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ORIGINAL PAPER

Finite element approximations of a nonlinear diffusion model with memory

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Abstract The convergence of a finite element scheme approximating a nonlinear system of integro-differential equations is proven. This system arises in mathematical modeling of the process of a magnetic field penetrating into a substance. Properties of existence, uniqueness and asymptotic behavior of the solutions are briefly described. The decay of the numerical solution is compared with both the theoretical and finite difference results.

Keywords System of nonlinear integro-differential equations · Finite element scheme

Mathematics Subject Classifications (2010) 45K05 · 65N30 · 35K55

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1 Introduction

Integro-differential models arise in many scientific and engineering disciplines. Such a model arises, for instance, in mathematical modeling of the process of a magnetic field penetrating into a substance. If the coefficient of thermal heat capacity and electroconductivity of the substance is highly dependent on the temperature, then the corresponding Maxwell system [1] can be rewritten in the following form [2]:

$$\frac{\partial W}{\partial t} = -\nabla \times \left[a \left(\int_0^t |\nabla \times W|^2 \, d\tau \right) \nabla \times W \right],$$

where $W = (W_1, W_2, W_3)$ is the vector of the magnetic field and the function $a = a(\sigma)$ is defined for $\sigma \in [0, \infty)$.

If the magnetic field has the form $W = (0, u_1, u_2)$ and $u_i = u_i(x, t)$, $i = 1, 2$, then we have

$$\nabla \times (a(\sigma) \nabla \times W) = \left(0, \; -\frac{\partial}{\partial x} \left(a(\sigma) \frac{\partial u_1}{\partial x} \right), \; -\frac{\partial}{\partial x} \left(a(\sigma) \frac{\partial u_2}{\partial x} \right) \right).$$

Therefore, we obtain the following system of nonlinear integro-differential equations:

$$\frac{\partial u_i}{\partial t} = \frac{\partial}{\partial x} \left[a \left(\int_0^t \left[\left(\frac{\partial u_1}{\partial x} \right)^2 + \left(\frac{\partial u_2}{\partial x} \right)^2 \right] d\tau \right) \frac{\partial u_i}{\partial x} \right], \quad i = 1, 2. \tag{1.1}$$

Note that the (1.1)-type model is complex, but special cases of it were investigated; see [2–8]. The existence of global solutions to initial-boundary value problems for such models has been proven in [2–5, 8] by using some modifications of the Galerkin method and compactness arguments [9, 10]. For solvability and uniqueness properties of initial-boundary value problems for (1.1)-type models, see also [6, 7] as well as many other scientific works.

Assume the temperature of the considered body is constant throughout the material, i.e., dependent on time, but independent of the space coordinates. If the magnetic field again has the form $W = (0, u_1, u_2)$ and $u_i = u_i(x, t)$, $i = 1, 2$, then the same process of the magnetic field penetrating into the material is modeled by the following system of integro-differential equations:

$$\frac{\partial u_i}{\partial t} = a \left(\int_0^t \int_0^1 \left[\left(\frac{\partial u_1}{\partial x} \right)^2 + \left(\frac{\partial u_2}{\partial x} \right)^2 \right] dx d\tau \right) \frac{\partial^2 u_i}{\partial x^2}, \quad i = 1, 2. \tag{1.2}$$

The existence and uniqueness of the solutions to (1.2)-type scalar models were studied in [8].

The asymptotic behavior of the solutions to the initial-boundary value problem for the (1.1) and (1.2)-type models have also been the subject of intensive research; see [8, 11–16]. For system (1.2) this issue is studied in [15].

Note that in [12, 16–19] and in a number of other works difference schemes for (1.1) and (1.2)-type models were investigated. Difference schemes and finite element approximations for a nonlinear parabolic integro-differential scalar model similar to (1.1) were studied in [20] and [21]. Finite difference schemes and finite element approximations for the scalar equation of (1.2)-type with $a(\sigma) = 1 + \sigma$ were studied in [16] and [22], respectively. The convergence of the finite difference approximations of system (1.2) for the case $a(\sigma) = 1 + \sigma$ was studied in [19].

