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## STABLE COUPLING OF NONCONFORMING, HIGH-ORDER FINITE DIFFERENCE METHODS\*

JEREMY E. KOZDON<sup>†</sup> AND LUCAS C. WILCOX<sup>†</sup>

**Abstract.** A methodology for handling block-to-block coupling of nonconforming, multiblock summation-by-parts finite difference methods is proposed. The coupling is based on the construction of projection operators that move a finite difference grid solution along an interface to a space of piecewise defined functions; we specifically consider discontinuous, piecewise polynomial functions. The constructed projection operators are compatible with the underlying summation-by-parts energy norm. Using the linear wave equation in two dimensions as a model problem, energy stability of the coupled numerical method is proven for the case of curved, nonconforming block-to-block interfaces. To further demonstrate the power of the coupling procedure, we show how it allows for the development of a provably energy stable coupling between curvilinear finite difference methods and a curved-triangle discontinuous Galerkin method. The theoretical results are verified through numerical simulations on curved meshes as well as eigenvalue analysis.

**Key words.** summation-by-parts, weak enforcement, high-order finite difference methods, coupling, stability, accuracy, projection operator, variational form, interface

**AMS subject classifications.** 65M06, 65M12, 65M50, 65M60, 65M70

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**1. Introduction.** Even though high-order multiblock finite difference methods are well suited for many problems, limitations arise for particularly complex geometries. For instance, most formulations require that grids conform at multiblock interfaces; that is, the grid lines must be continuous. This poses a challenge since resolution constraints in one portion of the domain can result in unnecessarily high resolution elsewhere in the domain. Furthermore, even though coordinate transforms enable the use of high-order finite difference methods for complex geometries, well-conditioned partitioning of complex domains into quadrilaterals in two dimensions and hexahedra in three dimensions can be challenging and/or impossible. The impact of this is a poorly conditioned Jacobian which has an adverse impact on the time step size and truncation error.

One approach to overcoming these complications is to relax the requirement that the grid and numerical methods conform across block interfaces. To do this, a variety of interpolation and projection techniques have been proposed, including the use of overlapping grids [2], strong enforcement using ghost points [18], or weak enforcement of continuity at block boundaries [11]. Here we particularly highlight the approach of Mattsson and Carpenter [11] as it is closely related to the work presented below on the discretization of hyperbolic equations using summation-by-parts (SBP) finite difference methods. In that paper compatibility conditions between interpolation operators and the underlying SBP finite difference method were presented which could be utilized to develop stable discretizations. The paper reported several compatible interpolation operators for fixed refinement ratio interfaces. The operators proposed by Mattsson and Carpenter required that at the block level the interfaces be con-

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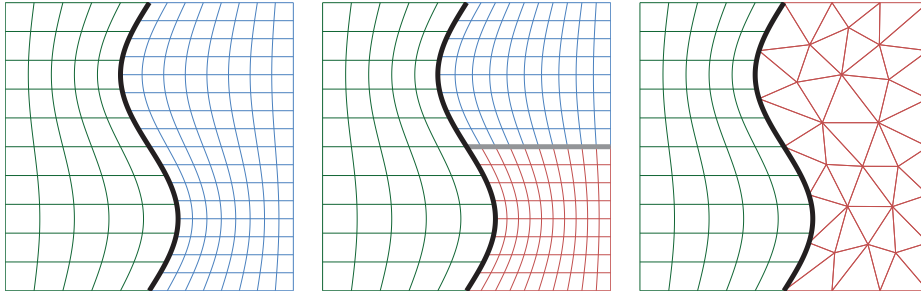


FIG. 1. Illustration of the computational grids supported by the projection operators described in this paper. On the left is a conforming two-block SBP grid. In the middle is a nonconforming ( $T$ -intersection) of three SBP blocks; here “nonconforming” refers to the SBP blocks and not the SBP grids. On the right is a coupled structured-unstructured grid; in this paper the methods used will be SBP finite differences and DG grid. In each illustration the thick line between blocks indicates coupling interfaces, i.e., the locations of the glue grids.

forming (i.e., the corners had to match), a restriction which has been removed by the work of Nissen et al. [13] (this later work retains the fixed refinement ratio requirement). To avoid geometric constraints, coupling high-order finite difference methods with unstructured grid methods has also been proposed. For example, Nordström and Gong [16] proposed coupling a high-order SBP method with an unstructured second-order finite volume method.

The class of finite difference methods to be considered in this work are SBP finite difference methods [8, 9, 19, 12]; see section 2 for the basic SBP ideas used in this work. One important feature of SBP methods is that the difference operator has an associated energy norm that discretely mimics integration by parts; this is referred to as the SBP property.

Here we present a general purpose technique for handling block-to-block coupling of nonconforming multiblock SBP finite difference methods. This technique also allows for the coupling with unstructured methods such as the discontinuous Galerkin (DG) method. A few possible grid couplings are illustrated in Figure 1. This coupling procedure uses projection operators that move finite difference grid solutions along the coupling interface to piecewise functions. It is with respect to the built-in norm of the SBP finite difference method that the projection operators are constructed; projection operators with the necessary properties are given in the electronic supplement. The SBP property alone is not sufficient to guarantee stability; SBP preserving boundary and interface closures are also required. In this work we will achieve this by enforcing all boundary and interface conditions weakly through the so-called simultaneous approximation term (SAT) method [1]; this is similar to the use of numerical flux terms in the DG method, a fact which will be exploited to couple SBP and DG methods stably.

Since the projection operators move the solution to a piecewise continuous representation (where projections are straightforward to construct), the operators only need to be constructed once for each finite difference operator. That is, projection operators can be constructed independently of the numerical method and grid on the other side of an interface. This independence of interface type is one of the features that enables development of provably stable couplings both between conforming and nonconforming SBP meshes and between SBP and DG methods. The fact that nonconforming grids can be accommodated enables the development of adaptive mesh

refinement codes using high-order, SBP finite difference methods.

For simplicity of presentation, we take as our model problem the two-dimensional linear acoustic wave equation in first-order form. We prove that the proposed coupling is stable for this system of equations as well as provide numerical evidence to confirm the analytical results. Since the projection operators are constructed based on the SBP operator and do not depend on the system of equations being solved, using standard techniques the extension to other linear symmetric hyperbolic systems should be possible.

**2. Definitions.** We begin by stating a few preliminary definitions that are at the heart of this work.

DEFINITION 1 (SBP property). A difference approximation  $\mathbf{D}$  is called a summation-by-parts (SBP) approximation to  $d/dx$  if it can be decomposed as  $\mathbf{D} = \mathbf{H}^{-1}\mathbf{Q}$  with  $\mathbf{H}$  being positive definite and  $\mathbf{Q}$  having the property  $\mathbf{Q} + \mathbf{Q}^T = \mathbf{B} = \text{Diag}[-1 \ 0 \ \cdots \ 0 \ 1]$ , i.e.,  $\mathbf{Q}$  is almost skew-symmetric.

To understand why such a difference approximation is called SBP, consider a grid function  $\mathbf{f} = [f_0 \ f_1 \ \cdots \ f_N]^T$ . The  $\mathbf{H}$ -weighted inner product of  $\mathbf{f}$  and  $\mathbf{D}\mathbf{f}$  gives

$$(1) \quad (\mathbf{f}, \mathbf{D}\mathbf{f})_{\mathbf{H}} = \mathbf{f}^T \mathbf{H} \mathbf{D} \mathbf{f} = \mathbf{f}^T \mathbf{Q} \mathbf{f} = \frac{1}{2} \mathbf{f}^T (\mathbf{Q} + \mathbf{Q}^T) \mathbf{f} = \frac{1}{2} (f_N^2 - f_0^2),$$

which is of the same form as the inner product of a continuously differentiable function  $f \in C^1[x_l, x_r]$  and  $df/dx$ :

$$(2) \quad \left( f, \frac{df}{dx} \right)_{L^2(x_l, x_r)} = \int_{x_l}^{x_r} f \frac{df}{dx} dx = \frac{1}{2} (f_r^2 - f_l^2),$$

where  $(x_l, x_r)$  is an open interval of the real line  $\mathbb{R}$ ,  $f_l = f(x_l)$ , and  $f_r = f(x_r)$ .

The difference operators commonly referred to as SBP methods are central difference operators in the interior (with orders 2, 4, 6, 8, ...) which transition to one-sided approximations near the boundary in such a way that the SBP property is achieved [8, 9, 12, 19]. This transition to one-sided approximations typically leads to a degradation in accuracy at the boundary. These SBP operators are subdivided into two classes: diagonal norm (diagonal  $\mathbf{H}$ ) and block norm (nondiagonal  $\mathbf{H}$ ) operators. For the diagonal norm operators the boundary accuracy can be at most half the interior accuracy; i.e., if the approximation is  $2q$ -accurate in the interior, it is at most  $q$ -accurate at the boundary. For the block norm operators it is possible to construct difference approximations that are  $(2q-1)$ -accurate at the boundary. In both cases the global accuracy of the scheme is one more than the boundary accuracy, i.e.,  $q+1$  for the diagonal norm operators and  $2q$  for the block norm operators [3]. For most practical calculations the diagonal norm operators are used, as they result in stable schemes for problems with coordinate transforms and variable coefficients [6, 7, 14, 15, 17]; a notable exception is the recent work of Mattsson and Almquist [10], where artificial dissipation is used to stabilize the block norm operators in complex geometries.

A key concept for this work is the definition of an SBP  $\mathbf{H}$ -compatible projection operator. This operator will allow us to move from a grid function to a space of piecewise continuous functions in a manner that is compatible (in an  $L^2(\Gamma)$  sense) with the SBP finite difference method. We call the space of piecewise continuous functions the glue grid since it allows us to “glue” together differing numerical methods.

To make this more concrete, given a finite difference grid  $[x_0 \ x_1 \ \cdots \ x_N]$ , let  $\mathcal{G}_h \subset L^2(\Gamma)$  be a finite dimensional space of functions, i.e., the space of functions the glue grid can represent. Let  $\boldsymbol{\psi}(\eta) = [\psi_0(\eta) \ \psi_1(\eta) \ \cdots \ \psi_K(\eta)]^T$  be a vector

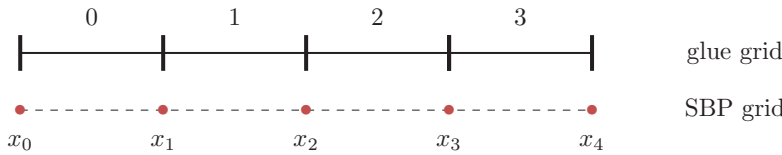


FIG. 2. Alignment of the glue grid, the line with interval boundaries denoted with hatch marks, and the finite difference grid, denoted with dots representing the grid points, at the leftmost boundary. The glue grid supports functions which are continuous on each interval, e.g., piecewise continuous functions. Note that the glue grid and the finite difference grid coincide spatially, but here we have separated them vertically for display purposes.

of linearly independent basis functions for  $\mathcal{G}_h$ ,  $\mathbf{f} = [f_0 \ f_1 \ \cdots \ f_N]^T$  be a grid function, and  $\mathbf{H}$  be an SBP norm. Our goal is to define a projection operator so that a set of coefficients  $\bar{\mathbf{f}} = [\bar{f}_0 \ \bar{f}_1 \ \cdots \ \bar{f}_K]^T$  can be defined from  $\mathbf{f}$  such that  $\bar{f}(\eta) = \sum_i \bar{f}_i \psi_i(\eta) = \bar{\mathbf{f}}^T \boldsymbol{\psi}(\eta)$  is a compatible representation of the grid function  $\mathbf{f}$  in the space  $\mathcal{G}_h$ . Note that throughout the paper we use the overline notation to represent quantities defined on the glue grid.

In order to define the  $\mathbf{H}$ -compatible projection operators we must define the mass matrix on the glue grid. Namely, the symmetric, positive definite mass matrix is  $\mathbf{M} = \int_{\Gamma} \boldsymbol{\psi}(\eta) \boldsymbol{\psi}^T(\eta) \, d\eta$ ; thus, given two functions  $\bar{f}(\eta) = \bar{\mathbf{f}}^T \boldsymbol{\psi}(\eta)$  and  $\bar{g}(\eta) = \bar{\mathbf{g}}^T \boldsymbol{\psi}(\eta)$  in  $\mathcal{G}_h$ , the inner product is  $(\bar{f}, \bar{g}) = \bar{\mathbf{f}}^T \mathbf{M} \bar{\mathbf{g}}$ .

DEFINITION 2 ( $\mathbf{H}$ -compatible projection operator). Let  $\mathbf{f}$  be a grid function and  $\bar{\mathbf{u}}(\eta) = \bar{\mathbf{u}}^T \boldsymbol{\psi}(\eta) \in \mathcal{G}_h$  be a glue grid function. We call the projection matrices  $\mathbf{P}_{f2g}$  and  $\mathbf{P}_{g2f}$   $\mathbf{H}$ -compatible if for all  $\mathbf{f}$  and  $\bar{\mathbf{u}}$

$$(3) \quad \mathbf{u}^T \mathbf{H} \mathbf{f} = \bar{\mathbf{u}}^T \mathbf{M} \bar{\mathbf{f}},$$

where  $\bar{\mathbf{f}} = \mathbf{P}_{f2g} \mathbf{f}$  and  $\mathbf{u} = \mathbf{P}_{g2f} \bar{\mathbf{u}}$ , or equivalently

$$(4) \quad \mathbf{P}_{g2f}^T \mathbf{H} = \mathbf{M} \mathbf{P}_{f2g}.$$

Here the subscript  $f2g$  signifies that the projection goes from the finite difference grid to the glue grid, and  $g2f$  signifies that the projection goes from the glue grid to the finite difference grid.

Notice that nothing in the definition implies that these solutions must be accurate representations of one another, and Definition 2 will only be used to guarantee stability. Furthermore, there is no statement that the functions can be moved between spaces without error; that is, we do not assume that  $\mathbf{P}_{g2f} \mathbf{P}_{f2g} \mathbf{v} = \mathbf{v}$  or that  $\mathbf{P}_{f2g} \mathbf{P}_{g2f} \bar{\mathbf{u}} = \bar{\mathbf{u}}$ .

It is natural to augment Definition 2 with a set of accuracy conditions based on the particular glue grid space  $\mathcal{G}_h$  being used. In this work, we let  $\mathcal{G}_h$  be the space of discontinuous, piecewise polynomials where the intervals over which the polynomials are defined align with the finite difference points as shown in Figure 2. Motivated by Mattsson and Carpenter [11], we require that the operators used in this work satisfy a set of polynomial accuracy conditions. Namely with a glue grid that can represent  $q$ th order polynomials exactly, we define the  $q$ th order polynomial grid function as  $\mathbf{f}_q = [0^q \ 1^q \ \cdots \ N^q]^T$  (with the convention that  $0^0 = 1$ ) and let  $\bar{g}_q(\eta) = \bar{\mathbf{g}}_q^T \boldsymbol{\psi}(\eta)$  be the same polynomial on the glue grid. We then require that the errors

$$(5) \quad \mathbf{e}_{g2f} = \mathbf{P}_{g2f} \bar{\mathbf{g}}_q - \mathbf{f}_q, \quad \mathbf{e}_{f2g} = \mathbf{P}_{f2g} \mathbf{f}_q - \bar{\mathbf{g}}_q$$

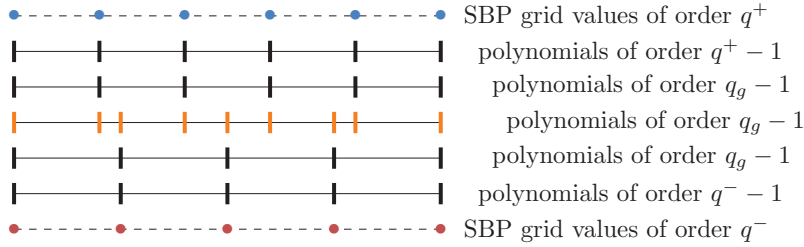


FIG. 3. Example of the alignment of a glue grid with two SBP finite difference grids. On the finite difference grids values are stored at the nodal locations, whereas the glue grids store polynomials over the indicated intervals. The projection operators associated with the SBP finite difference grid are defined to go to the glue grid that conforms to the SBP grid nodes. Since both sides of the interface do not have the same space of piecewise polynomials, additional projection operators are needed to move between the polynomial spaces.

be zero for all polynomials up to order  $q_i - 1$  everywhere except near the boundary, where it is required that the error be zero for polynomials up to order  $q_b - 1$ ; here  $q_i$  and  $q_b$  are the interior and boundary accuracy of the SBP finite difference method being used. In other words, we require the projection operators to mimic the accuracy of the SBP finite difference method. These are the same accuracy conditions used in the finite difference to finite difference operators of Mattsson and Carpenter [11]. The method we use to construct projection operators that satisfy the above accuracy and stability conditions using a space of discontinuous, piecewise polynomials is discussed in Appendix B. Additionally, the electronic supplement to this paper contains code to generate the operators as well as the operators themselves.

