2011-06

Optimal periodic patrolling trajectories of UUVs guarding a channel

Chung, H.

American Control Conference

2011 American Control Conference, June 29 - July 1, 2011, pp. 888-893
http://hdl.handle.net/10945/47550

2011 American Control Conference
on O'Farrell Street, San Francisco, CA, USA
June 29 - July 01, 2011

Optimal Periodic Patrolling Trajectories of UUVs Guarding a Channel

H. Chung, E. Polak, J. O. Royset, and S. S. Sastry

***Abstract*— Given a number of patrollers, the channel patrol problem consists of determining the periodic trajectories that the patrollers must trace out so as to maximize the probability of detection of the intruder. We formulate this problem as an optimal control problem. We assume that the patrollers' sensors are imperfect and that their motions are subject to turn-rate constraints, and that the intruder travels straight down a channel with constant speed.**

Using discretization of time and space, we approximate the optimal control problem with a large-scale nonlinear programming problem which we solve to obtain an approximately stationary solution and a corresponding optimized trajectory for each patroller. In numerical tests, we obtain new insight — not easily obtained using geometric calculations — into efficient patrol trajectory design for up to two patrollers in a narrow channel where interaction between the patrollers is unavoidable due to their limited turn rate.

I. Introduction

This paper deals with the optimal detection of an underwater intruder in a channel using one or more unmanned underwater vehicles (UUVs). In particular, it establishes optimal periodic patrol trajectories for the UUVs, which we refer to as patrollers, that maximize the probability of detection of an underwater intruder traveling straight down a channel at constant speed. While we focus on an underwater intruder and patrollers, our general approach may also be applicable in the case of other types of vehicles.

This problem is a multi-patroller extension of the classical "channel patrol problem" (also called the barrier patrol problem); see, e.g., Section 1.3 of [1] and Chapter 9 of [2]. The channel patrol problem for a single patroller was formulated by Koopman [3] during World War II and arises in naval operations where the channel may represent a relatively narrow body of water such as a strait or port entrance through which enemy vessels and submarines as well as smugglers and terrorists may attempt to pass. The need to consider multiple patrollers is apparent, especially in view of the development of small UUVs that may be used to guard channels. The channel patrol problem may also arise in anti-submarine warfare in an operating area around a carrier or naval expeditionary strike group [4] and then, typically, with multiple patrollers. With the proliferation of small diesel submarines and the advent of UUVs and self-propelled semi-submersibles the channel patrol problem has acquired new importance, since these vessels are difficult to detect.

The early studies by Koopman [3] as well as by Washburn [5] focus on the determination of the probability of intruder detection for a single patrol trajectory consisting of piecewise linear segments; see also Chapter 9 of [2].

These early studies ignore the limited turn-radius of the patroller or use coarse approximations. Moreover, they focus on a single patroller with the assumption that the case of multiple patrollers can be solved by dividing the channel into subchannels, with one patroller assigned to each subchannel. This policy may become problematic when there are many patrollers in a narrow channel. In that case, the limited turn radius of a patroller may force it to deviate greatly from the assigned, say, back-and-forth trajectory. We refer the reader to [6] for a broad review of other problems in search theory.

In this study, we consider one or more patrollers, account for turn-radius limits and imperfect sensors, and model the motion of the patrollers using ordinary differential equations. This formulation leads to an optimal control problem with solution trajectories that are executable by UUVs. Optimal control formulations of general search problems are found in [7] with later generalizations in [8]; see also references therein. However, these studies deal with the general situation where the intruder moves according to some diffusion process. We take advantage of the special structure of the channel patrol problem and derive significantly simpler expressions, which allow us to carry out a comprehensive numerical investigation of one and two patrollers.

In Section 2 we derive a formula for the detection probability, and in Section 3 we present the optimal control formulation of the channel patrol problem. Numerical results are found in Section 4, which is followed by our concluding remarks in Section 5. This paper is a summary of [11], which includes a more general formulation of the problem, details of discretization procedure, and a larger and more complete set of numerical results.

II. Detection Probability

We consider a scenario of patrolling a channel similar to the one in [5]: patrollers search a channel of width L looking for a single intruder which is moving straight down the channel with constant speed v_I (see Figure 1). The intruder is unaware of the patrollers, makes no attempt to evade them, and simply progresses straight down the channel. We assume

The first, second, and fourth authors were partially supported by ONR MURI "Computational Methods for Collaborative Motion" (CoMotion), and ARO MURI "Scalable SWARMS of Autonomous Robots and Mobile Sensors" (SWARMS). The third author is supported by AFOSR Young Investigator grant F1ATA08337G003.