Our main goals in the present paper are to study the finite element approximations of system (1.2), as well as to discuss the existence, uniqueness and asymptotic behavior of its solutions, to observe the asymptotic behavior obtained by our numerical experiments, and to carry out a comparative analysis of the finite element and finite difference methods. The rest of the paper is organized as follows. In the next section we briefly discuss the existence, uniqueness and asymptotic behavior of solutions to the initial boundary value problem. In Section 3 a variational formulation of the problem is presented. In Section 4 a finite element scheme for (1.2) is investigated. We close with a section on numerical implementation, where we present numerical results and compare the decay rate to the theoretical results and to the outcome of the finite difference scheme.

2 Statement of problem. Existence, uniqueness and asymptotic behavior of solutions

Consider the following initial-boundary value problem:

$$\frac{\partial u_i}{\partial t} = (1+\sigma)\frac{\partial^2 u_i}{\partial x^2}, \quad (x,t) \in (0,1) \times (0,\infty), \tag{2.1}$$

$$u_i(0,t) = u_i(1,t) = 0, \quad t \geq 0, \tag{2.2}$$

$$u_i(x,0) = u_{i0}(x), \quad x \in [0,1], \tag{2.3}$$

$$i = 1, 2,$$

where

$$\sigma(t) = \int_0^t \int_0^1 \left[\left(\frac{\partial u_1}{\partial x}\right)^2 + \left(\frac{\partial u_2}{\partial x}\right)^2 \right] dx d\tau$$

and $u_{i0} = u_{i0}(x)$, $i = 1, 2$ are given functions.

We use the usual spaces C^k, L_p, H^k and H_0^k.

Let us assume that $u_i = u_i(x,t)$, $i = 1, 2$ is a solution of problem (2.1)–(2.3) such that $u_i(\cdot,t)$, $\frac{\partial u_i(\cdot,t)}{\partial x}$, $\frac{\partial u_i(\cdot,t)}{\partial t}$, $\frac{\partial^2 u_i(\cdot,t)}{\partial t\, \partial x}$, $i = 1, 2$ are all in $C^0([0,\infty);\ L_2(0,1))$, while $\frac{\partial^2 u_i(\cdot,t)}{\partial t^2}$, $i = 1, 2$ are in $L_2((0,\infty);\ L_2(0,1))$.

It is easy to obtain the continuous dependence of solutions on initial data. Indeed, by multiplying equation (2.1) by u_i, $i = 1, 2$, after simple transformations, we get the following estimate

$$\|u_1\| + \|u_2\| \leq \|u_{10}\| + \|u_{20}\|.$$

Below we will show that stronger estimates guaranteeing the continuous dependence of solution on initial data are valid (see Theorems 2.1 and 2.2 below).

The following theorem holds [15].

Theorem 2.1 *If* $u_{i0} \in H_0^1(0,1)$, $i = 1, 2$, *then for the solution to problem* (2.1)–(2.3) *the following estimate is true*

$$\|u_1\| + \left\|\frac{\partial u_1}{\partial x}\right\| + \|u_2\| + \left\|\frac{\partial u_2}{\partial x}\right\| \le C \exp\left(-\frac{t}{2}\right).$$

Remark 1 Here and below in this section, C stands for positive constants which depend on u_{i0}, $i = 1, 2$ but are independent of t.

Note that Theorem 2.1 gives exponential stabilization of the solution to problem (2.1)–(2.3) in the norm of the space $H^1(0,1)$. The stabilization is also achieved in the norm of the space $C^1(0,1)$. In particular, we now show that the following theorem holds [15].

Theorem 2.2 *If* $u_{i0} \in H^4(0,1) \cap H_0^1(0,1)$, $i = 1, 2$, *then for the solution to problem* (2.1)–(2.3) *the following relations hold*:

$$\left|\frac{\partial u_i(x,t)}{\partial x}\right| \le C \exp\left(-\frac{t}{2}\right), \quad \left|\frac{\partial u_i(x,t)}{\partial t}\right| \le C \exp\left(-\frac{t}{2}\right), \quad i = 1, 2.$$

Here we will give a schematic proof of Theorem 2.2, but first we state and prove an auxiliary lemma [15].

