2\bar{\phi}}{\partial z^2}=0, \quad -\infty<z<\bar{\xi}$$
$$\bar{\phi}\to 0 \quad \text{as } z\to-\infty , \tag{2.20a}$$

while at $z=\bar{\xi}(\mathbf{r},t)$

$$\frac{\partial\bar{\xi}}{\partial t}+\nabla\bar{\phi}\cdot\nabla\bar{\xi}-\frac{\partial\bar{\phi}}{\partial z}=-\nabla\cdot\int \mathbf{k}n_k(\mathbf{r},t)d\mathbf{k} , \tag{2.20b}$$

and

$$\frac{\partial\bar{\phi}}{\partial t}+\frac{1}{2}(\nabla\bar{\phi})^2+\frac{1}{2}\left[\frac{\partial\bar{\phi}}{\partial z}\right]^2+g\bar{\xi} - \left[\tfrac{1}{2}\int\omega_k n_k(\mathbf{r},t)d\mathbf{k}\right]\nabla^2\bar{\xi}=0 \tag{2.20c}$$

where we have retained only the leading order isotropic and anisotropic reactions due to the high-frequency waves. These equations would, for instance, describe the effects of the short-wavelength noise on long-wavelength solitons and Stokes waves. For these nonlinear equations the mass flow term $\nabla\cdot\mathbf{J}$ is not equivalent to the virtual surface pressure p_s. The term proportional to $\nabla^2\bar{\xi}$ represents an effective surface tension due to the short waves.

C. Refraction of high-frequency waves in the field of a long coherent wave

Equations (2.16) determine the flow of a long coherent wave in a sea of random high-frequency waves. In particular (2.16b) shows the effect of the radiation stress. Correspondingly, one expects that the long wave does work on the short-wave radiation. Longuet-Higgins and Stewart[7] calculated the changes in wavelength and amplitude of a short-wave train superposed on a much longer wave of the same type. Their results will correctly yield the frequency of a short wave in the inhomogeneous deformation of the surface in the frame of reference moving with the surface executing the long-wave motion. This frequency corresponds to that of a deep gravity wave in modulated gravitational acceleration about the value $g=980$ cm/sec^2, where the modulation is due to the vertical particle acceleration of the long-wave field. To the same order, long-wave straining introduces short wave-number distortions, where the strain is due to gradients in the horizontal particle velocity of the long wave. In the laboratory frame, however, the effects due to the orbital velocities of liquid particles in the field of a long wave introduce a Doppler shift. The Doppler shift dominates the individual effects of a modulated g and surface strain by $O(\gamma^{1/2})$. However, the combined contribution of these last two terms to the intrinsic frequency of the short-wave motion in an inhomogeneous surface is $O(\gamma)$ less than the Doppler shift when the long wave is effectively in deep water.

In order to establish the validity of these claims we are required to consider the evolution equation of $n_k(\mathbf{r},t)$. To this end, consider the evolution equation of the correlator $\langle a_k a^*_{k-q}\rangle$. In order to look systematically at infinitesimal waves propagating on a nonuniform background we (temporarily) set $\epsilon^2\ll\gamma\ll\epsilon$. This will rule out collisions among the noise components. For $\langle a_k a^*_{k-q}\rangle$ we obtain[14]

$$\left[\frac{\partial}{\partial t}+i(\omega_k-\omega_{k-q})\right]\langle a_k a^*_{k-q}\rangle=-i\int(V^{(1)}_{k|12}\langle a_1a_2a^*_{k-q}\rangle\delta_{k-1-2}-V^{(1)}_{k-q|12}\langle a^*_1a^*_2a_k\rangle\delta_{k-q-1-2}$$
$$+2V^{(1)}_{1|2k}\langle a_1a^*_2a^*_{k-q}\rangle\delta_{1-k-2}-2V^{(1)}_{1|k-q,2}\langle a^*_1a_2a_k\rangle\delta_{1-k+q-2})d12 . \tag{2.21}$$

The leading order for the inhomogeneous correlator

$$\langle a^*_k a_1 a_2\rangle=4\pi^2 n^0_k(\langle a_1\rangle\delta_{k-2}+\langle a_2\rangle\delta_{k-1}) \tag{2.22}$$

yields for $k\gg q$

$$\left[\frac{\partial}{\partial t}+i\frac{\partial\omega_k}{\partial\mathbf{k}}\cdot\mathbf{q}\right]\langle a_k a^*_{k-q}\rangle -8\pi^2 i\frac{\partial n^0_k}{\partial\mathbf{k}}\cdot\mathbf{q}(V^{(1)}_{k|k,q}A_q+V^{(1)}_{k|k-q}A^*_{-q})=0 , \tag{2.23}$$

where n^0_k is the homogeneous contribution to the spectral wave action. The corresponding equation for $\langle a^*_k a_{k+q}\rangle$ is obtained by taking the complex conjugate and making the replacement $\mathbf{q}\to-\mathbf{q}$. Thus the equation for $n_k(\mathbf{r},t)$ becomes

$$\frac{\partial n_k}{\partial t}+\frac{\partial\omega_k}{\partial\mathbf{k}}\cdot\frac{\partial n_k}{\partial\mathbf{r}}-\frac{\partial n^0_k}{\partial\mathbf{k}}\cdot\frac{\partial\delta\omega_k}{\partial\mathbf{r}}=0 , \tag{2.24}$$

where

$$\delta\omega_k(\mathbf{r},t)=2\int(V^{(1)}_{k|k,q}A_a+V^{(1)}_{k|k-q}A^*_{-q})e^{i\mathbf{q}\cdot\mathbf{r}}d\mathbf{q}$$
$$=\mathbf{k}\cdot\nabla\bar{\psi} , \tag{2.25}$$

which corresponds to a Doppler shift of the high-frequency wave in the field of the long wave. Equation (2.24) is a linearized wave Vlasov equation for the noise components. The last term, a consequence of phase-space conservation, shows that the long coherent wave does work against the radiation stress of the ensemble of high-frequency random waves.