H. Chung is with the Department of Mechanical and Aerospace Enigneering, Monash University, Clayton, VIC 3800, Australia Hoam.Chung@monash.edu

E. Polak and S. S. Sastry are with the Department of Electrical Engineering and Computer Science, University of California, Berkeley, CA 94720, USA polak/sastry@eecs.berkeley.edu

J. O. Royset is with Operations Research Department, Naval Postgraduate School, Monterey, California joroyset@nps.edu

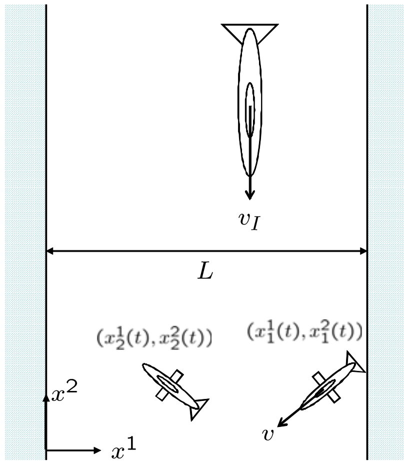

Fig. 1. Two patrollers (bottom) try to detect an intruder (top) in a channel

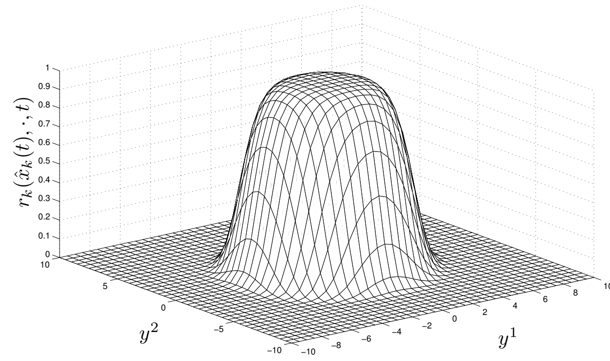

Fig. 2. Detection rate function based on Poisson Scan Model (1).

that the probability of detection, of the intruder by a patroller, depends on the positions of the patroller and the intruder, the quality of the patroller's sensor, and on the time allowed for observation. We also could easily let the probability of detection depend on the speeds of the intruder and patrollers, but ignore that possibility here to avoid complicated detection models.

Suppose that there are q patrollers looking independently for the intruder and let $\hat{x}_k(t) \triangleq (x_k^1(t), x_k^2(t)) \in \mathbb{R}^2$ be the position of the k-th patroller at time t, where $k = 1, 2, ..., q$. See Figure 1 for the case with $q = 2$. We use superscripts to denote components of a vector. Of course, UUVs can also vary their depth, but we ignore this possibility for simplicity of exposition. The formulation below can trivially be extended to three dimensions.

We derive the expression for the probability of detection in two steps. First, we derive the detection probability for a stationary intruder. Second, we extend that expression to the situation at hand with a moving intruder in a channel. Hence, temporarily assume that the intruder is stationary and located at $y \in \mathbb{R}^2$. Again, an extension of the following formulation to three dimensions is trivial. Let $r_k(\hat{x}_k(t), y, t) \geq 0$, $k = 1, 2, ..., q$, denote the detection rate at time t for the k-th patroller at $\hat{x}_k(t)$ when the intruder is located at y. The detection rates reflect the qualities of the patrollers' sensors as described in more details below and are defined so that the probability that the k-th patroller detects the intruder during a small time interval $[t, t + \Delta t)$ is $r_k(\hat{x}_k(t), y, t)\Delta t$. For theoretical and computational reasons, $r_k(\cdot, \cdot, \cdot)$, $k = 1, 2, ..., q$, must be smooth, but can otherwise take any form to reflect a variety of sensors. We focus on patrollers that are UUVs and intruders that are diesel-electric submarines, and assume that the patrollers' sensors are sonars. Hence, we adopt the Poisson Scan Model (see, e.g., [1] p. 3-1) and, for the k-th patroller, we set

$$r_k(\hat{x}_k(t), y, t) = \lambda \Phi[\{F_k - \rho(\hat{x}_k(t), y)\}/\sigma], \qquad (1)$$

where $\Phi(\cdot)$ is the standard normal cumulative distribution function, λ is the scan opportunity rate, F_k is the "figure of merit" (a sonar characteristic), σ reflects the variability in the "signal excess," and $\rho(\hat{x}_k(t), y)$ is the propagation loss, which depends on the distance between the patroller and the intruder, see, e.g, Figure 4.5 on page 93 in [2]. All these quantities may be time dependent. The typical shape of $r_k(\hat{x}_k(t), \cdot, t)$ is shown in Figure 2, where $\hat{x}(t) = (0, 0)$ and $\rho(\hat{x}_k(t), y) = a\|\hat{x}_k(t) - y\|^2 + b$, with $\lambda = 1$, $F_k = 70$, $\sigma = 5$, $a = 0.5$, and $b = 60$. We now define the probability that the k-th patroller does not detect the intruder during some time interval $[0, T]$ in terms of the detection rate.