Lemma 2.1 *For the solution of problem* (2.1)–(2.3) *the following estimate holds*:

$$\left\|\frac{\partial u_1(x,t)}{\partial t}\right\| + \left\|\frac{\partial u_2(x,t)}{\partial t}\right\| \le C \exp\left(-\frac{t}{2}\right).$$

Proof Differentiating equation (2.1) with respect to t for $i = 1$ and multiplying by $\partial u_1/\partial t$, we deduce after some transformations that

$$\frac{d}{dt}\int_0^1 \left(\frac{\partial u_1}{\partial t}\right)^2 dx + (1+\sigma)\int_0^1 \left(\frac{\partial^2 u_1}{\partial x \partial t}\right)^2 dx$$
$$\le (1+\sigma)^{-1}\left\{\int_0^1 \left[\left(\frac{\partial u_1}{\partial x}\right)^2 + \left(\frac{\partial u_2}{\partial x}\right)^2\right] dx\right\}^2 \int_0^1 \left(\frac{\partial u_1}{\partial x}\right)^2 dx. \quad (2.4)$$

Using Poincaré's inequality, Theorem 2.1, the nonnegativity of $\sigma(t)$ and relation (2.4), we arrive at

$$\left\|\frac{\partial u_1}{\partial t}\right\| \le C \exp\left(-\frac{t}{2}\right).$$

Analogously,

$$\left\|\frac{\partial u_2}{\partial t}\right\| \le C \exp\left(-\frac{t}{2}\right).$$

Now we turn to the proof of Theorem 2.2. □

Proof First we estimate $\partial^2 u_1/\partial x^2$ in the norm of the space $L_1(0,1)$. From (2.1) for $i=1$ we have

$$\frac{\partial^2 u_1}{\partial x^2} = (1+\sigma)^{-1}\frac{\partial u_1}{\partial t}. \tag{2.5}$$

Integrating (2.5) on $(0,1)$, using the Cauchy–Schwarz inequality, applying Lemma 2.1 and taking into account the nonnegativity of $\sigma(t)$, we derive

$$\int_0^1 \left|\frac{\partial^2 u_1}{\partial x^2}\right| dx \le C\exp\left(-\frac{t}{2}\right).$$

From this, taking into account the relation

$$\frac{\partial u_1(x,t)}{\partial x} = \int_0^1 \frac{\partial u_1(y,t)}{\partial y}dy + \int_0^1\int_y^x \frac{\partial^2 u_1(\xi,t)}{\partial \xi^2}d\xi dy$$

and the boundary conditions (2.2), it follows that

$$\left|\frac{\partial u_1(x,t)}{\partial x}\right| \le \int_0^1 \left|\frac{\partial^2 u_1(y,t)}{\partial y^2}\right| dy \le C\exp\left(-\frac{t}{2}\right).$$

Analogously,

$$\left|\frac{\partial u_2(x,t)}{\partial x}\right| \le C\exp\left(-\frac{t}{2}\right).$$

Next, we estimate $\partial u_1/\partial t$ in the norm of the space $C^1(0,1)$. First we multiply (2.1) for $i=1$ by $\partial^3 u_1/\partial x^2\partial t$.

Using Theorem 2.1, relation (2.5) and Lemma 2.1, after some transformations we arrive at

$$\|u_1\|^2 + \left\|\frac{\partial u_1}{\partial x}\right\|^2 + \left\|\frac{\partial^2 u_1}{\partial x^2}\right\|^2 + \left\|\frac{\partial^2 u_1}{\partial x\partial t}\right\|^2 \le C\exp(-t).$$

From this, taking into account Lemma 2.1 once again, it follows that

$$\begin{aligned}\left|\frac{\partial u_1(x,t)}{\partial t}\right| &= \left|\int_0^1 \frac{\partial u_1(y,t)}{\partial t}dy + \int_0^1\int_y^x \frac{\partial^2 u_1(\xi,t)}{\partial \xi\partial t}d\xi dy\right| \\ &\le \left[\int_0^1\left(\frac{\partial u_1(x,t)}{\partial t}\right)^2 dx\right]^{1/2} + \int_0^1\left|\frac{\partial^2 u_1(y,t)}{\partial y\partial t}\right| dy \le C\exp\left(-\frac{t}{2}\right).\end{aligned}$$

Analogously,

$$\left|\frac{\partial u_2(x,t)}{\partial t}\right| \le C\exp\left(-\frac{t}{2}\right).$$

This completes the proof of Theorem 2.2. □

Remark 2 The existence of globally defined solutions of problems (2.1)–(2.3) can be obtained by a routine procedure. One first establishes the existence of local solutions on a maximal time interval and then uses the derived a priori estimates to show that the solutions cannot escape in finite time. This approach is used very often; see, for example, [9] and [10].