We emphasize again that the only requirement for a projection operator to result in a stable discretization is Definition 2. The accuracy conditions (5) only pertain to the specific choice of  $\mathcal{G}_h$  in this paper, and other conditions may be required for different glue grid spaces.

The  $\mathbf{H}$ -compatible projection operators discussed in Appendix B move between an SBP finite difference solution and a given set of piecewise polynomials of order  $q - 1$ , where  $q$  is the accuracy of the finite difference method; see, for example, Figure 2. To make these operators useful in practice we need to be able to transition between glue grid spaces.

To understand why this is, consider the situation shown in Figure 3. Here an interface between two SBP operators with different grids and orders of accuracy is shown. Since the operators constructed in Appendix B move between finite difference grid values and a fixed set of piecewise continuous polynomials (fixed intervals and orders), the glue grids defined for either side of the interface in Figure 3 will not conform (i.e., the polynomials may be of a different order, and/or the locations of the interval boundaries may be different). Thus, additional projection operators are needed to move between the different polynomial orders and intervals.

The projection operators in this work are constructed in a hierarchical fashion where at each stage we need to construct projections between two different piecewise polynomial glue grid spaces where one is the subset of the other; i.e.,  $G_h^a \subset G_h^b$ . The projection operators between spaces are constructed to satisfy an analogue of (4), namely,

$$(6) \quad \mathbf{M}_a \mathbf{P}_{g_b, 2g_a} = \mathbf{P}_{g_a, 2g_b}^T \mathbf{M}_b,$$

where  $\mathbf{M}_a$  and  $\mathbf{M}_b$  are the respective glue grid mass matrices (which are each symmet-

ric positive definite given linearly independent basis functions). Since  $G_h^a \subset G_h^b$ , then  $\mathbf{P}_{g_a 2g_b}$  is a basis transformation operation which can be determined independently of  $\mathbf{P}_{g_b 2g_a}$ , and we have

$$(7) \quad \mathbf{P}_{g_b 2g_a} = \mathbf{M}_a^{-1} \mathbf{P}_{g_a 2g_b}^T \mathbf{M}_b.$$

LEMMA 3. *Suppose  $\mathbf{P}_{f 2g_a}$  and  $\mathbf{P}_{g_a 2f}$  satisfy (4) with  $\mathbf{M} = \mathbf{M}_a$ . Furthermore, suppose  $\mathbf{P}_{g_b 2g_a}$  and  $\mathbf{P}_{g_a 2g_b}$  satisfy (7); then*

$$(8) \quad \mathbf{P}_{f 2g_b} = \mathbf{P}_{g_a 2g_b} \mathbf{P}_{f 2g_a}, \quad \mathbf{P}_{g_b 2f} = \mathbf{P}_{g_a 2f} \mathbf{P}_{g_b 2g_a}$$

*satisfy (4) with  $\mathbf{M} = \mathbf{M}_b$ .*

*Proof.* By direct calculation we have

$$(9) \quad \begin{aligned} \mathbf{P}_{g_b 2f}^T \mathbf{H} &= \mathbf{P}_{g_b 2g_a}^T \mathbf{P}_{g_a 2f}^T \mathbf{H} = \mathbf{P}_{g_b 2g_a}^T \mathbf{M}_a \mathbf{P}_{f 2g_a} \\ &= \mathbf{M}_b \mathbf{P}_{g_a 2g_b} \mathbf{P}_{f 2g_a} = \mathbf{M}_b \mathbf{P}_{f 2g_b}. \end{aligned} \quad \square$$

Furthermore, it follows from the nesting of the glue spaces that the accuracy conditions (5) are also satisfied by these compositions of projection operators. Due to this and Lemma 3, we may assume without loss of generality that  $\mathbf{P}_{f 2g}$  and  $\mathbf{P}_{g 2f}$  project all the way through to the finest glue space (represented by the middle glue grid in Figure 3); that is, the intermediate spaces are not explicitly considered further in this work.

**3. Acoustic wave equation: SBP-SAT discretization.** As a model problem we consider the two-dimensional acoustic wave equation in first-order form:

$$(10) \quad \rho \frac{\partial v_i}{\partial t} + \frac{\partial p}{\partial x_i} = 0 \quad (i = 1, 2), \quad \frac{\partial p}{\partial t} + \lambda \left( \frac{\partial v_1}{\partial x_1} + \frac{\partial v_2}{\partial x_2} \right) = 0,$$

where  $v_1$  and  $v_2$  are the particle velocities in the  $x_1$  and  $x_2$  directions, respectively, and  $p$  is the pressure. Here,  $\rho$  is the material density, and  $\lambda$  is Lamé’s second parameter, where we assume that  $\rho, \lambda > 0$ .

We are interested in discretizing (10) on a domain  $\Omega$  which is the union of curvilinear, quadrilateral domains (blocks)  $\{\Omega_e\}$ . To do so, we transform each domain from the physical space  $\Omega_e$  to the reference space  $\tilde{\Omega} = [-1, 1] \times [-1, 1]$  via the coordinate transform  $x_i = x_i(\xi_1, \xi_2)$ ,  $i = 1, 2$ , with  $x_i$  being the coordinates in the physical domain and  $\xi_i$  being the coordinates in the reference domain; we assume that the inverse transforms  $\xi_i(x_1, x_2)$  also exist. (Note that for simplification of notation we suppress the geometrical terms’ dependence on each domain.) The Jacobian determinant is

$$(11) \quad J = \frac{\partial x_1}{\partial \xi_1} \frac{\partial x_2}{\partial \xi_2} - \frac{\partial x_2}{\partial \xi_1} \frac{\partial x_1}{\partial \xi_2},$$

which gives rise to the metric relations

$$(12) \quad J \frac{\partial \xi_1}{\partial x_1} = \frac{\partial x_2}{\partial \xi_2}, \quad J \frac{\partial \xi_1}{\partial x_2} = -\frac{\partial x_1}{\partial \xi_2}, \quad J \frac{\partial \xi_2}{\partial x_2} = \frac{\partial x_1}{\partial \xi_1}, \quad J \frac{\partial \xi_2}{\partial x_1} = -\frac{\partial x_2}{\partial \xi_1}.$$

With these definitions, the acoustic wave equation (10) can be written as

$$(13) \quad \rho J \frac{\partial v_i}{\partial t} + \frac{\partial}{\partial \xi_1} \left( J \frac{\partial \xi_1}{\partial x_i} p \right) + \frac{\partial}{\partial \xi_2} \left( J \frac{\partial \xi_2}{\partial x_i} p \right) = 0, \quad i = 1, 2,$$

$$(14) \quad J \frac{\partial p}{\partial t} + \lambda \left( J \frac{\partial \xi_1}{\partial x_1} \frac{\partial v_1}{\partial \xi_1} + J \frac{\partial \xi_2}{\partial x_1} \frac{\partial v_1}{\partial \xi_2} + J \frac{\partial \xi_1}{\partial x_2} \frac{\partial v_2}{\partial \xi_1} + J \frac{\partial \xi_2}{\partial x_2} \frac{\partial v_2}{\partial \xi_2} \right) = 0.$$

Notice that we have written the transformed equations in skew-symmetric form with the velocity equations (13) written using a conservative transform and the pressure equation (14) using a nonconservative transform. It is common to do both terms conservatively, but doing this splitting results in a provably stable scheme.

Before presenting an SBP discretization of the governing equations, we first introduce a variational form of the equations on each domain. This is done to highlight the close connection between SBP finite difference methods and DG methods. To do this, we introduce test functions  $w_i$ ,  $i = 1, 2$ , and  $\varphi$  which belong to some appropriately chosen space. Multiplying the velocity equation (13) by  $w_i$ , the pressure equation (14) by  $\varphi$ , and integrating over a domain  $\tilde{\Omega}$  gives

$$\begin{aligned}
 (15) \quad & \int_{\tilde{\Omega}} w_i \left[ \rho J \frac{\partial v_i}{\partial t} + \frac{\partial}{\partial \xi_1} \left( J \frac{\partial \xi_1}{\partial x_i} p \right) + \frac{\partial}{\partial \xi_2} \left( J \frac{\partial \xi_2}{\partial x_i} p \right) \right] dA \\
 & = - \int_{\partial \tilde{\Omega}} w_i S_J n_i (p^* - p) ds, \quad i = 1, 2, \\
 (16) \quad & \int_{\tilde{\Omega}} \varphi \left[ J \frac{\partial p}{\partial t} + \lambda \left( J \frac{\partial \xi_1}{\partial x_1} \frac{\partial v_1}{\partial \xi_1} + J \frac{\partial \xi_2}{\partial x_1} \frac{\partial v_1}{\partial \xi_2} + J \frac{\partial \xi_1}{\partial x_2} \frac{\partial v_2}{\partial \xi_1} + J \frac{\partial \xi_2}{\partial x_2} \frac{\partial v_2}{\partial \xi_2} \right) \right] dA \\
 & = - \int_{\partial \tilde{\Omega}} \varphi \lambda S_J (v^* - v) ds,
 \end{aligned}$$

where  $p^*$  and  $v^*$  are penalty terms (also known as numerical fluxes) that satisfy the boundary or interface conditions that connect the domains. As discussed below in the discretization, these values are derived from the numerical solution along the edge of the block; in the case of interfaces, the numerical solution from both sides of the interface is considered. For most methods, these values are the same (up to a possible sign change) on either side of an interface, though in this work we will need to relax this to account for a possible projection error, that is, the fact that  $\mathbf{P}_{g2f} \mathbf{P}_{f2g}$  is not an identity operation. Here,  $S_J$  is the surface Jacobian,  $n_1$  and  $n_2$  are the components of the outward pointing unit normal (in the  $x_1$  and  $x_2$  directions, respectively), and  $v = n_1 v_1 + n_2 v_2$  is the normal component of velocity. For the reference domain  $\tilde{\Omega} = [-1, 1] \times [-1, 1]$  (which will be used for the finite difference discretization) the surface Jacobian terms and outward pointing normals for the edge defined by  $\xi_i = \pm 1$  are

$$(17) \quad S_J = J \sqrt{\left( \frac{\partial \xi_i}{\partial x_2} \right)^2 + \left( \frac{\partial \xi_i}{\partial x_1} \right)^2}, \quad n_1 = \pm \frac{J}{S_J} \frac{\partial \xi_i}{\partial x_2}, \quad n_2 = \pm \frac{J}{S_J} \frac{\partial \xi_i}{\partial x_1}.$$

With the above definition, the boundary integrals can be rewritten as

$$\begin{aligned}
 (18) \quad & \int_{\partial \tilde{\Omega}} w_i S_J n_i (p^* - p) ds \\
 & = \int_{-1}^1 [w_i S_J n_i (p^* - p)]_{\xi_1=-1} d\xi_2 + \int_{-1}^1 [w_i S_J n_i (p^* - p)]_{\xi_1=1} d\xi_2 \\
 & \quad + \int_{-1}^1 [w_i S_J n_i (p^* - p)]_{\xi_2=-1} d\xi_1 + \int_{-1}^1 [w_i S_J n_i (p^* - p)]_{\xi_2=1} d\xi_1,
 \end{aligned}$$



$$\begin{aligned}
 & \int_{\partial\bar{\Omega}} \varphi \lambda S_J (v^* - v) ds \\
 (19) \quad &= \int_{-1}^1 [\varphi \lambda S_J (v^* - v)]_{\xi_1=-1} d\xi_2 + \int_{-1}^1 [\varphi \lambda S_J (v^* - v)]_{\xi_1=1} d\xi_2 \\
 & \quad + \int_{-1}^1 [\varphi \lambda S_J (v^* - v)]_{\xi_2=-1} d\xi_1 + \int_{-1}^1 [\varphi \lambda S_J (v^* - v)]_{\xi_2=1} d\xi_1.
 \end{aligned}$$

Going back to the differential form of the equations, we discretize the reference domain  $\bar{\Omega}$  with an  $(N_1 + 1) \times (N_2 + 1)$  grid of equally spaced points. The grid spacing in the  $\xi_i$  dimension is  $h_i = 2/N_i$  for  $i = 1, 2$ . Thus the  $(k, l)$  grid point is at  $(\xi_1, \xi_2) = (kh_1 - 1, lh_2 - 1)$  for  $k = 0, \dots, N_1$  and  $l = 0, \dots, N_2$ . We define the pressure solution vector on the grid as

$$(20) \quad \mathbf{p} = [p_{00} \quad p_{01} \quad \cdots \quad p_{0N_2} \quad p_{10} \quad \cdots \quad p_{N_1N_2}]^T,$$

where  $p_{kl}$  approximates the pressure  $p$  at grid point  $(k, l)$ ; the solution vectors  $\mathbf{v}_1$  and  $\mathbf{v}_2$  are similarly defined. An SBP-SAT semidiscretization (discretization only in space) of (13)–(14) on a domain  $\Omega_e$  using an  $(N_1 + 1) \times (N_2 + 1)$  grid is then

$$(21) \quad \rho \mathbf{J} \frac{d\mathbf{v}_i}{dt} + \mathbf{D}_1 \mathbf{J} \frac{\partial \xi_1}{\partial \mathbf{x}_i} \mathbf{p} + \mathbf{D}_2 \mathbf{J} \frac{\partial \xi_2}{\partial \mathbf{x}_i} \mathbf{p} = -\mathbf{H}^{-1} \mathcal{F}_{v_i}, \quad i = 1, 2,$$

$$(22) \quad \mathbf{J} \frac{\partial \mathbf{p}}{\partial t} + \lambda \left( \mathbf{J} \frac{\partial \xi_1}{\partial \mathbf{x}_1} \mathbf{D}_1 \mathbf{v}_1 + \mathbf{J} \frac{\partial \xi_2}{\partial \mathbf{x}_1} \mathbf{D}_2 \mathbf{v}_1 + \mathbf{J} \frac{\partial \xi_1}{\partial \mathbf{x}_2} \mathbf{D}_1 \mathbf{v}_2 + \mathbf{J} \frac{\partial \xi_2}{\partial \mathbf{x}_2} \mathbf{D}_2 \mathbf{v}_2 \right) = -\lambda \mathbf{H}^{-1} \mathcal{F}_p.$$

Here, we have defined the matrices

$$(23) \quad \mathbf{H} = \mathbf{H}_{N_1} \otimes \mathbf{H}_{N_2}, \quad \mathbf{D}_1 = \mathbf{D}_{N_1} \otimes \mathbf{I}_{N_2}, \quad \mathbf{D}_2 = \mathbf{I}_{N_1} \otimes \mathbf{D}_{N_2},$$

where  $\mathbf{I}_{N_i}$ ,  $\mathbf{H}_{N_i}$ , and  $\mathbf{D}_{N_i}$  are all matrices of size  $(N_i + 1) \times (N_i + 1)$  with  $\mathbf{I}_{N_i}$  being the identity matrix and  $\mathbf{H}_{N_i}$  and  $\mathbf{D}_{N_i}$  being the one-dimensional SBP finite difference operators in the  $\xi_i$  direction. The vectors  $\mathbf{v}_1$ ,  $\mathbf{v}_2$ , and  $\mathbf{p}$  are the unknown velocities and pressures at the finite difference grid points. The diagonal matrices  $\mathbf{J}$ ,  $\frac{\partial \xi_1}{\partial \mathbf{x}_1}$ ,  $\frac{\partial \xi_1}{\partial \mathbf{x}_2}$ ,  $\frac{\partial \xi_2}{\partial \mathbf{x}_1}$ , and  $\frac{\partial \xi_2}{\partial \mathbf{x}_2}$  have the respective geometric factors evaluated at the finite difference grid points along their diagonal. For example, letting  $J_{kl}$  denote the Jacobian determinant (or its approximation) at grid point  $(k, l)$ , we define

$$(24) \quad \mathbf{J} = \text{Diag} [J_{00} \quad J_{01} \quad \cdots \quad J_{0N_2} \quad J_{10} \quad \cdots \quad J_{N_1N_2}],$$

where  $\text{Diag}(\cdot)$  constructs a diagonal matrix from a given vector; the other diagonal matrices are defined similarly. For simplicity of the discussion we assume that the material parameters  $\rho$  and  $\lambda$  are constants in  $\Omega$ .