Given a trajectory $\{\hat{x}_k(t), 0 \leq t \leq T\}$ and an intruder at y, we denote the probability that the k-th patroller does not detect the intruder during $[0, t]$, $t \in [0, T]$, by $p_k(y, t)$. Assuming that events of detection in non-overlapping time intervals are all independent, we find that this probability can be computed recursively by solving the difference equation

$$p_k(y, t + \Delta t) = p_k(y, t)\,(1 - r_k(\hat{x}_k(t), y, t)\Delta t) \qquad (2)$$

with $p_k(y, 0) = 1$, or, as Δt tends to zero, by solving the parameterized differential equation

$$\frac{dp_k(y, t)}{dt} = -p_k(y, t) r_k(\hat{x}_k(t), y, t), \quad p_k(y, 0) = 1, \quad (3)$$

with solution

$$p_k(y, t) = \exp(-\eta_k(y, t)), \qquad (4)$$

where

$$\eta_k(y, t) = \int_0^t r_k(\hat{x}_k(s), y, s) ds. \qquad (5)$$

The above derivation follows standard arguments for Poisson processes and $\eta_k(y, t)$ is the mean value of the random number of detections at y, up to time t, by the k-th patroller, when that number is given by a Poisson law.

Now, let $\phi : \mathbb{R}^2 \to \mathbb{R}$ be the probability density function of the location of the (stationary) intruder at time 0, i.e., for any $B \subset \mathbb{R}^2$, $\int_B \phi(y) dy$ is the probability that the intruder is located in the area B at time 0. This information may be provided by exogenous intelligence sources and reflects the patrollers knowledge about the intruder prior to the start of the patrols. Then, the probability that the k-th patroller fails to detect a stationary intruder during the time period $[0, T]$ is given by

$$\int_{y\in\mathbb{R}^2} p_k(y,T)\phi(y)dy \tag{6}$$

$$= \int_{y\in\mathbb{R}^2} \exp\left(-\int_0^T r_k(\hat{x}_k(t), y, t)dt\right)\phi(y)dy. \tag{7}$$

The functions $p_k(\cdot, t)$, $k = 1, 2, ..., q$, reflect the patrollers' knowledge about the intruder's location at time t and can therefore be considered to be "information states" or "belief states" that augment the "physical state" $\hat{x}_k(t)$, $k = 1, 2, ..., q$.

The extension from a stationary intruder, as assumed above, to an intruder that moves straight down a channel at constant speed, see Figure 1, is accomplished by a linear transformation as described next.

As in [5], we fix the position of the intruder on a tape moving down the channel at the speed of the intruder, v_I. Hence, the intruder is stationary relative to the tape and the formulae derived above are applicable. We only need to measure the patroller's location relative to the tape. In this framework, the probability of detection relates to the ratio of the rate at which the patroller examines new area on the tape to the rate at which new tape area appears.

In order to utilize this approach, let $\hat{z}_k(t)' \triangleq (z_k^1(t), z_k^2(t))$ be the position vector of the k-th patroller at time t relative to the tape, where prime denotes the transpose of a vector. Then we have that for all $k = 1, 2, ..., q$,

$$\begin{aligned} z_k^1(t) &= x_k^1(t) \\ z_k^2(t) &= x_k^2(t) + v_I t. \end{aligned} \tag{8}$$

We refer to $\hat{x}_k(t)$ and $\hat{z}_k(t)$ as the absolute and relative positions of the k-th patroller at time t, respectively. We will use y for both the absolute and relative positions of the intruder as the meaning is clear from the context.