The uniqueness of solutions of problem (2.1)–(2.3) can be proven as well. Indeed, let $u_i, \bar{u}_i, i = 1, 2$, be two solutions of problem (2.1)–(2.3) and $z_i(x,t) = u_i(x,t) - \bar{u}_i(x,t)$. We have

$$\frac{\partial z_i}{\partial t} = [1+\sigma(t)]\,\frac{\partial^2 u_i}{\partial x^2} - [1+\bar{\sigma}(t)]\,\frac{\partial^2 \bar{u}_i}{\partial x^2}, \tag{2.6}$$

where

$$\bar{\sigma}(t) = \int_0^t \int_0^1 \left[\left(\frac{\partial \bar{u}_1}{\partial x}\right)^2 + \left(\frac{\partial \bar{u}_2}{\partial x}\right)^2\right] dx d\tau.$$

Multiplying (2.6) by z_i and integrating, we get

$$\begin{aligned}
&\int_0^1 z_i \frac{\partial z_i}{\partial t} dx + \int_0^1 \left(\frac{\partial u_i}{\partial x} - \frac{\partial \bar{u}_i}{\partial x}\right)\left(\frac{\partial u_i}{\partial x} - \frac{\partial \bar{u}_i}{\partial x}\right) dx \\
&\quad + \int_0^1 \left[\sigma(t)\frac{\partial u_i}{\partial x} - \bar{\sigma}(t)\frac{\partial \bar{u}_i}{\partial x}\right]\left(\frac{\partial u_i}{\partial x} - \frac{\partial \bar{u}_i}{\partial x}\right) dx \\
&\qquad = \frac{1}{2}\frac{d}{dt}\int_0^1 z_i^2 dx + \int_0^1 \left(\frac{\partial z_i}{\partial x}\right)^2 dx + \frac{1}{2}\int_0^1 [\sigma(t)+\bar{\sigma}(t)]\left(\frac{\partial u_i}{\partial x} - \frac{\partial \bar{u}_i}{\partial x}\right)^2 dx \\
&\quad + \frac{1}{2}\int_0^1 [\sigma(t)-\bar{\sigma}(t)]\left[\left(\frac{\partial u_i}{\partial x}\right)^2 - \left(\frac{\partial \bar{u}_i}{\partial x}\right)^2\right] dx.
\end{aligned}$$

Integrating with respect to t, we get the inequality

$$\int_0^1 z_i^2 dx + \int_0^t [\sigma(\tau)-\bar{\sigma}(\tau)] \int_0^1 \left[\left(\frac{\partial u_i}{\partial x}\right)^2 - \left(\frac{\partial \bar{u}_i}{\partial x}\right)^2\right] dx d\tau \le 0, \qquad i = 1, 2.$$

Summing these inequalities, we obtain

$$\int_0^1 \left(z_1^2 + z_2^2\right) dx + \int_0^t [\sigma(\tau)-\bar{\sigma}(\tau)]\frac{d}{d\tau}[\sigma(\tau)-\bar{\sigma}(\tau)]\, d\tau \le 0,$$

or

$$\int_0^1 \left(z_1^2 + z_2^2\right) dx + \frac{1}{2}[\sigma(t)-\bar{\sigma}(t)]^2 \le 0.$$

From this we immediately get $z_i(x,t) \equiv 0$, $i = 1, 2$, which proves the uniqueness of the solution.