Before stating the specific form for the penalty terms  $\mathcal{F}_{v_i}$  and  $\mathcal{F}_p$ , we note the similarities between the SBP-SAT discretization and a DG method based on the integral form. Namely, if  $\mathbf{H}$  is interpreted as an elemental mass matrix, then the left-hand sides of (21)–(22) correspond to the body integral terms in (15)–(16). Similarly, the right-hand side terms correspond to the boundary integrals. Thus, the SBP-SAT discretization and the DG method have the same discrete structure. Additionally, what are commonly referred to as penalty terms in SBP-SAT finite difference methods are

of the same form as the numerical flux terms in DG methods. This is important because even though the methods are developed using different continuous representations (i.e., differential versus variational form), these similarities will facilitate the stable coupling using the penalty and flux terms; see section 4.2.

The penalty terms in (21)–(22) are taken to be of the form

$$(25) \quad \mathcal{F}_{v_i} = \left( \mathbf{e}_W \otimes \mathcal{F}_{v_i}^W \right) + \left( \mathbf{e}_E \otimes \mathcal{F}_{v_i}^E \right) + \left( \mathcal{F}_{v_i}^S \otimes \mathbf{e}_S \right) + \left( \mathcal{F}_{v_i}^N \otimes \mathbf{e}_N \right),$$

$$(26) \quad \mathcal{F}_p = \left( \mathbf{e}_W \otimes \mathcal{F}_p^W \right) + \left( \mathbf{e}_E \otimes \mathcal{F}_p^E \right) + \left( \mathcal{F}_p^S \otimes \mathbf{e}_S \right) + \left( \mathcal{F}_p^N \otimes \mathbf{e}_N \right).$$

Here the subscripts and superscripts  $W$ ,  $E$ ,  $S$ , and  $N$  are used to denote which side of the domain the penalty term correspond to. For instance,  $W$  and  $E$  correspond to the *west* and *east* sides of the domain with  $\xi_1 = -1$  and  $\xi_1 = 1$ , respectively. Similarly,  $S$  and  $N$  correspond to the *south* and *north* with  $\xi_2 = -1$  and  $\xi_2 = 1$ . The vectors  $\mathbf{e}_W$  and  $\mathbf{e}_E$  have length  $N_1 + 1$  and are zero everywhere except the first and last entries, respectively, where they are 1, i.e.,

$$(27) \quad \mathbf{e}_W = [1 \ 0 \ \cdots \ 0]^T \quad \text{and} \quad \mathbf{e}_E = [0 \ \cdots \ 0 \ 1]^T;$$

the vectors  $\mathbf{e}_S$  and  $\mathbf{e}_N$  are defined analogously.

The vectors  $\mathcal{F}_{v_i}^W$  and  $\mathcal{F}_p^W$ , of length  $N_2 + 1$ , are the actual penalty terms (or flux differences) along the west side. These vectors are taken to have the form

$$(28) \quad \mathcal{F}_{v_i}^W = \mathbf{H}_2 \mathbf{S}_{JW} \mathbf{n}_i^W (\mathbf{p}_W^* - \mathbf{p}_W), \quad \mathcal{F}_p^W = \mathbf{H}_2 \mathbf{S}_{JW} \lambda (\mathbf{v}_W^* - \mathbf{v}_W).$$

Here  $\mathbf{S}_{JW}$  and  $\mathbf{n}_i^W$  are  $(N_2 + 1) \times (N_2 + 1)$  diagonal matrices with elements corresponding to the surface Jacobian terms and outward pointing unit normals along the west face; see (17). The vectors  $\mathbf{p}_W$  and  $\mathbf{v}_W$ , of length  $N_2 + 1$ , are the pressure and normal components of velocity ( $v = v_1 n_1 + v_2 n_2$ ), respectively, at grid points along the west face. Finally, the vectors  $\mathbf{p}_W^*$  and  $\mathbf{v}_W^*$  will be set based on the interface and/or boundary conditions for the block. In the case of interface conditions, these edge values will need to be set in a consistent manner across the interface. As noted above, these penalty terms correspond to the numerical fluxes in DG methods, and stability results through a judicious choice of penalty or flux.

Stability of the semidiscrete discretization, whether a pure multiblock SBP-SAT discretization or a coupled SBP-DG method, will be based on energy analysis. To do so, we define the energy in a single SBP block as

$$(29) \quad E = \frac{\rho}{2} \mathbf{v}_1^T \mathbf{JH} \mathbf{v}_1 + \frac{\rho}{2} \mathbf{v}_2^T \mathbf{JH} \mathbf{v}_2 + \frac{1}{2\lambda} \mathbf{p}^T \mathbf{JH} \mathbf{p}$$

and define the total energy in the solution as

$$(30) \quad \mathcal{E} = \sum_{blocks} E.$$

Since the governing equations (10) are energy conservative with the free surface boundary condition, it is appropriate to use the following definition of discrete stability [4].

DEFINITION 4 (energy stability). *The semidiscrete discretization is said to be stable if*

$$(31) \quad \frac{d\mathcal{E}}{dt} \leq 0.$$

For a single SBP block, the energy dissipation rate is characterized by the following lemma.

LEMMA 5. *The single SBP block discretization (21)–(22) with penalty terms (25)–(26) of the form of (28) has the energy dissipation rate*

$$(32) \quad \frac{dE}{dt} = \sum_{K=\{W,E,S,N\}} \mathcal{D}_K,$$

$$(33) \quad \mathcal{D}_K = -\mathbf{v}_K^T \mathbf{H}_K \mathbf{S}_{JK} \mathbf{p}_K^* + \mathbf{v}_K^T \mathbf{H}_K \mathbf{S}_{JK} \mathbf{p}_K - (\mathbf{v}_K^*)^T \mathbf{H}_K \mathbf{S}_{JK} \mathbf{p}_K,$$

where  $\mathcal{D}_K$  is the dissipation rate along edge  $K$  of the block with  $\mathbf{H}_W = \mathbf{H}_E = \mathbf{H}_2$  and  $\mathbf{H}_S = \mathbf{H}_N = \mathbf{H}_1$ .

*Proof.* See Appendix A. □

The implication of the lemma is that the energy dissipation rate for a single block is the sum of the dissipation rate for each of its edges. Thus, we can prove global semi-discrete energy stability by showing that energy is dissipated across every interface and boundary.

The crux of a stable coupling is then choosing  $\mathbf{p}^*$  and  $\mathbf{v}^*$  such that when (32) is summed over all blocks  $d\mathcal{E}/dt \leq 0$ . Before continuing on to present how the interface terms  $\mathbf{p}^*$  and  $\mathbf{v}^*$  are formulated when projection operators are used, we consider the form the penalty terms take for an exterior boundary and when the interface is conforming (matching grid and SBP finite difference scheme across interface).

**3.1. Exterior boundary treatment.** Since the focus of this work is interface treatment, we only consider the zero pressure boundary condition  $p = 0$ . Numerically, through the penalty terms, this enforcement is done through a linear combination of a central and upwind biased penalty; here by “central” we mean a penalty term that leads to no energy dissipation. If a block edge  $K \in \{W, E, S, N\}$  is an outer boundary, the penalty terms are taken to be of the form

$$(34) \quad \mathbf{p}_K^* - \mathbf{p}_K = -\mathbf{p}_K, \quad \mathbf{v}_K^* - \mathbf{v}_K = \alpha \frac{\mathbf{p}_K}{Z},$$

where  $Z = \sqrt{\rho/\lambda} > 0$  is the impedance of the material. Here the parameter  $\alpha \geq 0$  has been introduced, with  $\alpha = 0$  being the central boundary treatment and  $\alpha = 1$  the fully upwind boundary treatment. The following lemma ensures that the external boundary treatment is dissipative.

LEMMA 6. *If edge  $K \in \{W, E, S, N\}$  of an SBP block is an exterior boundary with penalty terms of the form (34), then the energy dissipation rate for the edge is*

$$(35) \quad \mathcal{D}_K = -\frac{\alpha}{Z} \mathbf{p}_K^T \mathbf{H}_K \mathbf{S}_{JK} \mathbf{p}_K,$$

which is nonpositive if  $\alpha \geq 0$ .

*Proof.* Solving penalty term (34) for  $\mathbf{p}_K^*$  and  $\mathbf{v}_K^*$  gives

$$(36) \quad \mathbf{p}_K^* = 0, \quad \mathbf{v}_K^* = \mathbf{v}_K + \alpha \frac{\mathbf{p}_K}{Z},$$

and (35) follows immediately after substituting  $\mathbf{p}_K^*$  and  $\mathbf{v}_K^*$  into the edge dissipation rate (33). The nonpositiveness of (35) follows from the fact that  $\mathbf{H}_K$  and  $\mathbf{S}_{JK}$  are diagonal, positive definite matrices. □

**3.2. Conforming interface treatment.** We call a block interface *conforming* when the grid and the  $\mathbf{H}$ -norm are the same on both sides of the interface; the latter condition typically implies that the same SBP finite difference method is being used on both sides of the interface. The interface conditions are continuity of pressure and the normal component of velocity:

$$(37) \quad p^+ = p^-, \quad v^+ = -v^-.$$

Here we have introduced the superscripts  $+$  and  $-$  to denote the two sides of the interface. Recall that  $v$  is the normal component of the velocity, and thus the minus sign in the velocity condition is due to the fact that the normals are equal and opposite on either side of the interface. For the minus side of the interface the penalty terms can then be written as a combination of the central and upwind penalties:

$$(38) \quad \mathbf{p}^* - \mathbf{p} = \mathbf{p}^* - \mathbf{p}^- = \frac{1}{2}(\mathbf{p}^+ - \mathbf{p}^-) + \alpha \frac{Z}{2}(\mathbf{v}^+ + \mathbf{v}^-),$$

$$(39) \quad \mathbf{v}^* - \mathbf{v} = \mathbf{v}^* - \mathbf{v}^- = -\frac{1}{2}(\mathbf{v}^+ + \mathbf{v}^-) - \alpha \frac{1}{2Z}(\mathbf{p}^+ - \mathbf{p}^-),$$

where stability results when  $\alpha \geq 0$ , and the central penalty (zero energy dissipation) is achieved when  $\alpha = 0$  and the upwind penalty with  $\alpha = 1$ .

LEMMA 7. Consider a single, conforming interface between two SBP blocks with penalty terms of the form (38)–(39). Let  $\mathcal{D}^-$  and  $\mathcal{D}^+$  be the energy dissipation rate along each side of the interface; then

$$(40) \quad \begin{aligned} \mathcal{D}^- + \mathcal{D}^+ = & -\alpha \frac{Z}{2}(\mathbf{v}^- + \mathbf{v}^+)^T \mathbf{H} \mathbf{S}_J (\mathbf{v}^- + \mathbf{v}^+) \\ & - \alpha \frac{1}{2Z}(\mathbf{p}^- - \mathbf{p}^+)^T \mathbf{H} \mathbf{S}_J (\mathbf{p}^- - \mathbf{p}^+) \end{aligned}$$

is nonpositive for  $\alpha \geq 0$ .

*Proof.* Solving (38)–(39) for  $\mathbf{p}^*$  and  $\mathbf{v}^*$  on the minus side of the interface gives

$$(41) \quad \mathbf{p}^* = \frac{1}{2}(\mathbf{p}^+ + \mathbf{p}^-) + \alpha \frac{Z}{2}(\mathbf{v}^+ + \mathbf{v}^-),$$

$$(42) \quad \mathbf{v}^* = \frac{1}{2}(\mathbf{v}^- - \mathbf{v}^+) + \alpha \frac{1}{2Z}(\mathbf{p}^- - \mathbf{p}^+).$$

Substituting  $\mathbf{p}^*$  and  $\mathbf{v}^*$  into (33) on the minus side of the interface results in (after some simplification)

$$(43) \quad \begin{aligned} \mathcal{D}^- = & -\frac{1}{2}(\mathbf{v}^-)^T \mathbf{H} \mathbf{S}_J \mathbf{p}^+ - \alpha \frac{Z}{2}(\mathbf{v}^-)^T \mathbf{H} \mathbf{S}_J (\mathbf{v}^+ + \mathbf{v}^-) \\ & + \frac{1}{2}(\mathbf{p}^-)^T \mathbf{H} \mathbf{S}_J \mathbf{v}^+ + \alpha \frac{1}{2Z}(\mathbf{p}^-)^T \mathbf{H} \mathbf{S}_J (\mathbf{p}^+ - \mathbf{p}^-). \end{aligned}$$

A similar calculation for the plus side of the interface gives

$$(44) \quad \begin{aligned} \mathcal{D}^+ = & -\frac{1}{2}(\mathbf{v}^+)^T \mathbf{H} \mathbf{S}_J \mathbf{p}^- - \alpha \frac{Z}{2}(\mathbf{v}^+)^T \mathbf{H} \mathbf{S}_J (\mathbf{v}^- + \mathbf{v}^+) \\ & + \frac{1}{2}(\mathbf{p}^+)^T \mathbf{H} \mathbf{S}_J \mathbf{v}^- + \alpha \frac{1}{2Z}(\mathbf{p}^+)^T \mathbf{H} \mathbf{S}_J (\mathbf{p}^- - \mathbf{p}^+). \end{aligned}$$

Edge energy dissipation (40) then follows since  $\mathbf{H}$  and  $\mathbf{S}_J$  are diagonal matrices. Similarly, the nonpositiveness of (40) when  $\alpha \geq 0$  follows from the diagonal, positive definiteness of  $\mathbf{H}$  and  $\mathbf{S}_J$ .  $\square$

**4. General interface treatment.** Our discussion of more general interfaces begins with the coupling of two SBP finite difference blocks that conform at the block level (i.e., no hanging multiblock nodes). Throughout we assume that both blocks have the same continuous coordinate transform along the interface. For example, consider the case shown in Figure 1 (left panel), where we assume that the block on the right side of the interface has been transformed with  $x_1^+(\xi_1, \xi_2)$  and  $x_2^+(\xi_1, \xi_2)$ , and similarly the block on the left side has been transformed with  $x_1^-(\xi_1, \xi_2)$  and  $x_2^-(\xi_1, \xi_2)$ . With this notation, both blocks see the same transform along the interface if  $x_1^+(-1, \xi) = x_1^-(1, \xi)$  and  $x_2^+(-1, \xi) = x_2^-(1, \xi)$ , where for simplicity we have assumed that the west face of the right block is connected to the east face of the left block. The glue grid is then parameterized by the variable  $-1 \leq \eta \leq 1$ . Note that we assume nothing about how many grid points are along this interface—only that they conform at the continuous level.