Since the channel has width L, it suffices to consider the relative intruder position $y \in A(T) \triangleq [0, L] \times [0, v_I T]$ for patrols of duration T time units. Hence, it follows from (7) that given a trajectory $\{\hat{z}_k(t), 0 \le t \le T\}$, the probability that the k-th patroller does not detect the intruder during time period $[0, T]$ is

$$P_k \triangleq \int_{y\in A(T)} \exp\left(-\int_0^T r_k(\hat{z}_k(t), y, t)dt\right)\phi(y)dy, \tag{9}$$

where the probability density function of the relative position of the intruder takes the specific form $\phi(y) = \phi^1(y^1)/(v_I T)$, with $\phi^1(\cdot)$ being the probability density function of the intruder's y^1-position (i.e., the intruder's horizontal position in Figure 1). For example, if the patrollers have no prior knowledge of the y^1-position of the intruder, then one can assume a uniform distribution across the channel, i.e., $\phi^1(y^1) = 1/L$ for all $y^1 \in [0, L]$. Note that we abuse the notation $r_k(\cdot,\cdot,\cdot)$ slightly, by using it to represent the detection rate function both in the absolute and in the relative positions.

We assume that the patrollers make independent detection attempts and hence it follows from (4) and (5) that the conditional probability that no patroller detects the intruder given a specific relative intruder position y is simply the product

$$\begin{aligned} &\prod_{k=1}^{q} \exp\left(-\int_0^T r_k(\hat{z}_k(t), y, t)dt\right) \\ &\qquad = \exp\left(-\int_0^T \sum_{k=1}^{q} r_k(\hat{z}_k(t), y, t)dt\right). \end{aligned} \tag{10}$$

Consequently, the probability that no patroller detects the intruder during $[0, T]$ takes the form

$$P \triangleq \int_{y\in A(T)} \exp\left(-\int_0^T \sum_{k=1}^{q} r_k(\hat{z}_k(t), y, t)dt\right)\phi(y)dy. \tag{11}$$

We use this expression in an optimal control problem for determining patrol trajectories as discussed next.

III. Optimal Control Problem

Our objective is to find optimal periodic trajectories for multiple patrollers that maximize the probability of detection of the intruder. In contrast to [5], we consider multiple patrollers whose turn radius is constrained by their dynamics, in differential equation form, and available control action. Thus we assume that the positions of the patrollers are states of a differential equation. Specifically, we assume that the kinematic equations of all the patrollers are the same and are of the form

$$\frac{dx_k(t)}{dt} = f(x_k(t), u_k(t)) \triangleq \begin{bmatrix} v\cos x_k^3(t) \\ v\sin x_k^3(t) \\ u_k(t) \end{bmatrix} \tag{12}$$

with $x_k(0) = \xi_k$, where k-th patroller's state $x_k(t)' = (x_k^1(t)\ x_k^2(t)\ x_k^3(t))$, $(x_k^1(t), x_k^2(t))$ represent the absolute location of the k-th patroller, and x_k^3 represents its heading as in Figure 1. This planar kinematic model describes underwater vehicles that navigate at a constant depth and a constant forward speed with variable yaw rate. In [9], a similar model was suggested for use with underwater vehicles, but they regarded the vehicle's yaw rate as a function of vehicle's forward speed and steering angle. We assume that all patrollers move at constant speed v. The control input for the k-th patroller $u_k \in \mathbb{R}$ is its yaw rate. We define $\hat{x}_k(t)' = (x_k^1(t), x_k^2(t))$ representing the absolute location of the k-th patroller.

The assumption that all patrollers are governed by the same kinematic equation is easily relaxed, but requires further notation and is therefore avoided here. Also it should be noted that our approach is applicable to general n-dimensional cases.

Next, referring to (8), let $e_2 \triangleq (0, 1, 0)$, and let $z_k(t) \triangleq x_k(t) + v_I t e_2$. Hence, $z_k(t) = (\hat{z}_k(t)', x_k^3(t))'$. We refer to $x_k(t)$ and $z_k(t)$ as absolute and relative states for the k-th patroller, respectively. Then we find that the k-th patroller's dynamics in the relative state become

$$\frac{dz_k(t)}{dt} = \tilde{f}(z_k(t), u_k(t)), \quad z_k(0) = \xi_k, \tag{13}$$

where

$$\tilde{f}(z_k(t), u_k(t)) \triangleq f(z_k(t) - v_I t e_2, u_k(t)) + v_I e_2. \tag{14}$$

We let the patrol duration T be a decision variable. Hence, we introduce the time transformation $t = Ts$ to enable us to define the channel patrol problem on the fixed time interval $[0, 1]$. For simplicity of notation, we use the same notation for states and controls defined on $[0, T]$ as on the normalized time interval $[0, 1]$. The meaning should be clear from the context. We now obtain the time-normalized kinematic equations

$$\frac{dz_k(s)}{ds} = T\tilde{f}(z_k(s), u_k(s)), \quad z_k(0) = \xi_k. \tag{15}$$

We denote the solution of (15) by $z_k(\cdot; T, u_k, \xi_k)$. Moreover, the probability P that no patroller detects the intruder during the interval $[0, T]$ (see (11)) can be written as $P(T, u, \xi)$, where $u' \triangleq (u_1, \ldots, u_q)$ and $\xi' \triangleq (\xi_1', \ldots, \xi_q')$.