3 Variational formulation

Consider the following initial-boundary value problem:

$$\frac{\partial u_i}{\partial t} = (1+\sigma(t))\frac{\partial^2 u_i}{\partial x^2} + f_i(x,t), \quad (x,t) \in (0,1)\times(0,T),$$
$$u_i(0,t) = u_i(1,t) = 0, \quad 0 \le t \le T,$$
$$u_i(x,0) = u_{i0}(x), \quad x \in [0,1],$$
$$i = 1,\ 2, \tag{3.1}$$

where

$$\sigma(t) = \int_0^t \int_0^1 \left[\left(\frac{\partial u_1}{\partial x}\right)^2 + \left(\frac{\partial u_2}{\partial x}\right)^2 \right] dx d\tau\,,$$

$T = Const. > 0$ and $u_{i0} = u_{i0}(x)$, $i = 1, 2$ are given functions.

One of the ingredients of the finite element method is a variational formulation of the problem. To provide this variational formulation, let us denote by H the linear space of functions u_i satisfying the boundary conditions in (3.1) and

$$||u_i(\cdot,t)||_1 < \infty,$$

where

$$||u_i(\cdot,t)||_r = \left\{ \int_0^1 \left[|u_i(x,t)|^2 + \sum_{j=1}^{r} \left| \frac{\partial^j u_i(x,t)}{\partial x^j} \right|^2 \right] dx \right\}^{1/2}, \quad i = 1,\ 2.$$

The variational formulation of the problem can now be stated as follows: Find a pair of functions $u_i(x,t) \in H$ for which

$$\left\langle v_i, \frac{\partial u_i}{\partial t} \right\rangle + \left\langle (1+\sigma(t))\frac{\partial u_i}{\partial x}, \frac{\partial v_i}{\partial x} \right\rangle = < f_i, v_i >, \qquad \forall v_i \in H, \tag{3.2}$$

and

$$< v_i, u_i(x,0) > = < v_i, u_{i0}(x) >, \qquad \forall v_i \in H, \quad i = 1,\ 2, \tag{3.3}$$

where

$$< p(x), q(x) > = \int_0^1 p(x)q(x)dx.$$

To approximate the solution of (3.2) and (3.3) we require that u_i and v_i lie in a finite-dimensional subspace S_h of H for each t and $i = 1, 2$. The following property concerning approximability in S_h can be readily verified for finite element spaces; see [23].

Approximation property There is an integer $r \geq 2$ and positive numbers C_0, C_1 independent of h such that for any $v \in H$, there exists a point $v^h \in S_h$ satisfying

$$||v - v^h||_\ell \leq C_\ell h^{j-\ell}||v||_j \qquad \text{for} \quad 0 \leq \ell \leq 1, \quad \ell < j \leq r. \tag{3.4}$$

The approximation $u_i^h \in S_h$ to u_i is defined by the following variational analog of (3.2), (3.3): Find a pair $u_i^h \in S_h$ such that

$$\left\langle v_i^h, \frac{\partial u_i^h}{\partial t} \right\rangle + \left\langle (1 + \sigma_h(t))\frac{\partial u_i^h}{\partial x}, \frac{\partial v_i^h}{\partial x} \right\rangle = < f_i, v_i^h >, \qquad \forall v_i^h \in S_h, \tag{3.5}$$

and

$$< v_i^h, u_i^h(x, 0) >=< v_i^h, u_{i0}(x) >, \qquad \forall v_i^h \in S_h, \qquad i = 1, 2, \tag{3.6}$$

where

$$\sigma_h(t) = \int_0^t \int_0^1 \left[\left(\frac{\partial u_1^h}{\partial x} \right)^2 + \left(\frac{\partial u_2^h}{\partial x} \right)^2 \right] dx d\tau.$$

Once a basis has been selected for S_h, (3.5) and (3.6) are equivalent to a set of N integro-differential equations, where N is the dimension of S_h. The solution of such a system will be discussed in Section 5.

4 Error estimates

In this section we shall estimate the error in the finite element approximation using the norm

$$|||E|||_r = \left[\int_0^T \int_0^1 \left(\sum_{j=0}^r \left| \frac{\partial^j E(x,t)}{\partial x^j} \right|^2 \right) dx dt \right]^{1/2}.$$

Whenever $r = 0$ we will omit the subscript for this norm as well.