The core idea behind the nonconforming interface treatment is that the penalty terms are computed on a *glue grid* between the two domains. An example glue grid between two finite difference methods is shown in Figure 3. As can be seen, the glue grid between the two domains is defined so that the grid points are nested with the glue grid interval boundaries.

To move values between the finite difference grid and the glue grid, the previously defined projection operators are used. Namely, the operators  $\mathbf{P}_{f2g}^-$  and  $\mathbf{P}_{f2g}^+$  move values from the grid on the minus and plus sides of the interface to the glue grid, and  $\mathbf{P}_{g2f}^-$  and  $\mathbf{P}_{g2f}^+$  move values from the glue grid to the finite difference grids on the plus and minus sides of the interface. We will see that since at the discrete level both sides of the interface may sample the geometry and metric terms differently, these geometry differences, specifically the surface Jacobian, must be taken into account in the projection to ensure discrete stability. To do this, we project the square root of the surface Jacobians along with the grid values to the glue grid; since the surface Jacobian matrices  $\mathbf{S}_J^\pm$  are positive, diagonal matrices, the square roots of these matrices are trivial to compute. Hence, the values that we work with on the glue grid are

$$(45) \quad \bar{\mathbf{p}}^\pm = \mathbf{P}_{f2g}^\pm (\mathbf{S}_J^\pm)^{1/2} \mathbf{p}^\pm, \quad \bar{\mathbf{v}}^\pm = \mathbf{P}_{f2g}^\pm (\mathbf{S}_J^\pm)^{1/2} \mathbf{v}^\pm;$$

we note that values on the glue grid are always scaled by the square root of the surface Jacobian. Here, the vectors  $\mathbf{p}^\pm$  and  $\mathbf{v}^\pm$  refer only to pressure values and the normal component of velocity along the interface of interest.

With this notation, the penalty terms along a nonconforming interface are

$$(46) \quad \mathbf{p}^* - \mathbf{p}^- = (\mathbf{S}_J^-)^{-1/2} \mathbf{P}_{g2f}^- (\bar{\mathbf{p}}^* - \bar{\mathbf{p}}^-) + \frac{1}{2} \left[ (\mathbf{S}_J^-)^{-1/2} \mathbf{P}_{g2f}^- \bar{\mathbf{p}}^- - \mathbf{p}^- \right],$$

$$(47) \quad \mathbf{v}^* - \mathbf{v}^- = (\mathbf{S}_J^-)^{-1/2} \mathbf{P}_{g2f}^- (\bar{\mathbf{v}}^* - \bar{\mathbf{v}}^-) + \frac{1}{2} \left[ (\mathbf{S}_J^-)^{-1/2} \mathbf{P}_{g2f}^- \bar{\mathbf{v}}^- - \mathbf{v}^- \right],$$

where  $\bar{\mathbf{p}}^* - \bar{\mathbf{p}}^-$  and  $\bar{\mathbf{v}}^* - \bar{\mathbf{v}}^-$  are defined by (38)–(39) using the values  $\bar{\mathbf{p}}^\pm$  and  $\bar{\mathbf{v}}^\pm$  for  $\mathbf{p}^\pm$  and  $\mathbf{v}^\pm$ , respectively. As in the conforming case, the parameter  $\alpha \geq 0$  controls the central versus upwind bias of the scheme. The second term on the right-hand side of (46) (and (47)) is a projection error which arises because  $\mathbf{P}_{g2f} \mathbf{P}_{f2g}$  is not an identity operation.

An important implication of the penalty terms (46)–(47) is that the projection operations for the two sides are independent of the scheme on either side of the interface and the underlying representation of the geometry. This latter fact means that the geometry does not need to be built into the projection operation. Also note

that if the interface is conforming, the conforming penalties (38)–(39) are equivalent to the nonconforming penalty terms (46)–(47) if the projection matrices are taken to be the identity matrix:  $\mathbf{P}_{f2g}^\pm = \mathbf{P}_{g2f}^\pm = \mathbf{I}$ .

We can now state the first major result of the paper.

**THEOREM 8.** *Consider a single, nonconforming interface between two SBP blocks with penalty terms of the form (46)–(47). Let  $\mathcal{D}^-$  and  $\mathcal{D}^+$  be the energy dissipation rates along each side of the interface; then*

$$\begin{aligned} \mathcal{D}^- + \mathcal{D}^+ &= -\alpha \frac{Z}{2} (\bar{\mathbf{v}}^- + \bar{\mathbf{v}}^+)^T \mathbf{M} (\bar{\mathbf{v}}^- + \bar{\mathbf{v}}^+) \\ &\quad - \alpha \frac{1}{2Z} (\bar{\mathbf{p}}^- - \bar{\mathbf{p}}^+)^T \mathbf{M} (\bar{\mathbf{p}}^- - \bar{\mathbf{p}}^+), \end{aligned} \tag{48}$$

which is nonpositive for  $\alpha \geq 0$ .

*Proof.* Solving (46)–(47) for  $\mathbf{p}^*$  and  $\mathbf{v}^*$  on the minus side of the interface gives

$$\mathbf{p}^* = (\mathbf{S}_J^-)^{-1/2} \mathbf{P}_{g2f}^- (\bar{\mathbf{p}}^* - \bar{\mathbf{p}}^-) + \frac{1}{2} \left[ (\mathbf{S}_J^-)^{-1/2} \mathbf{P}_{g2f}^- \bar{\mathbf{p}}^- + \mathbf{p}^- \right], \tag{49}$$

$$\mathbf{v}^* = (\mathbf{S}_J^-)^{-1/2} \mathbf{P}_{g2f}^- (\bar{\mathbf{v}}^* - \bar{\mathbf{v}}^-) + \frac{1}{2} \left[ (\mathbf{S}_J^-)^{-1/2} \mathbf{P}_{g2f}^- \bar{\mathbf{v}}^- + \mathbf{v}^- \right]. \tag{50}$$

Substituting  $\mathbf{p}^*$  and  $\mathbf{v}^*$  into (33) on the minus side of the interface results in (after some simplification)

$$\begin{aligned} \mathcal{D}^- &= -(\mathbf{v}^-)^T \mathbf{H}^- (\mathbf{S}_J^-)^{1/2} \mathbf{P}_{g2f}^- (\bar{\mathbf{p}}^* - \bar{\mathbf{p}}^-) - \frac{1}{2} (\mathbf{v}^-)^T \mathbf{H}^- (\mathbf{S}_J^-)^{1/2} \mathbf{P}_{g2f}^- \bar{\mathbf{p}}^- \\ &\quad - (\mathbf{p}^-)^T \mathbf{H}^- (\mathbf{S}_J^-)^{1/2} \mathbf{P}_{g2f}^- (\bar{\mathbf{v}}^* - \bar{\mathbf{v}}^-) - \frac{1}{2} (\mathbf{p}^-)^T \mathbf{H}^- (\mathbf{S}_J^-)^{1/2} \mathbf{P}_{g2f}^- \bar{\mathbf{v}}^-. \end{aligned} \tag{51}$$

Using property (4) of the projection operator, the energy dissipation on the minus side of the interface is

$$\begin{aligned} \mathcal{D}^- &= -(\mathbf{v}^-)^T (\mathbf{S}_J^-)^{1/2} (\mathbf{P}_{f2g}^-)^T \mathbf{M} (\bar{\mathbf{p}}^* - \bar{\mathbf{p}}^-) - \frac{1}{2} (\mathbf{v}^-)^T (\mathbf{S}_J^-)^{1/2} (\mathbf{P}_{f2g}^-)^T \mathbf{M} \bar{\mathbf{p}}^- \\ &\quad - (\mathbf{p}^-)^T (\mathbf{S}_J^-)^{1/2} (\mathbf{P}_{f2g}^-)^T \mathbf{M} (\bar{\mathbf{v}}^* - \bar{\mathbf{v}}^-) - \frac{1}{2} (\mathbf{p}^-)^T (\mathbf{S}_J^-)^{1/2} (\mathbf{P}_{f2g}^-)^T \mathbf{M} \bar{\mathbf{v}}^- \\ &= -(\bar{\mathbf{v}}^-)^T \mathbf{M} \bar{\mathbf{p}}^* + (\bar{\mathbf{v}}^-)^T \mathbf{M} \bar{\mathbf{p}}^- - (\bar{\mathbf{p}}^-)^T \mathbf{M} \bar{\mathbf{v}}^*, \end{aligned} \tag{52}$$

where we have used that  $\mathbf{H}$  and  $\mathbf{S}_J$  commute since they are diagonal as well as the definitions of  $\bar{\mathbf{p}}^-$  and  $\bar{\mathbf{v}}^-$ ; see (45). Solving (38) and (39), evaluated with  $\bar{\mathbf{p}}^\pm$  and  $\bar{\mathbf{v}}^\pm$ , for  $\bar{\mathbf{p}}^*$  and  $\bar{\mathbf{v}}^*$  and substituting these values into (52) gives (after minor algebraic manipulations)

$$\begin{aligned} \mathcal{D}^- &= -\frac{1}{2} (\bar{\mathbf{v}}^-)^T \mathbf{M} \bar{\mathbf{p}}^+ - \alpha \frac{Z}{2} (\bar{\mathbf{v}}^-)^T \mathbf{M} (\bar{\mathbf{v}}^+ + \bar{\mathbf{v}}^-) \\ &\quad + \frac{1}{2} (\bar{\mathbf{p}}^-)^T \mathbf{M} \bar{\mathbf{v}}^+ + \alpha \frac{1}{2Z} (\bar{\mathbf{p}}^-)^T \mathbf{M} (\bar{\mathbf{p}}^+ - \bar{\mathbf{p}}^-). \end{aligned} \tag{53}$$

A similar calculation for the plus side of the interface gives

$$\begin{aligned} \mathcal{D}^+ &= -\frac{1}{2} (\bar{\mathbf{v}}^+)^T \mathbf{M} \bar{\mathbf{p}}^- - \alpha \frac{Z}{2} (\bar{\mathbf{v}}^+)^T \mathbf{M} (\bar{\mathbf{v}}^- + \bar{\mathbf{v}}^+) \\ &\quad + \frac{1}{2} (\bar{\mathbf{p}}^+)^T \mathbf{M} \bar{\mathbf{v}}^- + \alpha \frac{1}{2Z} (\bar{\mathbf{p}}^+)^T \mathbf{M} (\bar{\mathbf{p}}^- - \bar{\mathbf{p}}^+). \end{aligned} \tag{54}$$

Summing (53) and (54) then gives (48). Similarly, the nonpositiveness of (48) when  $\alpha \geq 0$  follows from the positive definiteness of  $\mathbf{M}$ .  $\square$

*Comparison with Mattsson and Carpenter’s [11] interpolation operators.* As noted above, Mattsson and Carpenter have previously proposed a set of SBP-compatible operators for coupling conforming (at the block level) SBP finite difference methods with a fixed refinement ratio [11]. These operators, which Mattsson and Carpenter denote as  $\mathbf{I}_{F2C}$  and  $\mathbf{I}_{C2F}$  with  $F2C$  and  $C2F$  denoting fine to coarse and vice versa, move a solution all the way from one finite difference grid to the next. Thus, an important difference from the projection operators we employ here is that there is an intermediate glue grid which allows the projection operators to be defined independently of the coupling; Mattsson and Carpenter’s operators depend on both the refinement ratio and the SBP operator on either side of the interface. Additionally, Mattsson and Carpenter had to introduce additional constraints in order to ensure stability when upwind bias penalties/numerical fluxes are used; see equation (16) of Mattsson and Carpenter [11]. In their paper, Mattsson and Carpenter note that they were unable to construct operators which always satisfy these constraints and for some cases dissipation was introduced to stabilize the method. In this work the use of the glue grid allows us to overcome these extra constraints on the operators as well as the need to introduce dissipation on the interface; note that in the method we propose here there is dissipation on the interface, and it is controlled by the upwind parameter  $\alpha$ . Finally, the compound operators  $\mathbf{I}_{F2C} = \mathbf{P}_{g2f}^+ \mathbf{P}_{f2g}^-$  and  $\mathbf{I}_{C2F} = \mathbf{P}_{g2f}^- \mathbf{P}_{f2g}^+$  satisfy the consistency constraints (15) and the accuracy conditions of Definition 2.4 of Mattsson and Carpenter [11] (though it should be noted that they are numerically distinct).

**4.1. Many-to-many interfaces.** We now move on to the case when several finite difference blocks are coupled together along a single interface. As will be seen, the treatment for this case is identical to the one-to-one interface case except that the surface Jacobians of the blocks along the coupling interface must be scaled to put them into the glue grid space. An example of the sort of coupling considered is shown in the center panel of Figure 1, where we are interested in the treatment of the T-intersection (denoted with a thick line); without loss of generality we assume that the interface occurs in the  $\xi_2$  direction for all blocks.

As noted above, we parameterize the glue space with a variable  $-1 \leq \eta \leq 1$ . We let  $N^-$  be the number of blocks along the minus side of the interface, and each block  $k$ ,  $1 \leq k \leq N^-$ , overlaps the glue interface over  $\beta^{-(k-1)} \leq \eta \leq \beta^{-(k)}$  with  $\beta^{-(0)} = -1$  and  $\beta^{-(N^-)} = 1$ . We then define the affine interface transform to take each block interface (which runs from  $-1 \leq \xi \leq 1$ ) to the appropriate portion of the glue interface:

$$(55) \quad \eta^{-(k)} = \frac{\beta^{-(k-1)}(1 - \xi_2^{-(k)}) + \beta^{-(k)}(1 + \xi_2^{-(k)})}{2},$$

$$(56) \quad \xi_2^{-(k)} = \frac{(\eta^{-(k)} - \beta^{-(k)}) + (\eta^{-(k)} - \beta^{-(k-1)})}{\beta^{-(k)} - \beta^{-(k-1)}},$$

where  $1 \leq k \leq N^-$  denotes which of the blocks along this side of the interface we are considering. Since these are affine transforms, their effect on the surface Jacobian (see (17)) will be a constant scaling of

$$(57) \quad \frac{\partial \xi_2^{-(k)}}{\partial \eta^{-(k)}} = \frac{2}{\beta^{-(k)} - \beta^{-(k-1)}} = \frac{1}{\Delta^{-(k)}}.$$

Here  $\Delta^{-(k)}$  is the fraction of the interface which intersects block  $k$ . A similar construction is used for the  $N^+$  blocks on the other side of the interface.