The optimal periodic patrol problem (OPPP) consists of maximizing the probability of detecting the intruder during the time interval $[0, T]$, i.e., $1 - P(T, u, \xi)$, by choosing the best values of T, u, and ξ. This leads to the following optimal control problem formulation:

$$\textbf{OPPP}: \ \max\{1 - P(T, u, \xi)\} \tag{16}$$

$$\text{s.t. } z_k(1; T, u_k, \xi_k) = g(\xi_k), \ k = 1, 2, ..., q, \tag{17}$$

$$z_k(s; T, u_k, \xi_k) \le z_k^{\max}(s; T), k = 1, ..., q, s \in [0, 1], \tag{18}$$

$$z_k(s; T, u_k, \xi_k) \ge z_k^{\min}(s; T), k = 1, ..., q, s \in [0, 1], \tag{19}$$

$$T \in [T^{\min}, T^{\max}], \tag{20}$$

$$u \in \mathbf{U}, \tag{21}$$

$$\xi \in \mathbf{X}, \tag{22}$$

where $g : \mathbb{R}^3 \to \mathbb{R}^3$ is a function that describes the end-state constraints, $z_k^{\max}(s; \cdot) : \mathbb{R} \to \mathbb{R}$ and $z_k^{\min}(s; \cdot) : \mathbb{R} \to \mathbb{R}$ are upper and lower bounds on the state trajectories at scaled time s, respectively, $T^{\min}$ and $T^{\max}$ are the minimum and maximum durations of a patrol, respectively, $\mathbf{U}$ is the set of admissible controls, and $\mathbf{X} \subset \mathbb{R}^3 \times \cdots \times \mathbb{R}^3$ is the set of admissible initial conditions. We assume that

$$\begin{aligned} \mathbf{U} \triangleq \{&u = (u_1, u_2, ..., u_q) \mid u_k \in L_{\infty,2}[0, 1], \\ &u_k^{\min} \le u_k(s) \le u_k^{\max}, s \in [0, 1], k = 1, ..., q\}, \end{aligned} \tag{23}$$

where $u_k^{\min}$ and $u_k^{\max}$ are the minimum and maximum control input at any point in time for the k-th patroller.

We use the constraints (17) to ensure that the patrollers' trajectories are closed. The constraints (18) and (19) are set up to contain the trajectories of the patrollers to be within a time-varying box. The constraint (20) limits the duration of a patrol. The constraints (21) and (22) ensure that the control input and initial conditions satisfy specific constraints.

We replace the running cost in (11) with an end cost using an auxiliary information state $p(y, s)$ to facilitate the evaluation of this integral by the same numerical integration technique used to solve the dynamic equations (15). For any $y \in \mathbb{R}^2$, let $p(y, s)$ be the solution of the parameterized differential equation

$$\frac{dp(y, s)}{ds} = -Tp(y, s) \sum_{k=1}^{q} r_k(\hat{z}_k(s), y, Ts) \tag{24}$$

with $p(y, 0) = 1$. In view of (3), $p(y, s)$ is the probability that no patroller has detected the intruder during the time interval $[0, Ts]$ given the intruder is located at y. It generalizes the information state $p_k(y, t)$ to the case of multiple patrollers, relative locations, and scaled time.

In this notation,

$$P(T, u, \xi) = \int_{y \in A} p(y, 1)\phi(y) dy, \tag{25}$$

where $p(y, 1)$ is given by (24) and computed using T, u, and ξ, and $A \triangleq [0, L] \times [0, v_I]$.

The numerical solution of **OPPP** requires the discretization of the time interval [0,1] and of the area A. We use Euler's method for time discretization and a uniform grid of equally spaced points for the spatial discretization; see [11] for details.

IV. Numerical Results

In **OPPP**, for every patroller $k = 1, 2, ..., q$, we define the end-state constraint by

$$g(\xi_k) = (\xi_k^1, \xi_k^2 + v_I, \xi_k^3 + 2n\pi)' \tag{26}$$

for some $n = 0, 1, 2, ...$ This ensures that the absolute location and heading of the patroller at time T is the same as at time 0. The integer n is a variable that determines the number of 360-degree rotations that are required during a patrol and hence it largely determines the shape of the trajectory. In this paper, we will solve the problem for $n = 0$ and 1 only. The full results and discussions has been submitted for publication in [11].