Theorem 4.1 *The error in the finite element approximation* u_i^h *generated by* (3.5), (3.6) *satisfies the inequality*

$$|||u_i - u_i^h|||_1 \leq h^{j-1} \left\{ c_1 h^2 \, ||u_{i0}||^2 + c_2 h^2 \left|\left|\left| \frac{\partial u_i}{\partial t} \right|\right|\right|^2 + c_3|||u_i|||^2 + c_4 h^{2(j-1)} \sum_{m=1}^2 |||u_m|||^2 + c_5 \left[\sum_{m=1}^2 \left(c_6 h^{j-1} \, |||u_m|||^2 + c_7[]u_m[] \right) \right]^2 \right\}^{1/2}, \quad j > 1,$$

where

$$[]u[] = \int_0^T \int_0^1 |u|\, dx d\tau$$

and c_i*,* $i = 1, 2, ..., 7$*, denote various positive constants.*

Proof Subtracting (3.5) from (3.2) with v_i^h instead of v_i, we obtain

$$\left\langle v_i^h, \frac{\partial u_i^h}{\partial t} \right\rangle + \left\langle (1+\sigma_h(t))\frac{\partial u_i^h}{\partial x}, \frac{\partial v_i^h}{\partial x} \right\rangle = \left\langle v_i^h, \frac{\partial u_i}{\partial t} \right\rangle + \left\langle (1+\sigma(t))\frac{\partial u_i}{\partial x}, \frac{\partial v_i^h}{\partial x} \right\rangle,$$
$$\forall v_i^h \in S_h, \qquad i = 1,\ 2.$$

Let $\tilde{u}_i^h$ be any function in S_h. Then

$$\left\langle v_i^h, \frac{\partial \left(u_i^h - \tilde{u}_i^h\right)}{\partial t} \right\rangle + \left\langle \left[(1+\sigma_h(t))\frac{\partial u_i^h}{\partial x} - (1+\tilde{\sigma}_h(t))\frac{\partial \tilde{u}_i^h}{\partial x} \right], \frac{\partial v_i^h}{\partial x} \right\rangle$$
$$= \left\langle v_i^h, \frac{\partial \left(u_i - \tilde{u}_i^h\right)}{\partial t} \right\rangle + \left\langle \left[(1+\sigma(t))\frac{\partial u_i}{\partial x} - (1+\tilde{\sigma}_h(t))\frac{\partial \tilde{u}_i^h}{\partial x} \right], \frac{\partial v_i^h}{\partial x} \right\rangle,$$
$$\forall v_i^h \in S_h, \qquad i = 1,\ 2, \tag{4.1}$$

where

$$\tilde{\sigma}_h(t) = \int_0^t \int_0^1 \left[\left(\frac{\partial \tilde{u}_1^h}{\partial x} \right)^2 + \left(\frac{\partial \tilde{u}_2^h}{\partial x} \right)^2 \right] dx d\tau.$$

Define the errors as follows:

$$e_i(x,t) = u_i^h(x,t) - \tilde{u}_i^h(x,t),$$
$$E_i(x,t) = u_i(x,t) - \tilde{u}_i^h(x,t), \qquad i = 1,\ 2. \tag{4.2}$$

Since $e_i \in S_h$, we can let $v_i^h = e_i$ and (4.1) becomes

$$\left\langle e_i, \frac{\partial e_i}{\partial t} \right\rangle + \left\langle \frac{\partial e_i}{\partial x}, \frac{\partial e_i}{\partial x} \right\rangle + \left\langle \left[\sigma_h(t)\frac{\partial u_i^h}{\partial x} - \tilde{\sigma}_h(t)\frac{\partial \tilde{u}_i^h}{\partial x} \right], \frac{\partial e_i}{\partial x} \right\rangle$$
$$= \left\langle e_i, \frac{\partial E_i}{\partial t} \right\rangle + \left\langle \frac{\partial E_i}{\partial x}, \frac{\partial e_i}{\partial x} \right\rangle + \left\langle \left[\sigma(t)\frac{\partial u_i}{\partial x} - \tilde{\sigma}_h(t)\frac{\partial \tilde{u}_i^h}{\partial x} \right], \frac{\partial e_i}{\partial x} \right\rangle, \quad i = 1, 2. \tag{4.3}$$