When we replace the restriction $\epsilon^2\ll\gamma\ll\epsilon$ made above by simply $\epsilon,\gamma\ll 1$ we then allow high-frequency waves to interact among themselves adding a collision term to the right-hand side of (2.24). Collisions may be formally as large or larger than the streaming terms of (2.24). In the case of a wave turbulent sea, the presence of a stationary power spectrum strongly suggests the latter. Thus Eq. (2.26) below will describe departures of the far off equilibrium steady state for anisotropic and inhomogenous solutions

$$\frac{\partial n_k}{\partial t}+\frac{\partial \omega_k}{\partial \mathbf{k}}\cdot\frac{\partial n_k}{\partial \mathbf{r}}-\frac{\partial n_k^0}{\partial \mathbf{k}}\cdot\frac{\partial \delta\omega_k}{\partial \mathbf{r}}=\delta I\{n_k\} . \quad (2.26)$$

The meaning of the $\delta I\{n_k\}$ will become clear below. When terms of higher order in $\nabla\bar{\psi}$ are retained (2.26) becomes

$$\frac{\partial n_k}{\partial t}+\frac{\partial \tilde{\omega}_k}{\partial \mathbf{k}}\cdot\frac{\partial n_k}{\partial \mathbf{r}}-\frac{\partial n_k}{\partial \mathbf{k}}\cdot\frac{\partial \tilde{\omega}_k}{\partial \mathbf{r}}=I\{n_k\} , \quad (2.27)$$

where $\tilde{\omega}_k=\omega_k+\delta\omega_k$.

D. Evaluation of the collision term; wave turbulent solutions

For completeness we will derive the collision integral for weakly nonlinear gravity waves. The derivation was first carried out by Hasselmann[9] and Zakharov and Filonenko[4] who used as a closure approximation the assumption that the random waves were statistically Gaussian. It was later shown by Benney and Saffman[15] that closure of the hierarchy of equations for the cumulants for dispersive systems comes from assuming that there is an initial instant, at which they are smooth and the higher moments factorize into products of lower ones. Then on a time scale ϵ^{-4} times longer than a characteristic period of the waves (for the problem of gravity waves in deep water) factorization of higher cumulants by products of lower ones is maintained.[16] While accepting these conclusions we will use the equivalent cumulant discard approach. Here, correlations higher than the second develop slowly (see below). This scheme permits the hierarchy for the energy spectrum to be closed after two iterations.

The derivation of (2.24) was carried out subject to the assumption $\gamma \gg \epsilon^2$. The clearest way to remove this restriction is to consider first the opposite extreme $\gamma=0$ (i.e., homogeneous sea of random waves) and to include wave interaction for small but not infinitesimal amplitude ϵ.

For gravity waves in deep water, four waves resonantly interact subject to the condition

$$\mathbf{k}+\mathbf{k}_1=\mathbf{k}_2+\mathbf{k}_3 \text{ and } \omega_0+\omega_1=\omega_2+\omega_3 , \quad (2.28)$$

and the collision integral is cubic in the wave action n_k. For such a resonant condition, the Hamiltonian describing the interaction of waves is given by[3]

$$H=\int \omega_k a_k^* a_k d\mathbf{k}+\tfrac{1}{2}\int T_{01|23}a_0^* a_1^* a_2 a_3 \delta_{0+1-2-3} d0123 , \quad (2.29)$$

where the cubic terms in the initial Hamiltonian (2.10) have been eliminated by the canonical transformation[3]

$$a_k \to a_k - \int \left[\frac{V^{(1)}_{0|12}a_1 a_2 \delta_{0-1-2}}{\omega_0-\omega_1-\omega_2}+2\frac{V^{(1)}_{1|02}a_1 a_2^* \delta_{1-0-2}}{\omega_1-\omega_0-\omega_2} + \frac{V^{(2)}_{012}a_1^* a_2^* \delta_{0+1+2}}{\omega_0+\omega_1+\omega_2}\right] d12 , \quad (2.30)$$

and $T_{01|23}$ is the interaction matrix that results. It has the same symmetries as $W^{(1)}_{01|23}$ and it is a homogeneous function of third degree. The equations of motion that result from (2.29) have the form

$$\frac{\partial a_k}{\partial t}+i\omega_k a_k=-i\int T_{01|23}a_1^* a_2 a_3 \delta_{0+1-2-3} d123 . \quad (2.31)$$

Multiplying the above equation by a_k^*, adding it to the complex conjugate expression, and taking the ensemble average yields

$$\frac{\partial n_k}{\partial t}\delta(\mathbf{k}-\mathbf{k}')=\frac{-i}{4\pi^2}\int d123\big(T_{01|23}\langle a_{0'}^* a_1 a_2 a_3\rangle \delta_{0+1-2-3} - T_{0'1|23}\langle a_0 a_1 a_2^* a_3^*\rangle \times \delta_{0'+1-2-3}\big) . \quad (2.32)$$

The mean value $\langle a_0^* a_1^* a_2 a_3\rangle$ decomposes into the sum of products of spectral cumulants,

$$\langle a_0^* a_1^* a_2 a_3\rangle = \mathcal{J}_{01,23}\delta_{0+1-2-3}+\cdots \quad (2.33)$$

where the ellipses represents terms with null contribution inside the integral, yielding

$$\frac{\partial n_k}{\partial t}=\frac{1}{2\pi^2}\mathrm{Im}\int T_{01|23}\mathcal{J}_{01,23}\delta_{0+1-2-3} d123 . \quad (2.34)$$