We set the state-trajectory constraints to be $z_k^{\min}(s; T) = (0, v_I sT - \gamma, -\infty)'$ and $z_k^{\max}(s; T) = (L, v_I sT + \gamma, \infty)'$ for $k = 1, 2, ..., q$, where $\gamma > 0$ is a constant that we vary below. We note that the state-trajectory constraints ensure that $z_k^1(s) \in [0, L]$, i.e., the patrollers stay within the channel, and $z_k^2(s) \in [-\gamma + v_I sT, \gamma + v_I sT]$, i.e., $x_k^2(t) \in [-\gamma, \gamma]$. Hence, the last constraint limits how much the patrollers can travel up and down the channel. The control input limits $u_k^{\max} = 1$ and $u_k^{\min} = -1$ for $k = 1, 2, ..., q$. We let the constraint set on the initial conditions be given by $\mathbf{X} = \{\xi \in \mathbb{R}^3 \mid 0 \le \xi^1 \le L, \ \xi^2 = 0, \ \xi^3 \in \mathbb{R}\}$.

We set the channel width $L = 20$, where one unit of length equals 1000 yards, and the intruder speed $v_I = 3$, and the patroller speed $v = 1$. We assume that one unit of time equals 0.1 hours. Hence, the intruder and patrollers move at approximately 15 knots and 5 knots, respectively. We always use $T^{\min} = 5$ and hence we do not consider patrols of shorter duration that 0.5 hours. We vary $T^{\max}$. We use the detection rate function (1) with parameters as given below that equation. Hence, the detection rate function is as

Case	n	γ	$T^{\max}$	T^*	P^*
1	0	$L/10$	25	24.001	0.43348
2	1	$L/10$	25	23.568	0.43300
3	0	$L/5$	15	15.000	0.42462
4	1	$L/5$	15	15.000	0.42620

TABLE I

SUMMARY OF NUMERICAL RESULTS FOR A SINGLE PATROLLER AND VARYING NUMBER OF ROTATIONS n (SEE (26)), VERTICAL RANGE γ, AND PATROL-DURATION LIMIT $T^{\max}$. T^* AND P^* ARE OPTIMIZED PATROL DURATION AND PROBABILITY OF DETECTION, RESPECTIVELY.

in Figure 2. We assume that the distribution of the intruder's y^1-location is uniform, i.e., $\phi^1(y^1) = 1/L$. We use 128 time discretization points and 32 by 32 spatial discretization points.

Finally, we use SNOPT version 6.2 [12] in TOMLAB MATLAB toolbox [13] as our nonlinear programming solver, running on a desktop computer with two AMD Opertron 2.2GHz processors with 8GB RAM, running Linux 2.6.28. We use SNOPT default parameters.

Next we describe the results of several numerical studies involving one and two patrollers.

A. One Patroller

Table I provides numerical results for $q = 1$, several values of vertical trajectory constraint γ, and maximum patrol duration $T^{\max}$. In cases 1-2, $\gamma = L/10 = 2$, i.e., the patroller cannot move vertically (in Figure 1) more than two units above or below its starting point. Moreover, in cases 1-2, the patrol duration is limited to $T^{\max} = 25$. Case 1 requires the patroller to return to the same heading at the end of the patrol (i.e., the heading change during one patrol cycle should be zero and $n = 0$ in (26)) forcing the optimized trajectory to have a "bow-tie" shape, as displayed in Figure 3 (solid line). Since the discretized **OPPP** may be nonconvex, we cannot guarantee that the control input that generates this trajectory or those reported below are globally optimal. However, the optimized control inputs and corresponding trajectories satisfy the default stopping criterion of SNOPT and hence are close to a stationary solution of the discretization of **OPPP**. Figure 3 also displays the initial trajectory prior to optimization (dotted line). The arrows in Figure 3 as well as all other figures indicate the direction of travel for the patroller. Large white and black triangles denote initial positions and headings before and after optimization, respectively. Since the patroller's sensor range is roughly 5 units (see Figure 2), the optimized trajectory is stretched out so that the sensor effectively reaches both sides of the channel. The initial trajectory has probability of detection 0.42145 and length of patrol 15, while the corresponding optimized numbers are 0.43348 and 24.001 as listed under T^* and P^* in Table I.

Case 2 in Table I is identical to Case 1 but requires a 360-degree heading change at the end of one patrolling period (i.e., 360-degree change in heading during one patrol cycle is forced, $n = 1$). Hence, the patroller must return to a heading shifted 360 degrees from the initial heading, which excludes a "bow-tie" type trajectory, but is compatible with a "racetrack" type trajectory. Figure 4 shows the corresponding initial trajectory (dotted line, probability of detection is 0.42587) and optimized trajectory (solid line, probability of detection is 0.43300). We note that the optimized probability of detection is slightly worse for $n = 1$ than for $n = 0$, 0.43348 versus 0.43300.