Considering the last term on the left-hand side of (4.3) and denoting,

$$\omega_i = \frac{\partial u_i^h}{\partial x}, \qquad \eta_i = \frac{\partial \tilde{u}_i^h}{\partial x},$$

we have

$$\begin{aligned}
&\left\langle \left[\sigma_h(t) \frac{\partial u_i^h}{\partial x} - \tilde{\sigma}_h(t) \frac{\partial \tilde{u}_i^h}{\partial x} \right], \frac{\partial e_i}{\partial x} \right\rangle \\
&\quad = \frac{1}{2} \left\langle \int_0^t \int_0^1 \left[\omega_1^2 + \omega_2^2 \right] d\xi d\tau + \int_0^t \int_0^1 \left[\eta_1^2 + \eta_2^2 \right] d\xi d\tau, (\omega_i - \eta_i)^2 \right\rangle \\
&\qquad + \frac{1}{2} \left\langle \int_0^t \int_0^1 \left[\omega_1^2 + \omega_2^2 \right] d\xi d\tau - \int_0^t \int_0^1 \left[\eta_1^2 + \eta_2^2 \right] d\xi d\tau, \omega_i^2 - \eta_i^2 \right\rangle \\
&\quad \geq \frac{1}{2} \int_0^1 \left\{ \int_0^t \int_0^1 \left(\omega_1^2 + \omega_2^2 - \eta_1^2 - \eta_2^2 \right) d\xi d\tau \right\} \left(\omega_i^2 - \eta_i^2 \right) dx.
\end{aligned}$$

Now $\omega_1^2 - \eta_1^2 + \omega_2^2 - \eta_2^2 \geq 2 \min_{j=1,2} (\omega_j^2 - \eta_j^2) \equiv 2(\omega_k^2 - \eta_k^2)$. Also $\omega_i^2 - \eta_i^2 \geq \min_{j=1,2} (\omega_j^2 - \eta_j^2)$. So,

$$\begin{aligned}
&\left\langle \left[\sigma_h(t) \frac{\partial u_i^h}{\partial x} - \tilde{\sigma}_h(t) \frac{\partial \tilde{u}_i^h}{\partial x} \right], \frac{\partial e_i}{\partial x} \right\rangle \\
&\quad \geq \int_0^t \int_0^1 \left(\omega_k^2 - \eta_k^2 \right) d\xi d\tau \int_0^1 \left(\omega_k^2 - \eta_k^2 \right) dx = \frac{1}{2} \frac{d\phi_k^2}{dt},
\end{aligned}$$

where

$$\phi_k(t) \equiv \int_0^t \int_0^1 \left(\omega_k^2 - \eta_k^2 \right) d\xi d\tau.$$

Therefore the left-hand side of (4.3) can be rewritten as follows:

$$\begin{aligned}
&\left\langle e_i, \frac{\partial e_i}{\partial t} \right\rangle + \left\langle \frac{\partial e_i}{\partial x}, \frac{\partial e_i}{\partial x} \right\rangle + \left\langle \left[\sigma_h(t) \frac{\partial u_i^h}{\partial x} - \tilde{\sigma}_h(t) \frac{\partial \tilde{u}_i^h}{\partial x} \right], \frac{\partial e_i}{\partial x} \right\rangle \\
&\quad \geq \frac{1}{2} \frac{d}{dt} ||e_i||^2 + \left|\left| \frac{\partial e_i}{\partial x} \right|\right|^2 + \frac{1}{2} \frac{d\phi_k^2}{dt}, \qquad i = 1, 2.
\end{aligned}$$

Now consider the last term on the right-hand side of (4.3). Substituting for $\frac{\partial u_i}{\partial x}$ from (4.2), we have

$$\sigma(t)\frac{\partial u_i}{\partial x} - \tilde{\sigma}_h(t)\frac{\partial \tilde{u}_i^h}{\partial x}$$
$$= \frac{\partial E_i}{\partial x}\int_0^t\int_0^1 \left[\left(\frac{\partial E_1}{\partial x}+\frac{\partial \tilde{u}_1^h}{\partial x}\right)^2+\left(\frac{\partial E_2}{\partial x}+\frac{\partial \tilde{u}_2^h}{\partial x}\right)^2\right]dxd\tau$$
$$+\frac{\partial \tilde{u}_i^h}{\partial x}\int_0^t\int_0^1\left[\left(\frac{\partial E_1}{\partial x}\right)^2+\left(\frac{\partial E_2}{\partial x}\right)^2+2\frac{\partial E_1}{\partial x}\frac{\partial \tilde{u}_1^h}{\partial x}+2\frac{\partial E_2}{\partial x}\frac{\partial \tilde{u}_2^h}{\partial x}\right]dxd\tau, \quad i=1,2.$$