Using the assumption that correlations higher than the second develop at a very slow rate we obtain the expression for $\langle a_0^* a_1^* a_2 a_3\rangle$ by use of their evolution equation

$$i(\omega_0+\omega_1-\omega_2-\omega_3)\mathcal{J}_{01|23}\delta_{0+1-2-3}=128\pi^6 i\delta_{0+1-2-3}T_{01|23}(n_1n_2n_3+n_0n_2n_3-n_0n_1n_3-n_0n_1n_2) , \quad (2.35)$$

where we have made use of the factorization assumption. Hence

$$\mathrm{Im}\mathcal{J}_{01,23}=128\pi^7\delta(\omega_0+\omega_1-\omega_2-\omega_3)T_{01|23} \times(n_1n_2n_3+n_0n_2n_3-n_0n_1n_3-n_0n_1n_2) \quad (2.36)$$

which yields

$$\frac{\partial n_k}{\partial t}=64\pi^5\int |T_{01|23}|^2\delta_{0+1-2-3}\delta_{\omega+\omega_1-\omega_2-\omega_3} \times n_0n_1n_2n_3\left[\frac{1}{n_0}+\frac{1}{n_1}-\frac{1}{n_2}-\frac{1}{n_3}\right]d123 . \quad (2.37)$$

Zakharov and Filonenko[4] first showed that there are two stationary isotropic solutions n_k^0 to (2.37). The first corresponds to equipartition of energy in a closed system ($n_k^0 \sim \omega^{-1}$). The second is a far off equilibrium solution corresponding to a constant flux of energy

$$Q = -\rho \int I\{n_k^0\}\omega d\mathbf{k} , \qquad (2.38)$$

from the input at low frequency to the sink at high frequency. Here $I\{n_k^0\}$ is the collision integral defined by the right-hand side of Eq. (2.37). Formation of the turbulent spectrum takes place as a result of the interaction of waves in the inertial range of phase space where nonlinearities dominate molecular viscosity and where the nature of the (low-frequency) energy input is irrelevant. In this inertial range $n_k^0 \sim \omega_k^{-8}$.

Although this soluton can be obtained formally from (2.37) and (2.38), we present here a simpler Kolmogorov style approach based upon dimensional analysis and study the process whereby wave energy cascades from one length scale to the next.[5,6,17] We label the properties of successive length scales with subscript n, so that we have the wave numbers

$$k_n = 2^n / l_0 , \qquad (2.39)$$

where l_0 is the length scale at which energy is being injected into the wave turublence motion. The energy per unit area for gravity waves of length scale corresponding to n is

$$E_n = \rho g (\zeta_n)^2 . \qquad (2.40)$$

The key to the cascade argument is that the rate at which energy rolls over from one length scale to the next is a function of the energy contained in that length scale (locality). For deep gravity water waves with normal dispersion, the leading interaction effect is a four wave process, which yields a rollover time t_n for the wave energy given by

$$\frac{1}{t_n} \cong \omega_n (k_n \zeta_n)^4 . \qquad (2.41)$$

The symbol $\cong$ means equality except for a numerical factor. The stationary state then follows from setting the rollover rate equal to the input rate Q,

$$E_n / t_n = Q . \qquad (2.42)$$

The discrete stationary spectrum is then

$$E_n \cong (Q\rho^2)^{1/3} g^2 / \omega_n^3 , \qquad (2.43)$$

so that the continuous power spectral density is

$$e(\omega) \cong (Q\rho^2)^{1/3} g^2 / \omega^4 , \qquad (2.44)$$

in agreement with $n_k^0 \approx \omega_k^{-8}$ in the inertial region.

A welcome feature of the dimensional argument is that it also provides a physical picture for the saturated Phillips spectrum.[18] For the length scale n, the measure of the nonlinearity is given by the Mach number of the corresponding scale (the rms steepness of a wave defined by n)

$$M_n = k_n \zeta_n \cong (Q / g^3 \rho)^{1/6} \omega_n^{1/2} . \qquad (2.45)$$

Since $M_n \sim \omega_n^{1/2}$, the weakly nonlinear theory described by (2.37) will fail to characterize the whole inertial range of wave numbers. Saturation will require a higher number of waves interacting. An $m > 4$ wave process will have a rollover rate given by

$$\frac{1}{t_n} \cong \omega_n (k_n \zeta_n)^{2(m-2)} , \qquad (2.46)$$

which along with stationarity $Q = E_n / t_n$ yields as $m \to \infty$ the continuous power spectrum

$$e(\omega) \cong \rho g^3 \omega^{-5} , \qquad (2.47)$$

and $M_n < 1$ a constant. The transition from an ω^{-4} to an ω^{-5} spectrum should in principle be a smooth transition. It should be observed when both

$$t_n \ll g^2 / 2\nu\omega_n^4 \qquad (2.48)$$

and

$$M_n \to M_n (m \to \infty) \qquad (2.49)$$

are met. The first condition establishes that the energy rollover time has to be smaller than the lifetime of the wave due to molecular viscosity ($\equiv \rho\nu$), while the second asserts the saturation characteristic of the Phillips spectrum. In order to satisfy the above conditions it is required that there be a large energy input into the wave motion at low frequency. Measurements under hurricane conditions analyzed for Forristall[19] display both spectra with the Phillips spectrum at the high-frequency end starting on or about 1.88 rad/sec, with Mach number $M_n \cong 0.15$.