Projections to the glue and the penalty terms for each block are then defined as in (45), (46), and (47) except with the surface Jacobians scaled by  $1/\Delta^{\pm(k)}$ :

$$(58) \quad \bar{\mathbf{p}}^{\pm(k)} = \mathbf{P}_{f2g}^{\pm(k)} \left( \frac{\mathbf{S}_J^{\pm(k)}}{\Delta^{\pm(k)}} \right)^{1/2} \mathbf{p}^{\pm(k)}, \quad \bar{\mathbf{v}}^{\pm(k)} = \mathbf{P}_{f2g}^{\pm(k)} \left( \frac{\mathbf{S}_J^{\pm(k)}}{\Delta^{\pm(k)}} \right)^{1/2} \mathbf{v}^{\pm(k)},$$

$$(59) \quad \mathbf{p}^{*(k)} - \mathbf{p}^{-(k)} = \left( \frac{\mathbf{S}_J^{-}(k)}{\Delta^{-}(k)} \right)^{-1/2} \mathbf{P}_{g2f}^{-}(k) \left( \bar{\mathbf{p}}^{*(k)} - \bar{\mathbf{p}}^{-}(k) \right) + \frac{1}{2} \left[ \left( \frac{\mathbf{S}_J^{-}(k)}{\Delta^{-}(k)} \right)^{-1/2} \mathbf{P}_{g2f}^{-}(k) \bar{\mathbf{p}}^{-}(k) - \mathbf{p}^{-}(k) \right],$$

$$(60) \quad \mathbf{v}^{*(k)} - \mathbf{v}^{-(k)} = \left( \frac{\mathbf{S}_J^{-}(k)}{\Delta^{-}(k)} \right)^{-1/2} \mathbf{P}_{g2f}^{-}(k) \left( \bar{\mathbf{v}}^{*(k)} - \bar{\mathbf{v}}^{-}(k) \right) + \frac{1}{2} \left[ \left( \frac{\mathbf{S}_J^{-}(k)}{\Delta^{-}(k)} \right)^{-1/2} \mathbf{P}_{g2f}^{-}(k) \bar{\mathbf{v}}^{-}(k) - \mathbf{v}^{-}(k) \right].$$

Before going on to state the edge dissipation rates, we note that the change of variables (55)–(56) requires a slight modification to the  $\mathbf{H}$ -compatible definition (2); namely, we now use the definition

$$(61) \quad \Delta^{-}(k) \left( \mathbf{P}_{g2f}^{-}(k) \right)^T \mathbf{H}^{-}(k) = \mathbf{M}^{-}(k) \mathbf{P}_{f2g}^{-}(k).$$

Here,  $\mathbf{H}^{-}(k)$  is the one-dimensional SBP norm matrix for a grid from  $-1$  to  $1$ , and  $\mathbf{M}^{-}(k)$  is the mass matrix for the portion of the glue grid running from  $\beta^{-(k-1)}$  to  $\beta^{-}(k)$ . The difference between the domains of  $\mathbf{H}^{-}(k)$  and  $\mathbf{M}^{-}(k)$  gives rise to the  $\Delta^{-}(k)$  in the compatibility condition.

We define the solution vectors on the glue as

$$(62) \quad \bar{\mathbf{v}}^{\pm} = \begin{bmatrix} \bar{\mathbf{v}}^{\pm(1)} \\ \vdots \\ \bar{\mathbf{v}}^{\pm(N^{\pm})} \end{bmatrix}, \quad \bar{\mathbf{p}}^{\pm} = \begin{bmatrix} \bar{\mathbf{p}}^{\pm(1)} \\ \vdots \\ \bar{\mathbf{p}}^{\pm(N^{\pm})} \end{bmatrix}$$

and the block diagonal glue mass matrix as

$$(63) \quad \mathbf{M} = \begin{bmatrix} \mathbf{M}^{-}(1) & & \\ & \ddots & \\ & & \mathbf{M}^{-}(N^-) \end{bmatrix} = \begin{bmatrix} \mathbf{M}^{+}(1) & & \\ & \ddots & \\ & & \mathbf{M}^{+}(N^+) \end{bmatrix},$$

where we note that it is equivalent to define  $\mathbf{M}$  from either the plus or the minus side block mass matrices since they integrate the same space of function after stacking. With these definitions, we now have the following theorem which guarantees stability of the interface treatment.

**THEOREM 9.** *Consider a single, nonconforming interface with  $N^-$  and  $N^+$  SBP blocks on either side of the interface with penalty terms of the form (59)–(60). Let*



$\mathcal{D}^-$  and  $\mathcal{D}^+$  be the energy dissipation rate along each side of the interface; then

$$\begin{aligned}
 \mathcal{D}^- + \mathcal{D}^+ &= -\alpha \frac{Z}{2} (\bar{\mathbf{v}}^- + \bar{\mathbf{v}}^+)^T \mathbf{M} (\bar{\mathbf{v}}^- + \bar{\mathbf{v}}^+) \\
 &\quad - \alpha \frac{1}{2Z} (\bar{\mathbf{p}}^- - \bar{\mathbf{p}}^+)^T \mathbf{M} (\bar{\mathbf{p}}^- - \bar{\mathbf{p}}^+)
 \end{aligned}
 \tag{64}$$

is the nonpositive for  $\alpha \geq 0$ .

*Proof.* Solving the penalty terms for  $\mathbf{p}^{*(k)}$  and  $\mathbf{v}^{*(k)}$  and substituting them into (33) gives (after a calculation similar to (52)) the edge dissipation rate

$$\begin{aligned}
 \mathcal{D}^{-(k)} &= -\left(\bar{\mathbf{v}}^{-(k)}\right)^T \mathbf{M}^{-(k)} \bar{\mathbf{p}}^{*(k)} + \left(\bar{\mathbf{v}}^{-(k)}\right)^T \mathbf{M}^{-(k)} \bar{\mathbf{p}}^{-(k)} \\
 &\quad - \left(\bar{\mathbf{p}}^{-(k)}\right)^T \mathbf{M}^{-(k)} \bar{\mathbf{v}}^{*(k)}.
 \end{aligned}
 \tag{65}$$

Defining the vectors

$$\bar{\mathbf{v}}^* = \begin{bmatrix} \bar{\mathbf{v}}^{*(1)} \\ \vdots \\ \bar{\mathbf{v}}^{*(N^-)} \end{bmatrix}, \quad \bar{\mathbf{p}}^* = \begin{bmatrix} \bar{\mathbf{p}}^{*(1)} \\ \vdots \\ \bar{\mathbf{p}}^{*(N^-)} \end{bmatrix},
 \tag{66}$$

the sum of the contributions from all the blocks on the minus side of the interface can be written as

$$\mathcal{D}^- = \sum_{k=1}^{N^-} \mathcal{D}^{-(k)} = -(\bar{\mathbf{v}}^-)^T \mathbf{M} \bar{\mathbf{p}}^* + (\bar{\mathbf{v}}^-)^T \mathbf{M} \bar{\mathbf{p}} - (\bar{\mathbf{p}}^-)^T \mathbf{M} \bar{\mathbf{v}}^*.
 \tag{67}$$

Since this equation is identical to (53), the remainder of the proof is identical to that of Theorem 8. □

**4.2. Connecting with discontinuous Galerkin methods.** In addition to allowing for the stable coupling of general finite difference grids, the projection operators defined above can also be used to couple finite difference methods with numerical methods in variational form. To demonstrate this, we consider the coupling of SBP finite difference methods with a curvilinear, triangle-based DG method. We begin by introducing a triangular, curved element DG method and then proceed to view each DG element as a small SBP finite difference block, which leads immediately to a stable coupling between the methods. Though we use one particular DG method, coupling with other formulations is possible as the coupling is purely done at the numerical flux level, so any scheme that gives rise to similar interface terms will be stable. In what follows we only highlight the essential parts of the DG formulation that are necessary to couple it with SBP methods; for a more complete description of DG, the interested reader is directed to, for instance, Hesthaven and Warburton [5].

To introduce the DG method, we start with the variational form of the governing equations (15)–(16) for a DG element  $\Omega_e$  whose reference element is  $\tilde{\Omega}$ ; for the examples in this paper we use a triangular reference element. Applying integration by parts to the conservation of momentum (15) in order to move the spatial derivatives from the solution  $p$  to the test functions  $w_i$ , we get conservation of momentum in the form

$$\int_{\tilde{\Omega}} \left[ w_i \rho J \frac{\partial v_i}{\partial t} - \frac{\partial w_i}{\partial \xi_1} J \frac{\partial \xi_1}{\partial x_i} p - \frac{\partial w_i}{\partial \xi_2} J \frac{\partial \xi_2}{\partial x_i} p \right] dA = - \int_{\partial \tilde{\Omega}} w_i S J n_i p^* ds.
 \tag{68}$$

Notice that the right-hand side of (68) depends only on  $p^*$ , i.e., the value which will become the numerical flux. The form of (68) with the derivative on the test function and of (16) with the derivative on the trial function is sometimes referred to as the skew-symmetric form of the variational equations.

Discretizing the variational forms (68) and (16) in space using the DG method gives rise to the following semidiscretization on each element:

$$\begin{aligned}
 (69) \quad \rho M_J \frac{dv_i}{dt} &= D_1^T M_{1i} p + D_2^T M_{2i} p - \sum_{K=1}^3 L_K^T P_{bc}^T n_{iK} \Omega_{bc} S_{JK} p_K^*, \\
 M_J \frac{dp}{dt} &= -\lambda (M_{11} D_1 v_1 + M_{21} D_2 v_1 + M_{12} D_1 v_2 + M_{22} D_2 v_2) \\
 (70) \quad &- \sum_{K=1}^3 \lambda L_K^T P_{bc}^T \Omega_{bc} S_{JK} (v_K^* - v_K^-),
 \end{aligned}$$

where the vector  $v_K^-$  is the normal component of velocity along edge  $K$  of the element evaluated at the cubature points:

$$(71) \quad v_K^- = n_{1K} P_{bc} L_K v_1 + n_{2K} P_{bc} L_K v_2.$$

Here  $L_K$  takes the volume terms to edge  $K$  of the element, and  $L_K^T$  takes edge  $K$  terms to the volume; this is similar to the behavior of  $e_{W/E} \otimes I$  and  $I \otimes e_{N/S}$  in the SBP method. Also as in the SBP method,  $D_1$  and  $D_2$  are the reference element differentiation matrices for the two reference coordinate directions. Since we will be using curved triangular elements, integration is done using a cubature in the volume and quadratures along the edges of the elements. Thus we introduce the projection matrices  $P_c$  and  $P_{bc}$  that project from the volume and edge approximations to the volume and edge cubature points, respectively. At the cubature locations, the matrices  $\Omega_c$  and  $\Omega_{bc}$  are diagonal matrices of the integration weights for the volume and an edge, respectively. To ensure stability of the method, we will assume that  $\Omega_c$  and  $\Omega_{bc}$  are positive definite. The element mass matrices in the discretization are defined as

$$(72) \quad M_J = P_c^T \Omega_c J P_c, \quad M_{ij} = P_c^T \Omega_c J \frac{\partial \xi_i}{\partial x_j} P_c.$$

Here the diagonal matrices  $J$  and  $\frac{\partial \xi_i}{\partial x_j}$  are, respectively, the Jacobian determinant and metric derivatives defined at the cubature points. The diagonal matrices  $S_{JK}$  and  $n_{iK}$  are the surface Jacobian and the components of the unit normal for edge  $K$ , respectively, defined at the cubature points.

Defining the energy in a DG element as

$$(73) \quad E = \frac{\rho}{2} v_1^T M_J v_1 + \frac{\rho}{2} v_2^T M_J v_2 + \frac{1}{2\lambda} p^T M_J p$$

as well as the edge projected pressures as

$$(74) \quad p_K^- = P_{bc} L_K p,$$

the energy dissipation rate for a single DG element can be characterized by the following lemma.

LEMMA 10. *The single DG block discretization (69)–(70) has the energy dissipation rate*

$$(75) \quad \frac{dE}{dt} = \sum_{K=1}^3 \mathcal{D}_K,$$

$$(76) \quad \mathcal{D}_K = -(\mathbf{v}_K^-)^T \boldsymbol{\Omega}_{bc} \mathbf{S}_{JK} \mathbf{p}_K^* - (\mathbf{p}_K^-)^T \boldsymbol{\Omega}_{bc} \mathbf{S}_{JK} (\mathbf{v}_K^* - \mathbf{v}_K^-),$$

with energy as defined in (73).

*Proof.* Equation (76) follows directly by inserting (69)–(70) into the time derivative of (73),

$$(77) \quad \frac{dE}{dt} = \rho \mathbf{v}_1^T \mathbf{M}_J \frac{d\mathbf{v}_1}{dt} + \rho \mathbf{v}_2^T \mathbf{M}_J \frac{d\mathbf{v}_2}{dt} + \frac{1}{\lambda} \mathbf{p}^T \mathbf{M}_J \frac{d\mathbf{p}}{dt},$$

and simplifying using the definition of the edge projected pressures (74) and normal velocity (71). The nonpositiveness of (76) follows from the fact that  $\boldsymbol{\Omega}_{bc}$  and  $\mathbf{S}_{JK}$  are diagonal, positive definite matrices.  $\square$

**4.2.1. Boundary and DG-to-DG numerical flux.** When an edge occurs on the physical boundary, the boundary condition  $p = 0$  is enforced with the numerical flux

$$(78) \quad \mathbf{p}_K^* = 0, \quad \mathbf{v}_K^* - \mathbf{v}_K = \alpha \frac{\mathbf{p}_K}{Z}.$$

Similarly, the numerical flux between two DG elements is taken to be

$$(79) \quad \mathbf{p}^* = \frac{1}{2} (\mathbf{p}^+ + \mathbf{p}^-) + \alpha \frac{Z}{2} (\mathbf{v}^+ + \mathbf{v}^-),$$

$$(80) \quad \mathbf{v}^* - \mathbf{v}^- = -\frac{1}{2} (\mathbf{v}^+ + \mathbf{v}^-) - \alpha \frac{1}{2Z} (\mathbf{p}^+ - \mathbf{p}^-).$$

In both cases, as in the SBP case, the parameter  $\alpha$  controls the upwind nature of the numerical flux. For stability,  $\alpha \geq 0$ , with the central flux resulting if  $\alpha = 0$  and the fully upwind flux if  $\alpha = 1$ . Note that if  $\mathbf{p}^-$  is subtracted from  $\mathbf{p}^*$ , these are identical to the penalty terms previously defined for SBP boundaries (34) and conforming interfaces (38)–(39).

LEMMA 11. *If edge  $K$  of a DG element is an exterior boundary with penalty terms of the form (78), then the energy dissipation rate for the edge is*

$$(81) \quad \mathcal{D}_K = -\frac{\alpha}{Z} (\mathbf{p}_K^-)^T \boldsymbol{\Omega}_{bc} \mathbf{S}_{JK} \mathbf{p}_K^-,$$

which is nonpositive if  $\alpha \geq 0$ .

*Proof.* The proof follows directly by using (78) in (76).  $\square$

LEMMA 12. *Consider a single interface between two DG elements with penalty terms of the form (79)–(80). Let  $\mathcal{D}^-$  and  $\mathcal{D}^+$  be the energy dissipation rates along each side of the interface; then*

$$(82) \quad \begin{aligned} \mathcal{D}^- + \mathcal{D}^+ &= -\alpha \frac{Z}{2} (\mathbf{v}^- + \mathbf{v}^+)^T \boldsymbol{\Omega}_{bc} \mathbf{S}_J (\mathbf{v}^- + \mathbf{v}^+) \\ &\quad - \alpha \frac{1}{2Z} (\mathbf{p}^- - \mathbf{p}^+)^T \boldsymbol{\Omega}_{bc} \mathbf{S}_J (\mathbf{p}^- - \mathbf{p}^+) \end{aligned}$$

is nonpositive for  $\alpha \geq 0$ .