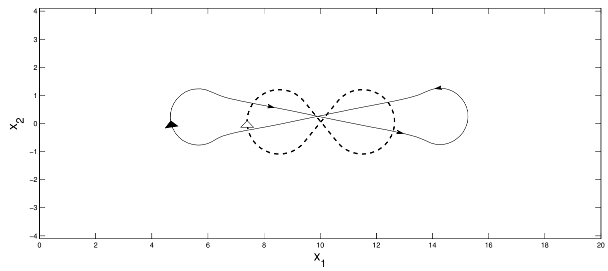

Fig. 3. Case 1: Initial trajectory (dotted line) and optimized trajectory (solid line) of a single patroller with no rotation ($n = 0$ in (26)). The arrows indicate direction of travel for the patroller. The white triangle denotes initial position and heading before the optimization, and the black triangle denotes the one after optimization.

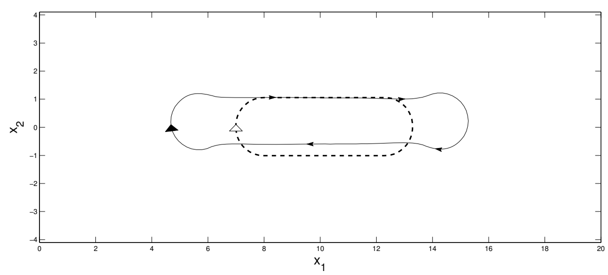

Fig. 4. Case 2: Initial trajectory (dotted line) and optimized trajectory (solid line) of a single patroller with 360-degree rotation ($n = 1$ in (26)).

In Cases 3 and 4 $T^{\max}$ is reduced to 15 and also the vertical movement restriction γ is relaxed to $L/5 = 4$. We see from Table I that these changes impose a restriction on the patroller and the probability of detection worsens.

B. Two Patrollers

Next we consider two patrollers, i.e., $q = 2$, and four additional cases as summarized in Table II. In all of these cases the patrol-duration limit $T^{\max} = 25$. Rows one and two of Table II give the optimized patrol duration and probability of detection for no rotation ($n = 0$) and 360-degree rotation ($n = 1$), respectively, using $\gamma = L/10$. We see again that no rotation (Case 5) results in better probability of detection. Figure 5 shows that the optimized trajectories are similar to "figure eights," even though the initial trajectories are similar to the infinity symbol. This effect is caused by the narrowness of the channel. The two patrollers obtain better probability of detection and less overlap in their "coverage" by moving along the channel instead of across. The probability of detection for the initial trajectory is 0.78003 and improves to 0.82037 after optimization.

We observe that the trajectories in Figure 5 are different for the two patrollers, which may be counterintuitive as the dis-

Case	n	γ	T^*	P^*
5	0	L/10	25.000	0.82037
6	1	L/10	11.633	0.79340
7	0	L/5	25.000	0.82354
8	1	L/5	25.000	0.81594

TABLE II

SUMMARY OF NUMERICAL RESULTS FOR TWO PATROLLERS

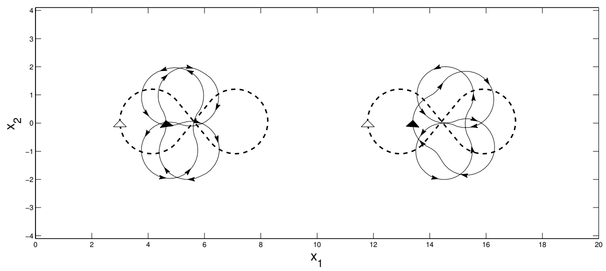

Fig. 5. Case 5: Initial trajectories (dotted line) and optimized trajectories (solid line) of two patrollers with no rotation ($n = 0$ in (26)).

tribution of the intruder's y^1-location is uniform. Additional calculations show that the trajectories in Figure 5 yield a larger probability of detection (0.82037) than patrol plans consisting of identical but translated trajectories for both patrollers. If the right-most patroller mimics the left-most patroller in Figure 5, but on the right side of the channel, then the probability of detection deteriorates to 0.81630. If the left-most patroller mimics the right-most patroller, then the probability of detection deteriorates to 0.81472. The probabilities deteriorate further when the patrollers carry out identical but mirror-imaged trajectories. These results provide new insight that is not easily obtained using the idealized calculations of [2], Chapter 9.