Coupling to the long coheren wave [viz. (2.16b)] requires a degree of anisotrpy . Due to the conservation of ω and $\mathbf{k}$ in a collision of gravity waves the equation $I\{n_k\} = 0$ has a slightly anisotropic generalized solution

$$n_k = n_k^0 + \delta n_k (\hat{\mathbf{k}} + \mathbf{u}, \mathbf{k}, \mathbf{r}, t) , \qquad (2.50)$$

where n_k^0 is the stationary off-equilibrium response to the isotropic injection of wave energy at wave number k_s. The isotropic distribution is determined by balancing the collision integral with a source at ω_s and a sink at ω_∞ so that

$$I\{n_k^0\} + \frac{Q}{2\pi\rho\omega_s}\delta(\mathbf{k} - \mathbf{k}_s) - \frac{Q}{2\pi\rho\omega_\infty}\delta(\mathbf{k} - \mathbf{k}_\infty) = 0 . \qquad (2.51)$$

In the internal range $n_k^0 \sim \omega_k^{-8}$ and the first-order anisotropic modification is[20]

$$\delta n_k = -(\theta\omega_k + \hat{\mathbf{k}} \cdot \mathbf{u}) \frac{\partial n_k^0}{\partial \omega_k} , \qquad (2.52)$$

where $\theta, \mathbf{u}$ are independent of k. This inertial range solution represents a symmetry breaking in the original set of variables. Outside the inertial region (namely in the injection region) (2.52) does not apply; but for small $\mathbf{u}$ one can set

$$\delta n_k = \delta n_k^0(\omega_k) + \mathbf{u} \cdot \hat{\mathbf{k}} \delta \tilde{n}_k (\omega_k) . \qquad (2.53)$$

While θ redefines the amplitude of the spectral wave action, it significance as well as the meaning of the symmetry breaking field $\mathbf{u}$ will become clear later (see Sec. III). For the turbulent state these solutions are valid only to first order in $\mathbf{u}$. As the collision integral is a cubic functional of the spectral wave action, the anisotropic turbulent solution (2.52) leads in the inertial regime to

$$\int \begin{Bmatrix} \omega_k \\ \mathbf{k} \end{Bmatrix} \delta I\{n_k\} d\mathbf{k}$$
$$-8\int \begin{Bmatrix} \omega_k \\ \mathbf{k} \end{Bmatrix} I\{n_k^0\} \left[3\theta + 2\alpha \frac{\mathbf{u}\cdot\hat{\mathbf{k}}}{\omega_k} \right] d\mathbf{k} \; . \tag{2.54}$$

As indicated, the above expression is meaningful only when taking ω or $\mathbf{k}$ moments. In this case (2.38) implies that, in addition to the constant energy flux along the spectrum, there is slight momentum flux associated with the anisotropic solutions. Here α is a number of $O(1)$ and it is the result of angle integration over the anisotropic part of the solution.

In contrast with (2.52) the anisotropic generalization of the global equilibrium state ($n_k^0 \sim 1/\omega_k$) is given by[10]

$$n_k = n_k^0[\beta(\omega_k - \mathbf{k}\cdot\mathbf{w})] \; , \tag{2.55}$$

where β, $\mathbf{w}$ are also independent of k. The local equilibrium solution is valid to all orders of $\beta, \mathbf{w}$ and

$$\int \begin{Bmatrix} \omega_k \\ \mathbf{k} \end{Bmatrix} I(n_k) d\mathbf{k} = 0 \; . \tag{2.56}$$

III. DISPERSION AND ATTENUATION OF DEEP GRAVITY WAVES

A. The hydrodynamic limit

The near steady state motion is obtained by letting δn and therefore also $\mathbf{u},\theta$ vary slowly $[O(\gamma)]$ with $\mathbf{r},t$. The equations for the relaxation of $\bar{\zeta}$, $\bar{\phi}$, $\mathbf{u}$, and θ come from (2.16) and the moments of (2.26) with respect to ω, $\mathbf{k}$. The evaluation of these moments is straightforward and involves the definitions

$$E_T = \rho \int \omega n_k(\mathbf{r},t) d\mathbf{k} \; ,$$
$$\sigma_T(\mathbf{v}_T - \mathbf{v}_s) = \rho \int \mathbf{k}\,\delta n_k(\mathbf{r},t) d\mathbf{k} \; , \tag{3.1}$$

$$(\mathbf{v}_T - \mathbf{v}_s) \int \omega_k^2 \frac{\partial n_k}{\partial \omega_k} d\mathbf{k} = -2g \int \hat{\mathbf{k}}\,\delta n_k d\mathbf{k} \; . \tag{3.2}$$

The surface momentum density carried by the excitations has been denoted by $\sigma_T(\mathbf{v}_T - \mathbf{v}_s)$ where $\mathbf{v}_s = \nabla\bar{\psi}$. From the definitions (3.1) and (3.2) we find for the surface density of excitations

$$\sigma_T = -\frac{\rho}{2g} \frac{\left[\int \omega_k^2 \frac{\partial n_k}{\partial \omega_k} d\mathbf{k} \right] \left| \int \mathbf{k}\,\delta n_k d\mathbf{k} \right|}{\left| \int \hat{\mathbf{k}}\,\delta n_k d\mathbf{k} \right|} \; . \tag{3.3}$$

The small amplitude ω, $\mathbf{k}$ moments of (2.26) lead to equations for the surface energy density and surface momentum given by

$$\frac{\partial \delta E_T}{\partial t} + \tfrac{5}{4} E_T^0 \nabla\cdot\mathbf{v}_T = -3\frac{Q_0}{E_T^0}\delta E_T \; , \tag{3.4}$$

$$\frac{\partial}{\partial t}[\sigma_T(\mathbf{v}_T - \mathbf{v}_s)] + \tfrac{1}{4}\nabla\delta E_T = -\alpha \frac{Q_0}{E_T^0}\sigma_T(\mathbf{v}_T - \mathbf{v}_s) \; , \tag{3.5}$$

where

$$E_T^0 = \rho \int \omega_k n_k^0 d\mathbf{k} \; , \tag{3.6a}$$

$$\delta E_T = \rho \int \omega_k \delta n_k d\mathbf{k} \; , \tag{3.6b}$$

and an integration by parts yields

$$\rho \int \omega_k^2 \frac{\partial n_k^0}{\partial \omega_k} d\mathbf{k} = -5E_T^0 \tag{3.7}$$

provided that at low ω, the product $\omega^5 n^0(\omega)$ vanishes.

The right-hand sides of (3.4) and (3.5) represents a damping due to the cascade.