*Proof.* Equation (82) follows directly by adding  $\mathcal{D}^+ + \mathcal{D}^+$  and using the definition of the numerical flux (79)–(86). The nonpositiveness of (82) follows from the fact that  $\Omega_{bc}$  and  $S_{JK}$  are diagonal, positive definite matrices.  $\square$

**4.2.2. SBP-to-DG interface.** We now consider the case of an edge corresponding to an interface with an SBP block. Note that in general the edge of an SBP block will be connected to many DG elements, as shown in Figure 1, and thus a procedure similar to that used in section 4.1 will be required for connecting many SBP blocks across one interface. For simplicity, we assume that the DG element only connects to a single SBP block and that the coupling occurs along the east face of the SBP block (as shown in the right panel of Figure 1). We index the glue grid using the SBP coordinate transform, so  $\eta = \xi_2$  in Figure 1, where  $\xi_2$  is the second metric coordinate of the SBP block. Let the DG element intersect the glue grid over the interval  $[\eta_1, \eta_2]$ . Since the surface Jacobian for the DG element is defined for the element’s reference space on the boundary of length  $\gamma$ , we have to scale the surface Jacobian before projecting to the glue space, as was done in the many-to-one SBP case of section 4.1. Thus we define the scaled and projected DG solution as

$$(83) \quad \bar{\mathbf{v}}^- = \mathbf{P}_{f2g}^- \left( \frac{\mathbf{S}_J^-}{\Delta^-} \right)^{1/2} \mathbf{v}^-,$$

$$(84) \quad \bar{\mathbf{p}}^- = \mathbf{P}_{f2g}^- \left( \frac{\mathbf{S}_J^-}{\Delta^-} \right)^{1/2} \mathbf{p}^-,$$

where  $\Delta^- = (\eta_2 - \eta_1)/\gamma$  and  $\mathbf{P}_{f2g}^-$  is the projection from the DG element edge cubature points to the portion of the glue grid it overlaps. Similarly, we define the projection back from the glue to the DG element edge as  $\mathbf{P}_{g2f}^-$ .

If we use polynomial basis functions of order  $q$  for DG and set the glue grid space to a higher-order polynomial space, then we have that  $\mathbf{P}_{g2f}^- \mathbf{P}_{f2g}^- = \mathbf{I}$ ; that is, there is no projection error as there was for the SBP solution; note that the converse is not true, as the glue grid is a higher-order space. With this assumption, we can now define the DG numerical flux when connected to the SBP finite difference solution as

$$(85) \quad \mathbf{p}^* = \left( \frac{\mathbf{S}_J^-}{\Delta^-} \right)^{-1/2} \mathbf{P}_{g2f}^- \bar{\mathbf{p}}^*,$$

$$(86) \quad \mathbf{v}^* - \mathbf{v}^- = \left( \frac{\mathbf{S}_J^-}{\Delta^-} \right)^{-1/2} \mathbf{P}_{g2f}^- (\bar{\mathbf{v}}^* - \bar{\mathbf{v}}^-),$$

with  $\bar{\mathbf{p}}^*$  and  $\bar{\mathbf{v}}^* - \bar{\mathbf{v}}^-$  defined as in (38)–(39) using the values  $\bar{\mathbf{p}}^\pm$  and  $\bar{\mathbf{v}}^\pm$  for  $\mathbf{p}^\pm$  and  $\mathbf{v}^\pm$ , respectively. As in the conforming case, comparing these numerical flux expressions with the SBP penalty terms (46)–(47), we see that they are identical since there is no projection error going to the glue and back for the DG solution, in particular since

$$(87) \quad \mathbf{P}_{g2f}^- \mathbf{P}_{f2g}^- \mathbf{p}^- = \mathbf{P}_{g2f}^- \bar{\mathbf{p}}^- = \mathbf{p}^-.$$

Additionally, these numerical fluxes are the same as those in (79)–(80) since in the case of connecting two DG elements the projection operators are identity operations, i.e.,  $\mathbf{P}_{f2g}^- = \mathbf{P}_{g2f}^- = \mathbf{I}$ , and the surface Jacobians are the same for both sides of the interface. With a high enough boundary cubature order, it can be assumed that

$$(88) \quad \Delta^- \left( \mathbf{P}_{g2f}^- \right)^T \Omega_{bc}^- = \mathbf{M}^{-(k)} \mathbf{P}_{f2g}^-,$$

and stability of the SBP-DG coupling is characterized by the following corollary to Theorem 9.

**COROLLARY 13.** *Consider a single, nonconforming interface between an SBP finite difference method and a DG method. If the SBP finite difference method has interface penalty terms of the form (59)–(60) and the DG method has numerical fluxes of the form (85)–(86), then the coupling interface satisfies the dissipation rates of Theorem 9.*

*Proof.* To prove that this corollary is true, we will show that a single DG cell, indexed by  $^{(k)}$ , satisfies (65). The single edge dissipation rate for a DG cell comes from substituting (85)–(86) into (76) and using

$$(89) \quad \begin{aligned} \mathcal{D}^{-(k)} = & - \left( \mathbf{v}^{-(k)} \right)^T \boldsymbol{\Omega}_{bc} \left( \Delta^{-(k)} \mathbf{S}_J^{-(k)} \right)^{1/2} \mathbf{P}_{g2f}^{-(k)} \bar{\mathbf{p}}^{*(k)} \\ & - \left( \mathbf{p}^{-(k)} \right)^T \boldsymbol{\Omega}_{bc} \left( \Delta^{-(k)} \mathbf{S}_J^{-(k)} \right)^{1/2} \mathbf{P}_{g2f}^{-(k)} \left( \bar{\mathbf{v}}^{*(k)} - \bar{\mathbf{v}}^{-(k)} \right). \end{aligned}$$

Using (88), this can be simplified to (65). Thus, the rest of the proof for the dissipation rates follows the same procedure as in the proof of Theorem 9.  $\square$

**5. Numerical results.** Here we confirm the above theoretical stability results as well as explore the accuracy of the coupling technique.<sup>1</sup> A method-of-lines approach is used to discretize the acoustic wave equation where the spatial scheme is as described in this paper and an explicit fourth-order Runge–Kutta method is used for the temporal discretization. The test problem is the discretization of (13)–(14) on the domain  $\Omega = [-1, 1] \times [-1, 1]$  with  $\rho = \lambda = 1$ . Zero pressure, i.e., free surface, boundary conditions are used on all boundaries. We use the initial condition

$$(90) \quad p(x_1, x_2, 0) = \cos(k_1 x_1) \cos(k_1 x_2) + \sin(k_2 x_1) \sin(k_2 x_2),$$

$$(91) \quad v_i(x_1, x_2, 0) = 0, \quad i = 1, 2,$$

where  $k_1 = \pi/2$  and  $k_2 = \pi$ . With this the exact solution is

$$(92) \quad p(x_1, x_2, t) = \cos(\omega_1 t) \cos(k_1 x_1) \cos(k_1 x_2) + \cos(\omega_2 t) \sin(k_2 x_1) \sin(k_2 x_2),$$

$$(93) \quad v_1(x_1, x_2, t) = \frac{k_1}{\omega_1} \sin(\omega_1 t) \sin(k_1 x_1) \cos(k_1 x_2) - \frac{k_2}{\omega_2} \sin(\omega_2 t) \cos(k_2 x_1) \sin(k_2 x_2),$$

$$(94) \quad v_2(x_1, x_2, t) = \frac{k_1}{\omega_1} \sin(\omega_1 t) \cos(k_1 x_1) \sin(k_1 x_2) - \frac{k_2}{\omega_2} \sin(\omega_2 t) \sin(k_2 x_1) \cos(k_2 x_2),$$

where  $\omega_j = k_j \sqrt{2}$  for  $j = 1, 2$ .

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<sup>1</sup>The MATLAB code for constructing the interpolation operators used in this section is available in the electronic supplement and at [https://github.com/bfam/sbp\\_projection\\_operators](https://github.com/bfam/sbp_projection_operators). The simulation code used to produce the results is available at [https://github.com/bfam/sbp\\_projection\\_2d](https://github.com/bfam/sbp_projection_2d). For DG, when coupling with SBP-SAT, we use the code from Hesthaven and Warburton [5] available at <https://github.com/tcew/nodal-dg>.

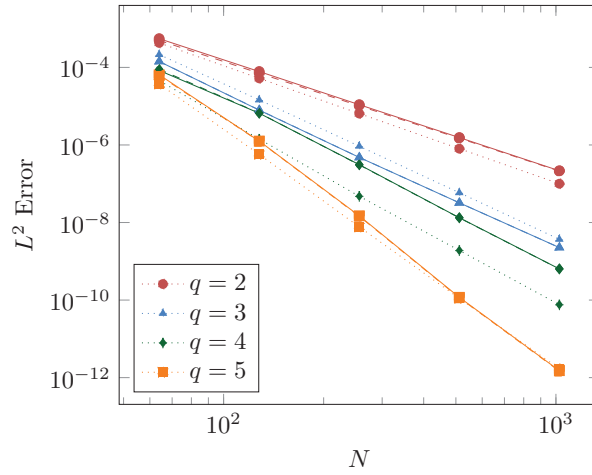


FIG. 4. Log-log plot of  $N$  versus the  $L^2$  error (measured with the energy norm) for the coupling of two SBP blocks (see Figure 1). Three different interface types are shown: a conforming interface (dotted line), a nested interface with a two-to-one refinement ratio (solid line), and an unnested interface with  $N_{fine} = 2N_{coarse} + 1$  (dashed line). In the figure axis  $N$  refers to the coarse block; i.e.,  $N_{coarse} = N$ .

**5.1. One-to-one SBP coupling.** We first test the coupling of two SBP blocks, as shown in the left panel of Figure 1. The coordinate transforms for the two blocks are

$$(95) \quad x_1^{(L)}(\xi_1, \xi_2) = \left(\frac{1 + \xi_1}{10}\right) \sin(\pi(\xi_2 + 1)) - \left(\frac{1 - \xi_1}{2}\right), \quad x_2^{(L)}(\xi_1, \xi_2) = \xi_2,$$

$$(96) \quad x_1^{(R)}(\xi_1, \xi_2) = \left(\frac{1 - \xi_1}{10}\right) \sin(\pi(\xi_2 + 1)) + \left(\frac{1 + \xi_1}{2}\right), \quad x_2^{(R)}(\xi_1, \xi_2) = \xi_2,$$

where the superscript  $(L)$  and  $(R)$  correspond to the left and right blocks, respectively. Notice that the coordinate transform along the interface is conforming  $x_1^{(L)}(1, \xi_2) = x_1^{(R)}(-1, \xi_2) = \frac{1}{5} \sin(\pi(\xi_2 + 1))$  and  $x_2^{(L)}(1, \xi_2) = x_2^{(R)}(-1, \xi_2) = \xi_2$ .

We discretize the left block by using an  $(N/2 + 1) \times (N + 1)$  grid where  $N = 2^k$  with  $k = 6, 7, 8, 9, 10$ . For the right block we use an  $(M/2 + 1) \times (M + 1)$  grid of points where  $M$  is chosen so the interface is conforming ( $M = N$ ), nested ( $M = 2N$ ), or unnested ( $M = 2N + 1$ ). In the conforming case no projection operator is used; i.e., this is the traditional SBP-SAT coupling. We run the simulation using SBP orders  $q = 2, 3, 4, 5$ , where here  $q$  refers to the boundary order; i.e., the interior finite difference method is of order  $p_i = 2q$ , the boundary finite difference order is  $p_b = q$ , and the expected rate of convergence for conforming multiblock SBP is  $q + 1$ . Note that in this work we will exclusively consider the diagonal norm SBP operators. The final time of the simulations is  $t = 1$ . For the penalties we use  $\alpha = 1$ ; thus the interface and boundaries are fully upwinded.

In Figure 4 and Table 1 we report the error for each of the three cases. We measure the error using the  $L^2$  norm

$$(97) \quad \varepsilon^2 = \sum_{\{\Omega_e\}} \left( \frac{\rho}{2} \Delta v_1^T J \bar{H} \Delta v_1 + \frac{\rho}{2} \Delta v_2^T J \bar{H} \Delta v_2 + \frac{1}{2\lambda} \Delta p^T J \bar{H} \Delta p \right),$$

TABLE 1

Table of the error and estimated convergence rates for the coupling of two SBP blocks (see simulations shown in Figure 4).

| $N$                               | $q = 2$<br>Error (rate)    | $q = 3$<br>Error (rate)    | $q = 4$<br>Error (rate)     | $q = 5$<br>Error (rate)     |
|-----------------------------------|----------------------------|----------------------------|-----------------------------|-----------------------------|
| Conforming meshes (no projection) |                            |                            |                             |                             |
| 64                                | $4.3 \times 10^{-4}$       | $2.1 \times 10^{-4}$       | $4.6 \times 10^{-5}$        | $3.7 \times 10^{-5}$        |
| 128                               | $5.2 \times 10^{-5}$ (3.0) | $1.4 \times 10^{-5}$ (3.9) | $1.4 \times 10^{-6}$ (5.0)  | $5.7 \times 10^{-7}$ (6.0)  |
| 256                               | $6.5 \times 10^{-6}$ (3.0) | $9.3 \times 10^{-7}$ (4.0) | $4.8 \times 10^{-8}$ (4.9)  | $7.7 \times 10^{-9}$ (6.2)  |
| 512                               | $8.0 \times 10^{-7}$ (3.0) | $5.9 \times 10^{-8}$ (4.0) | $1.9 \times 10^{-9}$ (4.6)  | $1.1 \times 10^{-10}$ (6.1) |
| 1024                              | $1.0 \times 10^{-7}$ (3.0) | $3.7 \times 10^{-9}$ (4.0) | $7.6 \times 10^{-11}$ (4.6) | $1.7 \times 10^{-12}$ (6.0) |
| Nested meshes                     |                            |                            |                             |                             |
| 64                                | $5.5 \times 10^{-4}$       | $1.4 \times 10^{-4}$       | $8.3 \times 10^{-5}$        | $6.5 \times 10^{-5}$        |
| 128                               | $7.8 \times 10^{-5}$ (2.8) | $7.9 \times 10^{-6}$ (4.2) | $6.6 \times 10^{-6}$ (3.7)  | $1.3 \times 10^{-6}$ (5.7)  |
| 256                               | $1.1 \times 10^{-5}$ (2.8) | $4.8 \times 10^{-7}$ (4.0) | $3.1 \times 10^{-7}$ (4.4)  | $1.5 \times 10^{-8}$ (6.4)  |
| 512                               | $1.6 \times 10^{-6}$ (2.8) | $3.2 \times 10^{-8}$ (3.9) | $1.3 \times 10^{-8}$ (4.5)  | $1.2 \times 10^{-10}$ (7.0) |
| 1024                              | $2.2 \times 10^{-7}$ (2.8) | $2.3 \times 10^{-9}$ (3.8) | $6.4 \times 10^{-10}$ (4.4) | $1.5 \times 10^{-12}$ (6.3) |
| Unnested meshes                   |                            |                            |                             |                             |
| 64                                | $5.0 \times 10^{-4}$       | $1.4 \times 10^{-4}$       | $8.9 \times 10^{-5}$        | $6.2 \times 10^{-5}$        |
| 128                               | $7.1 \times 10^{-5}$ (2.8) | $8.0 \times 10^{-6}$ (4.2) | $6.7 \times 10^{-6}$ (3.7)  | $1.2 \times 10^{-6}$ (5.6)  |
| 256                               | $1.0 \times 10^{-5}$ (2.8) | $4.8 \times 10^{-7}$ (4.1) | $3.1 \times 10^{-7}$ (4.4)  | $1.4 \times 10^{-8}$ (6.4)  |
| 512                               | $1.5 \times 10^{-6}$ (2.8) | $3.2 \times 10^{-8}$ (3.9) | $1.3 \times 10^{-8}$ (4.5)  | $1.1 \times 10^{-10}$ (7.0) |
| 1024                              | $2.1 \times 10^{-7}$ (2.8) | $2.3 \times 10^{-9}$ (3.8) | $6.3 \times 10^{-10}$ (4.4) | $1.5 \times 10^{-12}$ (6.3) |

where  $\Delta \mathbf{v}_1$ ,  $\Delta \mathbf{v}_2$ , and  $\Delta \mathbf{p}$  are the difference between the discrete solution and the exact solution at all the grid points; note that this is the same norm used in the stability analysis (29). In all the cases the error is decreasing with mesh refinement. For both of the cases where the projection operators are used, the overall error level is comparable and slightly higher than the conforming (no-projection) case. Also given in Table 1 are estimates of the convergence rate between two successive resolutions measured using

$$(98) \quad \frac{\log(\varepsilon_f) - \log(\varepsilon_c)}{\log(N_c) - \log(N_f)},$$

where here subscript  $f$  refers to the finer solution and  $c$  the coarser solution. As the table shows, the coupling does show higher-order convergence, though it is interesting to note that when the projections are used the rates are more sporadic than the conforming case.