The optimized trajectories of Case 6 with the constraint of one rotation (i.e., $n = 1$) yield a probability of detection of 0.79340, which is worse than in Case 5 (i.e., $n = 0$). We also examined the configuration with one patroller constrained to no rotation ($n = 0$) and the other one to a 360-degree rotation ($n = 1$). However, the resulting probability of detection (0.81234) is worse than in Case 5.

Cases 7 and 8 in Table II show results similar to those for cases 5 and 6, but for $\gamma = L/5$. With this relaxation of the vertical movement constraint for the patrollers, we obtain slightly better probability of detection. The relaxation allows for more complicated patrol trajectories as shown in Figure 6. We see that the patrollers stagger vertically their trajectories to avoid overlap and therefore increase the probability of detection.

Such insight about the coordination between multiple patrollers cannot be reached through single-patroller analysis. The initial trajectories in Case 8 result in a probability of detection of 0.77806, which is improved to 0.81594 after optimization.

V. CONCLUSIONS

We formulated the channel patrol problem for multiple patrollers subject to turn-rate constraints as an optimal control problem. In this problem, the patrollers aim to maximize the probability of detecting an intruder that travels straight down a channel with constant speed. Using discretization of time and space, we obtained a large-scale nonlinear programming approximation of that problem which we solved to obtain an approximately stationary solution and a corresponding optimized trajectory for each patroller.

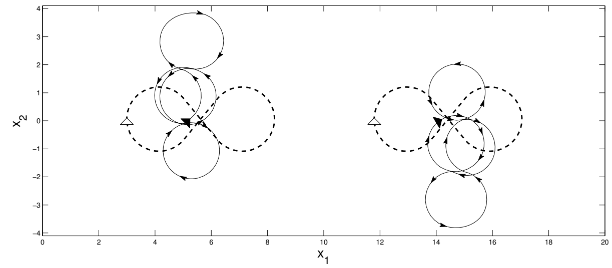

Fig. 6. Case 7: Initial trajectories (dotted line) and optimized trajectories (solid line) of two patrollers with no rotation ($n = 0$ in (26)) and relaxed vertical trajectory constraint.

In numerical tests, we found that simple "back-and-forth" trajectories across the channel are inferior to more complicated, optimized trajectories.

The results of this study provide new insight, not easily obtained using geometric calculations, into efficient patrol trajectory design for multiple patrollers in a narrow channel where interaction between the patrollers is unavoidable due to their limited turn rate.

REFERENCES

[1] A. R. Washburn, *Search and Detection*, 4th ed. Linthicum, Maryland: INFORMS, 2002.
[2] D. Wagner, W. C. Mylander, and T. Sanders, *Naval Operations Analysis*. Annapolis, MD: Naval Institute Press, 1999.
[3] B. Koopman, "Search and screening," Center for Naval Analysis, Alexandria, Virginia, Operations Evaluation Group Report 56, 1946.
[4] U. S. o. A. The Department of Navy, "The navy unmanned undersea vehicle (UUV) master plan," 2004.
[5] A. Washburn, "On patrolling a channel," *Naval Research Logistics*, vol. 29, pp. 609–615, 1982.
[6] S. J. Benkoski, M. G. Monticino, and J. R. Weisinger, "A survey of the search theory literature," *Naval Research Logistics*, vol. 38, no. 4, pp. 469–494, 1991.
[7] O. Hellman, "On the optimal search of a randomly moving target," *SIAM J. Applied Mathematics*, vol. 22, no. 4, pp. 545–552, 1972.
[8] A. Ohsumi, "Optimal search for a markovian target," *Naval Research Logistics*, vol. 38, pp. 531–554, 1991.
[9] G. Indiveri, "Kinematic time-invariant control of a 2d nonholonomic vehicle," in *IEEE Conference on Decision and Control*, 1999, pp. 2112–2117.
[10] E. Polak, *Optimization: Algorithms and Consistent Approximations*, ser. Applied Mathematical Sciences. Springer, 1997, vol. 124.
[11] H. Chung, E. Polak, J. O. Royset, and S. S. Sastry, "On the optimal detection of an underwater intruder in a channel using unmanned underwater vehicles," 2010, submitted to Naval Research Logistics.
[12] W. Murray, P. E. Gill, and M. A. Saunders, "SNOPT: An SQP algorithm for large-scale constrained optimization," *SIAM Journal on Optimization*, vol. 12, pp. 979–1006, 2002.
[13] K. Holmström, A. O. Göran, and M. M. Edvall, *User's Guide for TOMLAB*, Tomlab Optimization Inc., December 2006.