Substitution of the solutions

$$\delta E_T = E_T' e^{i(\mathbf{q}\cdot\mathbf{r} - \Omega t)}, \quad \mathbf{v}_T = \mathbf{v}_T' e^{i(\mathbf{q}\cdot\mathbf{r} - \Omega t)} \; ,$$
$$\bar{\phi} = \bar{\phi}' e^{qz} e^{i(\mathbf{q}\cdot\mathbf{r} - \Omega t)}, \quad \bar{\zeta} = \bar{\zeta}' e^{i(\mathbf{q}\cdot\mathbf{r} - \Omega t)} \; , \tag{3.8}$$

into (2.16), (3.4), and (3.5) gives a fourth-order dispersion law in the frequency Ω and wave number q

$$[\Omega^2(1 + i\alpha_1/\Omega) - c_T^2 q^2](\Omega^2 - gq) = \frac{5gE_T}{16\rho} q^4 \; , \tag{3.9}$$

where we have assumed the real part of Ω to be larger than its imaginary part ($\Omega_R \gg \Omega_I$) and set

$$c_T^2 = \frac{5E_T}{16\sigma_T}, \quad \alpha_1 = \frac{Q}{E_T}(3+\alpha) \; , \tag{3.10}$$

where the superscript "0" has now been dropped.

For $\Omega_R \gg \Omega_I$ this quadratic equation in Ω^2 gives two branches for the propagation of surface waves in the equilibrium range as well as the attenuation for each branch. Thus, for each value of the wave vector, two frequencies are allowed.

At low q, which means

$$\sigma_T q/\rho \ll 1 \; , \tag{3.11}$$

the dispersion laws for the two branches are given by

$$\Omega^2 = c_T^2 q^2 \; ,$$
$$\Omega^2 = gq \; . \tag{3.12}$$

The extrapolation of these relations to a higher wave number q_c would lead to a level crossing where

$$q_c = g/c_T^2 \; . \tag{3.13}$$

Above wave numbers q_D defined by

$$q_D = \rho/\sigma_T \; , \tag{3.14}$$

the product of the roots of (3.9) becomes negative so that the motion is either diffusive or unstable. However, this regime (3.14) will not concern us since it is surely beyond the region of applicability of the approximations which led to (3.9). Using q_D and q_c the dispersion law can be written in an excitation-independent form:

$$[\bar{\Omega}^2(1+i\bar{\alpha}_1/\bar{\Omega})-\bar{q}^2](\bar{\Omega}^2-\bar{q})=\bar{q}^4(q_c/q_D)\ , \quad (3.15)$$

where $\bar{\Omega}^2=\Omega^2/gq_c$, $\bar{q}=q/q_c$, and $\bar{\alpha}_1=\alpha_1/(gq_c)^{1/2}$. Figures 1 and 2 show the dispersion relation(s) and phase velocities for two values of q_c/q_D. Deviations from the linear law are apparent. In a laboratory experiment the wave turbulent steady state will be characterized by the energy input rate Q and the injection frequency ω_s, which is roughly speaking the low-frequency cutoff to the spectrum. Instead of Q we will characterize the flow by the Mach number at the injection frequency

$$M_0 \simeq \left[\frac{Q}{\rho}\right]^{1/6}\left[\frac{\omega_s}{g}\right]^{1/2}\ , \quad (3.16)$$

then estimating all integrals via the use of cutoffs we find

$$q_c/q_D \cong M_0^2\ ,$$
$$c_T^2=\frac{3}{40}\frac{g^2}{\omega_s^2}\ ,$$
$$q_D=\frac{\omega_s^2}{gM_0^2}\ , \quad (3.17)$$
$$\bar{\alpha}_1 \cong M_0^4\ ,$$
$$\frac{\sigma_T\omega_s^2}{g\rho}\cong M_0^2\ .$$

Figure 3 is a plot of $\bar{\Omega}_I/\bar{\Omega}_R$ versus $\bar{\Omega}_R$ for $\bar{\alpha}_1=10^{-4}$.

The dynamics of these two modes can be understood by solving the eigenvector problem associated with the two branches. For simplicity assume that the wave vector coincides with the x axis, and that the long-wavelength attenuation can be neglected. As we shall see, v_T is nonzero for each branch of excitation. Therefore we relate v_s and $\bar{\zeta}$ to v_T from the equations of motion according to

$$v_s=\left[1-\frac{c_T^2q^2}{\Omega^2}\right]v_T\ , \quad (3.18a)$$

$$\bar{\zeta}=\frac{(\Omega^2-c_T^2q^2)}{g\Omega q}v_T\ . \quad (3.18b)$$

For long wavelength and $\Omega^2=gq$ we find $v_T \approx v_s$ so that the turbulent motion moves with the background and the surface response is the same as for an ordinary gravity wave. When the second-sound-type mode propagates, $\Omega^2=c_T^2q^2$, and we find $v_s=0$ so that for this mode the excitations oscillate while the surface remains undistorted.

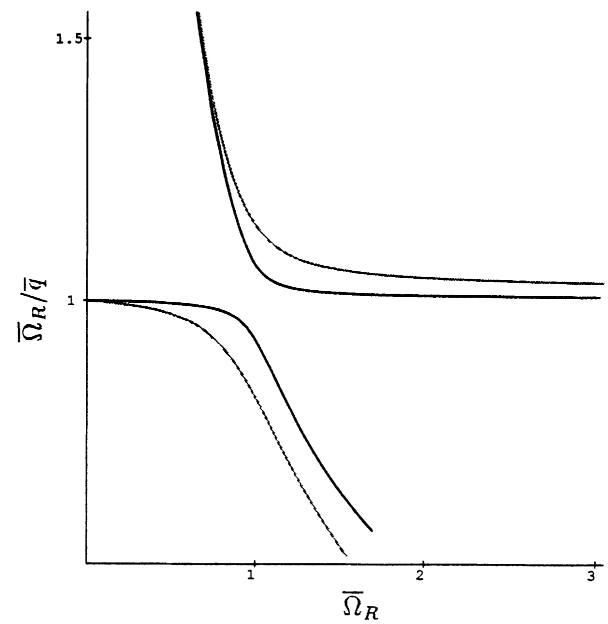

FIG. 2. Phase velocities of the two branches of propagation of surface waves in a "stormy" sea for Mach numbers of 0.01 (solid lines) and 0.05 (gray lines).