**5.2. Two-to-one SBP coupling.** We now test the use of the projection operators to couple multiple SBP blocks along a single interface. That is, we have a single block on the left side of the interface and two blocks on the right side of the interface, as in the center panel of Figure 1. The block on the left side of the interface has the coordinate transform (95), whereas the top and bottom right blocks have the coordinates transforms

$$(99) \quad x_1^{(RT)}(\xi_1, \xi_2) = x_1^{(R)}\left(\frac{\xi_1 + 1}{2}, \xi_2\right), \quad x_2^{(RT)}(\xi_1, \xi_2) = x_2^{(R)}\left(\frac{\xi_1 + 1}{2}, \xi_2\right),$$

$$(100) \quad x_1^{(RB)}(\xi_1, \xi_2) = x_1^{(R)}\left(\frac{\xi_1 - 1}{2}, \xi_2\right), \quad x_2^{(RB)}(\xi_1, \xi_2) = x_2^{(R)}\left(\frac{\xi_1 - 1}{2}, \xi_2\right),$$

where superscript  $(RT)$  and  $(RB)$  refer to the right-top and right-bottom blocks, respectively, and  $x_{1,2}^{(R)}$  are defined by (96).

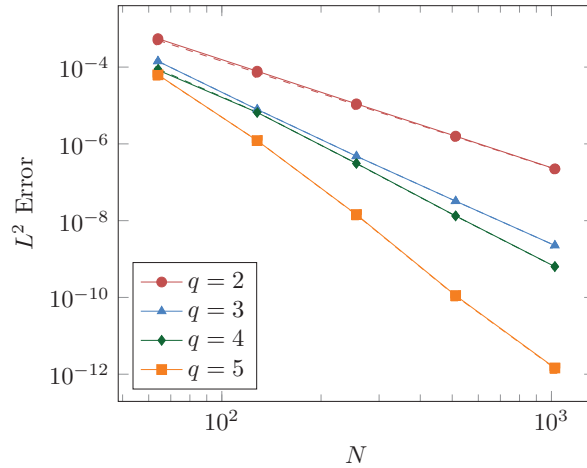


FIG. 5. Log-log plot of  $N$  versus the  $L^2$  error (measured with the energy norm) for the coupling of three SBP blocks (see Figure 1). Two different interface types are shown: a nested interface with a two-to-one refinement ratio (solid line) and an unnested interface with  $N_{\text{fine}} = 2N_{\text{coarse}} + 1$  (dashed line). Also shown for reference is the two block conforming interface test (dotted line) from Figure 4. In the figure axis  $N$  refers to the coarse block; i.e.,  $N_{\text{coarse}} = N$ .

TABLE 2

Table of the error and estimated convergence rates for the coupling of three SBP blocks (see simulations shown in Figure 5).

| $N$             | $q = 2$<br>Error (rate)    | $q = 3$<br>Error (rate)    | $q = 4$<br>Error (rate)     | $q = 5$<br>Error (rate)     |
|-----------------|----------------------------|----------------------------|-----------------------------|-----------------------------|
| Nested meshes   |                            |                            |                             |                             |
| 64              | $5.5 \times 10^{-4}$       | $1.4 \times 10^{-4}$       | $8.4 \times 10^{-5}$        | $6.3 \times 10^{-5}$        |
| 128             | $7.8 \times 10^{-5}$ (2.8) | $7.9 \times 10^{-6}$ (4.2) | $6.6 \times 10^{-6}$ (3.7)  | $1.2 \times 10^{-6}$ (5.7)  |
| 256             | $1.1 \times 10^{-5}$ (2.8) | $4.8 \times 10^{-7}$ (4.0) | $3.1 \times 10^{-7}$ (4.4)  | $1.4 \times 10^{-8}$ (6.4)  |
| 512             | $1.6 \times 10^{-6}$ (2.8) | $3.2 \times 10^{-8}$ (3.9) | $1.3 \times 10^{-8}$ (4.5)  | $1.1 \times 10^{-10}$ (7.0) |
| 1024            | $2.2 \times 10^{-7}$ (2.8) | $2.3 \times 10^{-9}$ (3.8) | $6.4 \times 10^{-10}$ (4.4) | $1.5 \times 10^{-12}$ (6.3) |
| Unnested meshes |                            |                            |                             |                             |
| 64              | $5.0 \times 10^{-4}$       | $1.4 \times 10^{-4}$       | $8.9 \times 10^{-5}$        | $6.2 \times 10^{-5}$        |
| 128             | $7.1 \times 10^{-5}$ (2.8) | $7.9 \times 10^{-6}$ (4.2) | $6.7 \times 10^{-6}$ (3.8)  | $1.2 \times 10^{-6}$ (5.7)  |
| 256             | $1.0 \times 10^{-5}$ (2.8) | $4.8 \times 10^{-7}$ (4.1) | $3.1 \times 10^{-7}$ (4.4)  | $1.4 \times 10^{-8}$ (6.4)  |
| 512             | $1.5 \times 10^{-6}$ (2.8) | $3.2 \times 10^{-8}$ (3.9) | $1.3 \times 10^{-8}$ (4.5)  | $1.1 \times 10^{-10}$ (7.0) |
| 1024            | $2.2 \times 10^{-7}$ (2.8) | $2.3 \times 10^{-9}$ (3.8) | $6.3 \times 10^{-10}$ (4.4) | $1.4 \times 10^{-12}$ (6.3) |

We discretize the left block with an  $(N/2 + 1) \times (N + 1)$  grid of points, with  $N = 2^k$  for  $k = 6, 7, 8, 9, 10$ . The right two blocks are discretized using grids of size  $(N + 1) \times (M + 1)$  where  $M$  is chosen so that the interface is nested with a 1 : 2 refinement ratio ( $M = N$ ) or fully unnested ( $M = N + 1$ ); in both cases the interface between the two right blocks is conforming. As before, we let the final time be  $t = 1$  and use  $\alpha = 1$  in the penalty terms.

In Figure 5 and Table 2 we report the error (as measured by (97)) for SBP orders  $q = 2, 3, 4, 5$ , where as before  $q$  refers to the boundary order of the SBP method. In Table 2 we also report the convergence rate as calculated using (98). As these results show, the method maintains the high-order accuracy of the SBP finite difference method when nonconforming block interfaces are used. As in the case of the one-to-



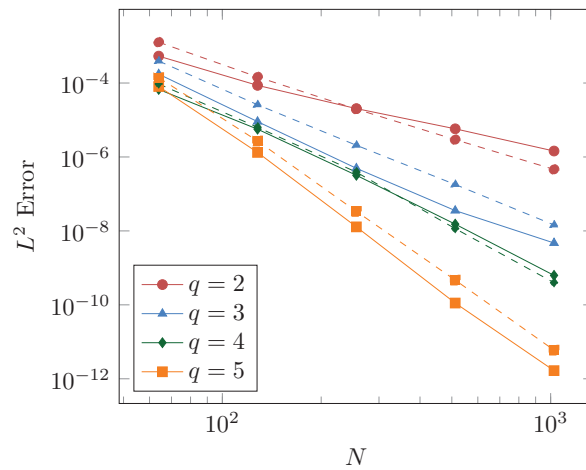


FIG. 6. Log-log plot of  $N$  versus the  $L^2$  error (measured with the energy norm) for the coupling of SBP and DG (see Figure 1). We plot different lines for runs of different orders where  $q$  is both the SBP boundary order and the DG polynomial order. The solid line shows the results with  $\alpha = 1$ , and the dashed line shows the results with  $\alpha = 0$ .

one coupling, the rates are more sporadic than the conforming one-to-one coupling case.

**5.3. SBP-DG coupling.** Here we consider the coupling between SBP finite difference methods and DG finite element methods as discussed in section 4.2. We specifically use the curvilinear nodal DG method on triangles, as described in Hesthaven and Warburton [5]. The configuration is as shown in the right panel of Figure 1 with a curvilinear SBP block on the left side of the coupling interface and an unstructured, curvilinear DG mesh on the right side of the coupling interface. The SBP block is transformed according to (95). For the DG mesh, the element edges along the curved interface are curved by moving the interpolation nodes to the interface, and the interior interpolation points are then moved using transfinite blending; elements that are not on the interface remain straight-sided. Refinement for the unstructured mesh is performed in a hierarchical fashion, with each triangle split into four nested triangles.

As before, we run the SBP mesh with SBP orders  $q = 2, 3, 4, 5$  and use a polynomial order for the DG elements of  $q$ . Similarly, we discretize the left SBP block with an  $(N/2 + 1) \times (N + 1)$  grid of points, with  $N = 2^k$  for  $k = 6, 7, 8, 9, 10$ . The initial DG mesh is chosen so that the number of unique degrees of freedom along the interface roughly matches the number of finite difference grid points. That is, we choose a base mesh for each order that has  $\lceil 2^6 / (q + 1) \rceil$  edges along the coupling interface which is then refined in a hierarchical manner. The final time for the simulations is  $t = 1$ , and now we consider both  $\alpha = 1$  and  $\alpha = 0$  for both the SBP penalties and the DG numerical flux terms.

Shown in Figure 6 and Table 3 are the error and convergence results for this test problem for both values of  $\alpha$ . As can be seen, the method is converging at high-order accuracy in all cases. As in the purely SBP-to-SBP coupling the convergence rates are rather sporadic.

TABLE 3  
*SBP-DG coupling where  $q$  is the SBP boundary order and the DG polynomial order.*

| $N$          | $q = 2$<br>Error (rate)    | $q = 3$<br>Error (rate)    | $q = 4$<br>Error (rate)     | $q = 5$<br>Error (rate)     |
|--------------|----------------------------|----------------------------|-----------------------------|-----------------------------|
| $\alpha = 1$ |                            |                            |                             |                             |
| 64           | $5.3 \times 10^{-4}$       | $1.7 \times 10^{-4}$       | $6.7 \times 10^{-5}$        | $8.1 \times 10^{-5}$        |
| 128          | $8.6 \times 10^{-5}$ (2.6) | $9.0 \times 10^{-6}$ (4.3) | $5.7 \times 10^{-6}$ (3.6)  | $1.3 \times 10^{-6}$ (5.9)  |
| 256          | $2.0 \times 10^{-5}$ (2.1) | $4.9 \times 10^{-7}$ (4.2) | $3.2 \times 10^{-7}$ (4.1)  | $1.3 \times 10^{-8}$ (6.7)  |
| 512          | $5.8 \times 10^{-6}$ (1.8) | $3.5 \times 10^{-8}$ (3.8) | $1.5 \times 10^{-8}$ (4.4)  | $1.1 \times 10^{-10}$ (6.8) |
| 1024         | $1.4 \times 10^{-6}$ (2.0) | $4.7 \times 10^{-9}$ (2.9) | $6.3 \times 10^{-10}$ (4.6) | $1.7 \times 10^{-12}$ (6.1) |
| $\alpha = 0$ |                            |                            |                             |                             |
| 64           | $1.3 \times 10^{-3}$       | $3.9 \times 10^{-4}$       | $9.5 \times 10^{-5}$        | $1.4 \times 10^{-4}$        |
| 128          | $1.5 \times 10^{-4}$ (3.1) | $2.6 \times 10^{-5}$ (3.9) | $6.4 \times 10^{-6}$ (3.9)  | $2.7 \times 10^{-6}$ (5.7)  |
| 256          | $2.0 \times 10^{-5}$ (2.9) | $2.1 \times 10^{-6}$ (3.6) | $3.9 \times 10^{-7}$ (4.0)  | $3.4 \times 10^{-8}$ (6.3)  |
| 512          | $2.9 \times 10^{-6}$ (2.7) | $1.8 \times 10^{-7}$ (3.5) | $1.1 \times 10^{-8}$ (5.1)  | $4.7 \times 10^{-10}$ (6.2) |
| 1024         | $4.6 \times 10^{-7}$ (2.7) | $1.4 \times 10^{-8}$ (3.6) | $4.0 \times 10^{-10}$ (4.9) | $6.0 \times 10^{-12}$ (6.3) |

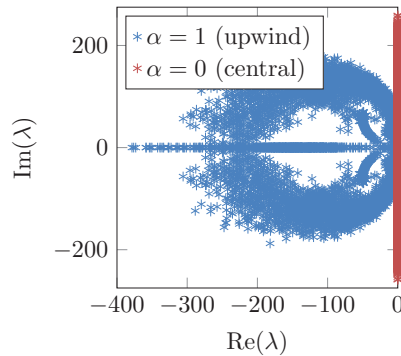


FIG. 7. *Eigenvalues of the coupled SBP-DG spatial discretization operator for the coupling of the SBP operator with  $q = 5$  with the DG using polynomial order 5; the mesh configuration used is the first resolution for this coupling from section 5.3. We show results for both upwind  $\alpha = 1$  and central  $\alpha = 0$  penalties.*

*Eigenvalue spectrum.* Here we confirm the stability results by looking at the eigenvalue spectrum of the SBP-DG coupling. To do this, we write the fully coupled system as a linear equation

$$(101) \quad \frac{du}{dt} = \mathbf{A}u$$

and then numerically compute the eigenvalues of  $\mathbf{A}$ . The energy stability analysis implies that all the eigenvalues should have a nonpositive real part. Furthermore, when the penalty/numerical flux parameter is chosen to be  $\alpha = 0$ , the eigenvalues should be purely imaginary. To confirm this, in Figure 7 we show the eigenvalue spectrum for the coupling of the SBP operator with  $q = 5$  with the DG using polynomial order 5; the mesh configuration used is the first resolution for this coupling from section 5.3. The maximum real part of the eigenvalue spectrum is  $8.16 \times 10^{-13}$  for the upwind penalty ( $\alpha = 1$ ), and the maximum magnitude real part of the eigenvalue spectrum is  $2.67 \times 10^{-13}$  for the central penalty ( $\alpha = 0$ ), thus confirming the theoretical stability results.

**6. Conclusions.** In this paper we have presented a new approach to coupling high-order finite difference methods across nonconforming grid interfaces as well as with DG methods. The core idea behind the proposed methodology is the construction of a projection operator that moves the grid solution from the finite difference points to a finite dimension subspace of the Hilbert space  $L^2(\Gamma)$ . The value of this is that once the solution is in this subspace, it can be projected using  $L^2$  integral projections to other finite dimensions subspaces of the Hilbert space  $L^2(\Gamma)$  and then projected back to the finite difference grid. Since the projection operators are consistent with the SBP  $H$ -matrix, the fully coupled method is provably stable through the use of weak enforcement of boundary conditions. In addition to enforcing the boundary conditions weakly, it was necessary to account for the error in the projection operator (namely, the fact that the finite difference grid space and the finite dimensional subspace are not hierarchical spaces).

In this work we chose a subspace of piecewise polynomial functions which conformed with the finite difference points. The order of the polynomials used matched the structure of the SBP finite difference method. That is, we required that the resulting projection operators be exact in the interior for polynomials of order  $q_i - 1$  and near the boundary for polynomials of order  $q_b - 1$ , with  $q_i$  and  $q_b$  being the interior and boundary accuracy of the SBP finite difference method.

This choice of piecewise polynomial functions as the intermediate space is tangential to the stability results, and any other finite dimensional subspace of the Hilbert space  $L^2(\Gamma)$  could have been chosen. The reason for this is that the projection operators we have constructed to move to polynomials can be used as an intermediate step in moving to any other finite dimensional subspace. That said, the value of another subspace (including a different set of piecewise polynomials) would lie in the ability to enforce different accuracy conditions which may be used to improve the error.