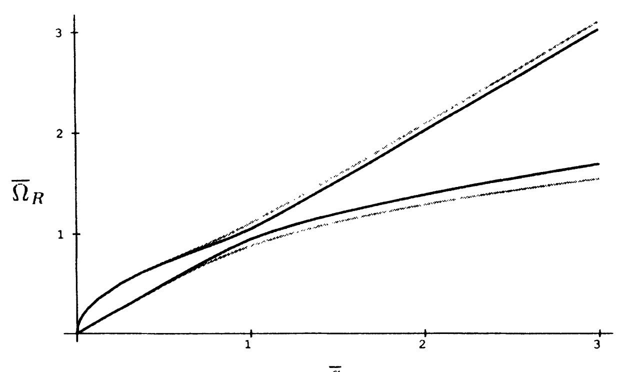

FIG. 1. Dispersion law for the two branches of propagation of surface waves in "stormy" waters for Mach numbers of 0.01 (solid lines) and 0.05 (gray lines). The second-sound-like branch is the low-frequency branch below $\bar{q}=1$ and it becomes the high-frequency branch above $\bar{q}=1$.

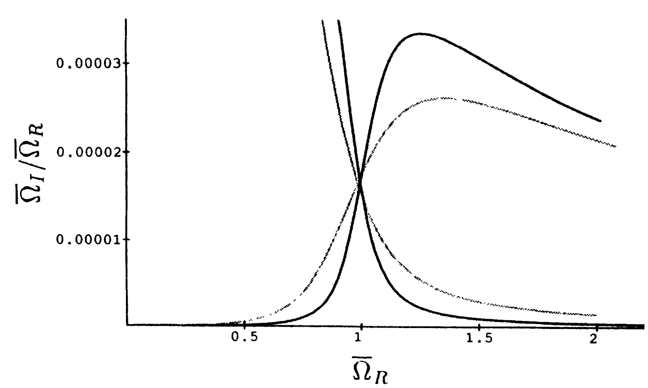

FIG. 3. Dimensionless attenuation rate of the two surface mode branches for Mach numbers of 0.01 (solid lines) and 0.05 (gray lines). The curves which diverge at low $\bar{\Omega}$ correspond to the lower branch.

B. The diffusive and ballistic regimes

The presence of an isotropic distribution of noise leads to a renormalization of the surface wave dispersion law. Let the renormalized frequency be denoted by $(\omega_q)_R$, then, according to (2.29),

$$(\omega_q)_R=\omega_q+8\pi^2\int T_{qk|qk}n_k d\mathbf{k} , \tag{3.19}$$

where $\omega_q^2=gq$. Inclusion of this term leads to a modification of the dispersion law (3.15). Its consequences are small compared to the effects which we emphasized above.

The theoretical framework developed in Sec. II was based on the assumption that $q \ll k$. From the energy spectral distribution one can estimate the mean wave number k_{mean}. It corresponds to about three times the spectral peak wave number. For q values greater than k_{mean} the dispersion relations given by (3.9) are not valid. In this regime the elasticity of the wave turbulence is lost and one might expect that the two branches coalesce into a single one where the effects of nonuniformities are negligible compared to the effects of nonlinearity. For these high values of q the dispersion law should be given by (3.19).

Another regime of interest corresponds to the limit where the mean free path of the short waves is larger than the wavelength of the collective oscillations. Such a situation arises in the singular limit for which the energy input into the wave motion is negligibly small. Obviously, under these conditions the existence of a local steady state solution (2.52) is not applicable. However, the properties of such a gas can be described by the kinetic equation (2.26). Now, however, the effects of nonlinearity are negligible compared to the effects of nonuniformity and so as a first approximation the collision integral is dropped. The properties of the mean flow of the long wave play the role of an external condition for the short gravity waves, and are determined by Eqs. (2.16). Looking for solutions of the form $\exp(i\mathbf{q}\cdot\mathbf{r}-i\Omega t)$ we obtain

$$\begin{aligned}&(\Omega-\mathbf{q}\cdot\mathbf{C}_g)n_k'+\mathbf{q}\cdot\mathbf{C}_g\frac{\partial n_k^0}{\partial\omega_k}\mathbf{k}\cdot\mathbf{v}_s'=0 ,\\ &\Omega\bar{\zeta}'-iq\bar{\psi}'-\mathbf{q}\cdot\int\mathbf{k}n_k' d\mathbf{k}=0 ,\\ &-i\Omega\bar{\psi}'+g\bar{\zeta}'=0 ,\end{aligned}\tag{3.20}$$

where $\mathbf{C}_g=\frac{1}{2}(g/k)^{1/2}\hat{\mathbf{k}}$ and where $\mathbf{v}_s=i\mathbf{q}\bar{\psi}$. The compatibility condition yields the dispersion relation

$$\Omega^2-gq+gq^2\int k^3\frac{\partial n_k^0}{\partial\omega_k}C_g\frac{\cos^3\xi}{\Omega/q-C_g\cos\xi}dk\,d\xi=0 , \tag{3.21}$$

where ξ is the angle between $\mathbf{q}$ and $\mathbf{k}$, and n_k^0 is otherwise undetermined.

C. Towards an experimental test of the theory

In the following we consider possible experimental arrangements that could detect the presence of second sound in wave turbulence. In the simplest arrangement, waves will be excited by an oscillating solid boundary so that at the wall

$$v_T=v_s=v , \tag{3.22}$$

where v is the velocity of the paddle. Denoting the two branches by subscripts "1" and "2" we have

$$\begin{aligned}&v_{s,1}+v_{s,2}=v ,\\ &v_{T,1}+v_{T,2}=v ,\end{aligned}\tag{3.23}$$

so that the ratio of surface motion amplitudes for the two waves is

$$\frac{\bar{\zeta}_2}{\bar{\zeta}_1}=\frac{u_2}{u_1}\left[\frac{c_T^2-u_2^2}{u_1^2-c_T^2}\right] , \tag{3.24}$$

where the phase velocity is $u\equiv\Omega/q$. In the region of the "level repulsion" u_1 and u_2 are approximately equal so that the paddle oscillating at a frequency given approximately by $(gq_c)^{1/2}$ will excite both modes. As these modes propagate away from the source beating will occur as a result of the small difference between u_1 and u_2. This beating phenomenon should provide a sensitive means of demonstrating the presence of second sound in wave turbulence in gravity surface waves, near this frequency region.

Far from the level repulsion regime the paddle preferentially excites the gravity type ($\Omega^2=gq_1$) mode. An estimate of the (small) extent to which the paddle excites the second-sound-type mode ($q_2=\Omega/c_T$) can be obtained at low Ω from the expansion of the dispersion law in powers of q_c/q_D. From the dispersion relation (3.15) we find

$$\begin{aligned}&\bar{q}_1=\bar{\Omega}^2-\frac{\bar{\Omega}^6}{1-\bar{\Omega}^2}\frac{q_c}{q_D}+\cdots ,\\ &\bar{q}_2=\bar{\Omega}+\frac{\bar{\Omega}^2}{2(1-\bar{\Omega})}\frac{q_c}{q_D}+\cdots ,\end{aligned}\tag{3.25}$$

where we reiterate that $\bar{\Omega}$ is sufficiently less than unity so that the corrections are small. Therefore a paddle oscillating with a frequency Ω excites a second sound type mode with amplitude $\bar{\zeta}_2$ determined at low $\bar{\Omega}$ by

$$\frac{\bar{\zeta}_2}{\bar{\zeta}_1}=\frac{\bar{\Omega}^4}{(1-\bar{\Omega})(1-\bar{\Omega}^2)}\frac{q_c}{q_D} . \tag{3.26}$$

A similar situation arises in superfluid ^{4}He, that is the ratio of intensities of second and first sound emitted by a plane oscillating in a direction perpendicular to itself is small. If, however, the surface's temprature is made to oscillate, this ratio becomes large. The combination of heater and thermometer can serve as a direct means of exciting and detecting second sound in superfluid ^{4}He.

For both superfluid ^{4}He and the present situation of wave turbulent deep gravity waves, second sound can be looked upon as a compressional wave in the gas of excitations. In the near thermodynamic equilibrium superfluid ^{4}He, oscillations of temperature bring about oscillations in the density of excitations. In the far off equilibrium wave turbulence, the concept of temperature is replaced by the energy flux over the spectrum.

The above analogies can provide means for direct excitations and detection of second sound in a wave turbulent environment. For instance, by launching pulse-modulated noise from a paddle, the propagation characteristics and the phase velocity of second sound can in principle be determined by the long time evolution of the integrated energy spectrum measured at two points.

A third way of investigating the properties of the oscillation spectrum (3.15), would be by the measurements of the reflection coefficient at the boundary of two regions, one of which is quiescent as regards the short-wave noise. This could be accomplished, for instance, in an L-shaped laboratory tank, the leg being a long rectangular section and the foot an anechoic square termination. The waves in the leg being excited by a prevailing wind would enter the short foot section without much diffraction allowing a quiescent region in the anechoic end. From this region or to it, ordinary gravity waves or second sound and ordinary gravity waves can be sent, and its reflection coefficient measured as a function of the angle of incidence at the boundary between the two regions.

Phase-velocity measurements in a large experimental tank ($70\times 8\times 3$ m^3) analyzed by Rikiishi[21,22] showed a roughly constant value among frequencies near and above the dominant frequency, which, however, varied with fetch in proportion to the wind speed over the water surface. To us, the merits of the experiment reside in Rikiishi's ability to discriminate unwelcome experimental points in a quantitative manner by means of the spectral condition number (μ in his notation). Thus the question of lowest Nyquist sampling period allowed by the matrix array used in the experiment was properly addressed for frequencies larger than a given value determined by upper bounds in μ. Also the question of sensitivity was quantitatively handled for frequencies lower than a given value determined from the same upper bound in μ. These two aspects translate in phase-velocity measurements for a finite band of frequencies with a determined width (see Fig. 13 in Rikiishi[22]). On the subjective side, his measurements were free of theoretical prejudices.

While further experimental investigations are required we note here that a possible connection between Rikiishi's phase-velocity measurements and the upper branch of Fig. 2 could be made. Rikiishi's multipoint correlations are, in a way, equivalent to measurement of second sound in superfluid helium by means of the cross correlation of temperature fluctuations rather than the direct excitation and detection of the propagating thermal wave.

IV. PERSPECTIVES

We have derived, starting from the inviscid, irrotational, and incompressible flow governing a one-component fluid, the two-fluid equations for surface waves on the surface of a liquid, that are otherwise nonlinearly driven. The elasticity of this wave turbulent solution yielded an extra propagating mode with similarities to second sound in superfluid liquid helium. However, the derivation of the equations was based on purely classical reasoning and the fact that the inertia of the turbulent noise is convergent.

As in the case of liquid helium where second sound consists of an adiabatic wave in the density of thermally excited phonons which can be detected as a temperature wave, the collective oscillation corresponding to the spectrum (3.9) is an adiabatic wave in the energy density of wave turbulent deep gravity waves. This, in fact, constitutes the most important conceptual difference between ^{4}He and wave turbulence. ^{4}He superfludity is an equilibrium phenomenon, while wave turbulence is a signature of far off equilibrium behavior.