In addition to proposing a new class of SBP-compatible projection operators, we also showed how these projection operators can be used to account for differences in the discrete geometry seen by different blocks on either side of an interface. Namely, we showed that the stability of the numerical method could be preserved if the geometry terms were projected through with the grid values. In the work presented here it is assumed that at the continuous level the coordinate transforms are conforming. The stability results carry over to the case of dissimilar continuous coordinate transforms, but the approach outlined in this paper reduces the results to first-order accuracy. It may be possible to preserve high-order accuracy in the more general case where accurate projection operators are constructed between the coordinate transforms themselves, but this was not explored in this work.

**Appendix A. Proof of Lemma 5.** Taking the time derivative of the energy norm (29) and substituting it into the discretization (21)–(22) gives the energy dissipation rate

$$\begin{aligned}
 (102) \quad \frac{dE}{dt} = & -v_1^T Q_1 J \frac{\partial \xi_1}{\partial x_1} p - v_1^T Q_2 J \frac{\partial \xi_2}{\partial x_1} p - v_2^T Q_1 J \frac{\partial \xi_1}{\partial x_2} p - v_2^T Q_2 J \frac{\partial \xi_2}{\partial x_2} p \\
 & - p^T J \frac{\partial \xi_1}{\partial x_1} Q_1 v_1 - p^T J \frac{\partial \xi_2}{\partial x_1} Q_2 v_1 - p^T J \frac{\partial \xi_1}{\partial x_2} Q_1 v_2 - p^T J \frac{\partial \xi_2}{\partial x_2} Q_2 v_2 \\
 & - v_1^T \mathcal{F}_{v_1} - v_2^T \mathcal{F}_{v_2} - p^T \mathcal{F}_p,
 \end{aligned}$$

where  $Q_1 = Q_{N_1} \otimes H_{N_2}$  and  $Q_2 = H_{N_1} \otimes Q_{N_2}$ . Using the SBP property  $Q + Q^T = B = \text{Diag}[-1, 0, \dots, 0, 1]$  and the fact that  $J$  and  $\frac{\partial \xi_i}{\partial x_i}$  are diagonal matrices (and thus

commute), the volume terms can be transformed into boundary terms:

$$\begin{aligned}
 \frac{dE}{dt} &= -\mathbf{v}_1^T \mathbf{B}_1 \mathbf{J} \frac{\partial \xi_1}{\partial x_1} \mathbf{p} - \mathbf{v}_1^T \mathbf{B}_2 \mathbf{J} \frac{\partial \xi_2}{\partial x_1} \mathbf{p} - \mathbf{v}_2^T \mathbf{B}_1 \mathbf{J} \frac{\partial \xi_1}{\partial x_2} \mathbf{p} - \mathbf{v}_2^T \mathbf{B}_2 \mathbf{J} \frac{\partial \xi_2}{\partial x_2} \mathbf{p} \\
 &\quad - \mathbf{v}_1^T \mathcal{F}_{v_1} - \mathbf{v}_2^T \mathcal{F}_{v_2} - \mathbf{p}^T \mathcal{F}_p \\
 (103) \quad &= -\mathbf{v}_W^T \mathbf{H}_2 \mathbf{S}_{JW} \mathbf{p}_W - \mathbf{v}_E^T \mathbf{H}_2 \mathbf{S}_{JE} \mathbf{p}_E - \mathbf{v}_S^T \mathbf{H}_1 \mathbf{S}_{JS} \mathbf{p}_S - \mathbf{v}_N^T \mathbf{H}_1 \mathbf{S}_{JN} \mathbf{p}_N \\
 &\quad - \mathbf{v}_1^T \mathcal{F}_{v_1} - \mathbf{v}_2^T \mathcal{F}_{v_2} - \mathbf{p}^T \mathcal{F}_p,
 \end{aligned}$$

where  $\mathbf{B}_1 = \mathbf{B}_{N_1} \otimes \mathbf{H}_{N_2}$  and  $\mathbf{B}_2 = \mathbf{H}_{N_1} \otimes \mathbf{B}_{N_2}$ . Note that here we have also used the fact that along the block boundaries

$$(104) \quad \mathbf{J} \frac{\partial \xi_i}{\partial x_2} = \pm n_1 S_J, \quad \mathbf{J} \frac{\partial \xi_i}{\partial x_1} = \pm n_2 S_J,$$

with the positive sign being taken on the “north” ( $\xi_2 = 1$ ) and “east” ( $\xi_1 = 1$ ) boundaries and the negative sign being taken on the “south” ( $\xi_2 = -1$ ) and “west” ( $\xi_1 = -1$ ) boundaries; see (17). Recall also that  $\mathbf{v}_W, \mathbf{v}_E, \mathbf{v}_N,$  and  $\mathbf{v}_S$  are the normal components of the velocity along the west, east, north, and south edges; see (28). Using the definition of the penalty terms (25)–(26) along with form (28) allows us to rewrite the penalty terms in (103) as

$$\begin{aligned}
 (105) \quad \mathbf{v}_1^T \mathcal{F}_{v_1} + \mathbf{v}_2^T \mathcal{F}_{v_2} &= \mathbf{v}_W^T \mathbf{H}_2 \mathbf{S}_{JW} (\mathbf{p}_W^* - \mathbf{p}_W) + \mathbf{v}_E^T \mathbf{H}_2 \mathbf{S}_{JE} (\mathbf{p}_E^* - \mathbf{p}_E) \\
 &\quad + \mathbf{v}_S^T \mathbf{H}_1 \mathbf{S}_{JS} (\mathbf{p}_S^* - \mathbf{p}_S) + \mathbf{v}_N^T \mathbf{H}_1 \mathbf{S}_{JN} (\mathbf{p}_N^* - \mathbf{p}_N), \\
 (106) \quad \mathbf{p}^T \mathcal{F}_p &= \mathbf{p}_W^T \mathbf{H}_2 \mathbf{S}_{JW} (\mathbf{v}_W^* - \mathbf{v}_W) + \mathbf{p}_E^T \mathbf{H}_2 \mathbf{S}_{JE} (\mathbf{v}_E^* - \mathbf{v}_E) \\
 &\quad + \mathbf{p}_S^T \mathbf{H}_1 \mathbf{S}_{JS} (\mathbf{v}_S^* - \mathbf{v}_S) + \mathbf{p}_N^T \mathbf{H}_1 \mathbf{S}_{JN} (\mathbf{v}_N^* - \mathbf{v}_N).
 \end{aligned}$$

These penalty terms can then be used in (103) to write

$$\begin{aligned}
 (107) \quad \frac{dE}{dt} &= -\mathbf{v}_W^T \mathbf{H}_2 \mathbf{S}_{JW} \mathbf{p}_W^* + \mathbf{v}_W^T \mathbf{H}_2 \mathbf{S}_{JW} \mathbf{p}_W - (\mathbf{v}_W^*)^T \mathbf{H}_2 \mathbf{S}_{JW} \mathbf{p}_W \\
 &\quad - \mathbf{v}_E^T \mathbf{H}_2 \mathbf{S}_{JE} \mathbf{p}_E^* + \mathbf{v}_E^T \mathbf{H}_2 \mathbf{S}_{JE} \mathbf{p}_E - (\mathbf{v}_E^*)^T \mathbf{H}_2 \mathbf{S}_{JE} \mathbf{p}_E \\
 &\quad - \mathbf{v}_S^T \mathbf{H}_1 \mathbf{S}_{JS} \mathbf{p}_S^* + \mathbf{v}_S^T \mathbf{H}_1 \mathbf{S}_{JS} \mathbf{p}_S - (\mathbf{v}_S^*)^T \mathbf{H}_1 \mathbf{S}_{JS} \mathbf{p}_S \\
 &\quad - \mathbf{v}_N^T \mathbf{H}_1 \mathbf{S}_{JN} \mathbf{p}_N^* + \mathbf{v}_N^T \mathbf{H}_1 \mathbf{S}_{JN} \mathbf{p}_N - (\mathbf{v}_N^*)^T \mathbf{H}_1 \mathbf{S}_{JN} \mathbf{p}_N,
 \end{aligned}$$

which is (32) with the substitution of (33).

**Appendix B. Projection operators.** Here we discuss the construction of projection operators that satisfy Definition 2 and the accuracy conditions (5). We will first consider the construction of the operators in the interior of the domain and then the operators near the boundary. We only consider diagonal norm SBP operators that have boundary accuracy  $p_b$  and interior accuracy  $p_i = 2p_b$ . Thus, from (5), we are seeking projection operators  $\mathbf{P}_{f2g}$  and  $\mathbf{P}_{g2f}$  which exactly project interior polynomials of order  $p_i - 1$  and boundary polynomials of order  $p_b - 1$ .

Let one interval of the glue grid be represented by the basis  $\{\phi_i\}_{i=0}^n$ , where  $n = p_i - 1$ ; we use Legendre polynomials defined on  $[-1, 1]$  in our codes. The piecewise polynomial on the  $k$ th glue grid interval (i.e., the glue grid interval  $[x_k, x_{k-1}]$ ) is then  $p_n^{(k)}(x) = \sum_{i=0}^n \omega_i^{(k)} \phi_i^{(k)}(x)$  with  $\{\omega_i^{(k)}\}_{i=0}^n$  being the modal weights which define the function on the glue. Note that here (and in the following) the superscript  $(k)$

on  $\phi_i^{(k)}(x)$  is used to signify that the polynomial has been shifted to the interval  $[x_k, x_{k+1}]$ ; i.e.,

$$(108) \quad \phi_i^{(k)}(x) = \phi_i \left( \frac{x - x_k}{x_{k+1} - x_k} - \frac{x_{k+1} - x}{x_{k+1} - x_k} \right).$$

Note that the basis functions  $\psi_i(\eta)$  discussed in section 2 are basis functions over the whole glue grid, whereas the basis functions  $\phi_i^{(k)}(x)$  used here are only over a single glue grid interval.

In the interior,  $m = 2l$  glue grid intervals are projected to a single grid point. Thus, if we consider grid solution  $q_k$  (i.e., the solution at grid point  $k$ ), we use intervals  $k-l$  through  $k+l$ . It is also natural to impose symmetry conditions on the operator, which then leads to a projection of the form

$$(109) \quad q_k = \sum_{j=1}^l \sum_{i=0}^n \beta_{ij} \left[ (-1)^i \omega_i^{(k-j)} + \omega_i^{(k+l+j)} \right],$$

where the  $(-1)^i$  comes from the fact that even modes are symmetric and odd modes are skew-symmetric. Here the coefficient  $\beta_{ij}$  is the contribution of mode  $i$  of intervals  $k+l-j$  and  $k+j$  to the grid point value  $k$ . Note that  $\{\beta_{ij}\}$  are the values that we are seeking to construct.

For the  $s+1$  grid points near the boundary, i.e., grid values  $q_k$  for  $k = 0, \dots, s$ , the intervals 1 through  $r$  are used to construct the projection; by the conditions for stability given in Definition 2 it is required that  $s+1 = r+1-l$ . The projection at the boundary then takes the form

$$(110) \quad q_k = \sum_{i=0}^n \sum_{j=0}^{r-1} \alpha_{ij}^{(k)} \omega_i^{(j)}, \quad k = 0, \dots, s,$$

where we allow the boundary polynomials to be of order  $n$  even though the accuracy conditions at the boundary are for polynomials of degree  $p_b - 1 = (n - 1)/2$ . Hence in total we have  $n_d = m(n + 1) + r(s + 1)(n + 1)$  coefficients  $\{\beta_{ij}\}$  and  $\{\alpha_{ij}^{(k)}\}$  to determine.

It is important to remember that, by Definition 2, the structure of the projection from the finite difference grid to the glue grid is given by  $\mathbf{P}_{f2g} = \mathbf{M}^{-1} \mathbf{P}_{g2f}^T \mathbf{H}$ . Thus, once the structure of  $\mathbf{P}_{g2f}$  is specified,  $\mathbf{P}_{f2g}$  is also fixed; i.e., no new degrees of freedom are introduced into the problem.

To determine value  $\{\alpha_{ij}^{(k)}\}$  and  $\{\beta_{ij}\}$ , (109) and (110) along with Definition 2 are used as constraints. To understand how this is done, consider first (109). Let  $\xi(x)$  be a polynomial of degree less than  $p_i$ . Let  $\xi(x)$  be represented on the glue grid by the modal coefficients  $\omega_i^{(j)}$  for  $i = 0, \dots, n$  and  $j = k-l, \dots, k+l-1$ . We want to find values for  $\{\beta_{ij}\}$  such that  $q_k = \xi(x_k)$ . Additionally, we want the  $\{\beta_{ij}\}$  values to have the property that if on the finite difference grid we have  $q_j = \xi(x_j)$  for  $j = k+l-l, \dots, k+l$ , then the projection  $\mathbf{P}_{f2g}$  results in all  $n+1$  the modal coefficients being exact for glue grid interval  $k$ . The requirement that these two constraints hold for all polynomials  $\xi(x)$  of degree less than  $p_i$  results in  $p_i + p_i(n+1)$  linear constraints; note that in our code we take  $\xi(x)$  to be the Legendre polynomials. Constraints for the  $\{\alpha_{ij}^{(k)}\}$  values are derived in an analogous way from (110) except that we require exactness for polynomials of degree less than  $p_b$ , and thus this introduces an additional

TABLE 4

Parameters used in construction of the SBP compatible projection operators. Here  $n_c$  is the number of constraints and  $n_d$  is the number of degrees of freedom.

| $p_i$ | $p_b$ | $l$ | $s+1$ | $r$ | $n_c$ | $n_d$ |
|-------|-------|-----|-------|-----|-------|-------|
| 2     | 1     | 1   | 1     | 1   | 11    | 6     |
| 4     | 2     | 2   | 4     | 5   | 76    | 96    |
| 6     | 3     | 3   | 6     | 8   | 222   | 324   |
| 8     | 4     | 4   | 9     | 12  | 524   | 928   |
| 10    | 5     | 5   | 12    | 16  | 1020  | 2020  |

$(s+1)p_b+rp_b(n+1)$  constraint (the factors  $r$  and  $s+1$  arise because we have a separate constraint for all  $r$  boundary intervals and  $s+1$  boundary grid points). There are an additional  $l(n+1)$  constraints due to the symmetry conditions on  $\{\beta_{ij}\}$ , and thus there are a total of constraints  $n_c = l(n+1) + p_i + p_i(n+1) + (s+1)p_b + (s+1)p_b(n+1)$  to enforce.

We solve these constraint equations numerically using the MATLAB codes included as an electronic supplement to this paper. The values of  $s$  and  $l$  needed for the SBP operators used in this work are given in Table 4 along with the total number of constraints,  $n_c$ , and degrees of freedom ( $\{\alpha_{ij}^{(k)}\}$  and  $\{\beta_{ij}\}$ ),  $n_d$ ; in the case of  $p_i = 2$ , even though we have more constraints than coefficients, a solution does exist, and in all cases there is redundancy in the constraints; that is, the linear system does not have full row rank. In all cases but the  $p_i = 2$  case, the coefficients of the projection are underdetermined, and we use MATLAB's optimization library to minimize the distance between the nearest eigenvalues of  $\mathbf{B} = \mathbf{P}_{g2f}\mathbf{P}_{f2g}$  for a finite difference grid of size  $N$ . The motivation for this optimization is to (in some sense) minimize the projection error by using the degrees of freedom to make  $\mathbf{B}$  closer to an identity operation. We use the value of  $N = 64$ , which has been chosen to make the optimization problem tractable. It is important to note that the optimization is being used to fix the remaining degrees of freedom after the stability condition (4) is satisfied; i.e., stability does not depend on the choice of the optimization objective function, and other types of optimization could be considered.

MATLAB routines as well as the final optimized coefficients can be found in the electronic supplement to this paper as well as in the github repository located at [https://github.com/bfam/sbp\\_projection\\_operators](https://github.com/bfam/sbp_projection_operators).

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