In the present situation, the normal mode analysis [via. Eq. (3.18)] reveals that the second sound does not exhibit the counterflow nature characteristic of second sound in bulk ^{4}He. This is in no way fundamental but simply a consequence of the surface mass being zero.

The spectrum of oscillations of first and second sound in bulk ^{4}He leads to unambiguous normal modes: one is a pressure wave while the other corresponds to a temperature wave. In the present study, the dispersive nature of the system and the fact that the dispersion relations for the two modes intersect lead to a richer behavior, and mode mixing. This latter property was indeed considered in the discussion of the spatial beating phenomena as a means to detect (indirectly) the presence of a second sound mode.

While the present investigation predicts second sound in surface gravity wave turbulence, this phenomenon is general and thus it should also appear in other situations like drift wave turbulence in a plasma.[12] There, fluctuations in the spectrum could propagate as second sound, thus working against confinement.

On the speculative side, the present results might find some application to the problem of understanding the mechanism whereby wind waves are generated. Experiments by Plant and Wright[23] have shown that for waves in the gravity range the mechanisms proposed by Phillips[2] and Miles[1] are unable to account for the observed high values of energy transfer from wind to waves of about 10 cm and longer. An enhanced (but still smaller than observed) energy transfer into the long-wave motion may be due to interaction with short wind-generated waves, as Landahl *et al.*[24] have shown. This naturally raises the question: Which branch of the oscillation spectrum is favored by wind as regards this indirect energy transfer? That is, given the presence of higher-frequency waves that have already grown to equilibrium what role does the second sound branch play in the energy transfer to long waves?

ACKNOWLEDGMENTS

We wish to acknowledge valuable discussions with P. H. Roberts. This research was supported in part by the Department of Energy, Office of Basic Energy Sciences, Division of Engineering and Geosciences. During part of this project A. L. was supported by the National Research Council. S. L. G. was supported by the Naval Postgraduate School Direct Funded Research Program.

APPENDIX: ANOMALOUS CORRELATORS

In obtaining Eq. (2.14) we did not include the contribution of terms that Kontorovich *et al.*[14] called anomalous correlators. In this appendix we will show that if the evolution equation for the long-wave motion is carried to $O(\epsilon^3)$, the anomalous correlators and $O(\epsilon^3)$ terms cancel each other out.

From Hamilton's equations up to $O(\epsilon^3)$ we have

$$\frac{\partial A_a}{\partial t}+i\omega_q A_q=-2i\int V^{(1)}_{k|q,k-q}\langle a_k a^*_{k-q}\rangle d\mathbf{k}-i\int\langle V^{(1)}_{q|k,q-k}\langle a_k a_{q-k}\rangle+V^{(2)}_{q,k,-q-k}\langle a^*_k a^*_{-q-k}\rangle)d\mathbf{k}$$
$$-i\int(W^{(1)}_{qk|12}\langle a^*_k a_1 a_2\rangle\delta_{q+k-1-2}+W^{(2)}_{k|q12}\langle a_k a^*_1 a^*_2\rangle\delta_{k-q-1-2})d012\ . \quad (\text{A1})$$

The second integral is the contribution from the anomalous correlators, while the last one represents the $O(\epsilon^3)$ contribution.

The evolution equation for the correlator $\langle a_k a_{q-k}\rangle$ is, to leading order in γ,

$$\left[\frac{\partial}{\partial t}+2i\omega_k\right]\langle a_k a_{q-k}\rangle=-16\pi^2 i n_k(V^{(1)}_{q|-k,k}A_q+V^{(2)}_{k,-k,-q}A^*_{-q})\ , \quad (\text{A2})$$

which follows from Hamilton's equation. Equation (A2) represents the statistical response for $\langle a_k a_{q-k}\rangle$ to waves in the inhomogeneous field of a longer (coherent) wave. It has the solution

$$\langle a_k a_{q-k}\rangle=\frac{2\pi n_k}{\sqrt{2}}\frac{kq}{\omega_k}\left[\frac{\omega_q}{q}\right]^{1/2}(A_q-A^*_{-q})\ . \quad (\text{A3})$$

The solution for $\langle a^*_k a^*_{q-k}\rangle$ is obtained by taking the complex conjugate of the above expression and making the replacement $\mathbf{q}\to-\mathbf{q}$. Thus to leading order in γ we have

$$\int d\mathbf{k}(V^{(1)}_{q|k,q-k}\langle a_k a_{q-k}\rangle+V^{(2)}_{q,k,-q-k}\langle a^*_k a^*_{-q-k}\rangle)=-\tfrac{1}{2}\int\frac{\omega_q}{\omega_k}k^2 q n_k(A_q-A^*_{-q})d\mathbf{k}+O(\gamma^2)\ . \quad (\text{A4})$$

For the cubic terms, and to leading order in γ we have

$$W^{(1)}_{qk|12}\langle a^*_k a_1 a_2\rangle\delta_{q+k-1-2}=8\pi^2 W^{(1)}_{qk|qk}n_k A_q\delta_{q-1}\delta_{k-2}\ , \quad (\text{A5a})$$

and

$$W^{(2)}_{k|q12}\langle a_k a^*_1 a^*_2\rangle\delta_{k-q-1-2}=8\pi^2 W^{(2)}_{k|q,k,-q}n_k A^*_{-q}\delta_{k-1}\delta_{q+2}\ . \quad (\text{A5b})$$

Since

$$W^{(1)}_{qk|qk}=\frac{1}{16\pi^2}\frac{\omega_q}{\omega_k}k^2 q+O(\gamma^2)=-W^{(2)}_{k|q,k,-q}\ , \quad (\text{A6})$$

the terms given by (A4) cancel out with the leading-order contribution of the cubic terms.

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