Taking this into account in the right-hand side of (4.3), we get

$$\left\langle e_i, \frac{\partial E_i}{\partial t}\right\rangle + \left\langle \frac{\partial E_i}{\partial x}, \frac{\partial e_i}{\partial x}\right\rangle + \left\langle \frac{\partial E_i}{\partial x}, \frac{\partial e_i}{\partial x}\right\rangle \int_0^t\int_0^1 \left[\left(\frac{\partial \tilde{u}_1^h}{\partial x}\right)^2 + \left(\frac{\partial \tilde{u}_2^h}{\partial x}\right)^2\right]dxd\tau$$
$$+\left\langle \frac{\partial E_i}{\partial x}, \frac{\partial e_i}{\partial x}\right\rangle \int_0^t\int_0^1 \left[\left(\frac{\partial E_1}{\partial x}\right)^2 + \left(\frac{\partial E_2}{\partial x}\right)^2\right]dxd\tau$$
$$+\left\langle \frac{\partial \tilde{u}_i^h}{\partial x}, \frac{\partial e_i}{\partial x}\right\rangle \int_0^t\int_0^1 \left[\left(\frac{\partial E_1}{\partial x}\right)^2 + \left(\frac{\partial E_2}{\partial x}\right)^2\right]dxd\tau$$
$$+\left\langle \frac{\partial E_i}{\partial x}+\frac{\partial \tilde{u}_i^h}{\partial x}, \frac{\partial e_i}{\partial x}\right\rangle \int_0^t\int_0^1 2\left(\frac{\partial E_1}{\partial x}\frac{\partial \tilde{u}_1^h}{\partial x}+\frac{\partial E_2}{\partial x}\frac{\partial \tilde{u}_2^h}{\partial x}\right)dxd\tau \le \left\langle e_i, \frac{\partial E_i}{\partial t}\right\rangle$$
$$+\left\langle \left|\frac{\partial E_i}{\partial x}\right|, \left|\frac{\partial e_i}{\partial x}\right|\right\rangle \left\{1+\int_0^t\int_0^1\left[\left(1+\frac{1}{\epsilon_1}\right)\left(\frac{\partial \tilde{u}_1^h}{\partial x}\right)^2+\left(1+\frac{1}{\epsilon_2}\right)\left(\frac{\partial \tilde{u}_2^h}{\partial x}\right)^2\right]dxd\tau\right.$$
$$\left.+\int_0^t\int_0^1\left[(1+\epsilon_1)\left(\frac{\partial E_1}{\partial x}\right)^2+(1+\epsilon_2)\left(\frac{\partial E_2}{\partial x}\right)^2\right]dxd\tau\right\}$$
$$+\left\langle \left|\frac{\partial \tilde{u}_i^h}{\partial x}\right|, \left|\frac{\partial e_i}{\partial x}\right|\right\rangle \left\{\int_0^t\int_0^1\left[\left(\frac{\partial E_1}{\partial x}\right)^2+\left(\frac{\partial E_2}{\partial x}\right)^2\right]dxd\tau\right.$$
$$+2\left|\sup_{x,t}\left(\frac{\partial \tilde{u}_1^h}{\partial x}\right)\right|\int_0^t\int_0^1\left|\frac{\partial E_1}{\partial x}\right|dxd\tau$$
$$\left.+2\left|\sup_{x,t}\left(\frac{\partial \tilde{u}_2^h}{\partial x}\right)\right|\int_0^t\int_0^1\left|\frac{\partial E_2}{\partial x}\right|dxd\tau\right\},$$

where ϵ_1 and ϵ_2 come from the Cauchy–Schwarz inequality.

Now incorporate these estimates into (4.3) to obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\,||e_i(\cdot,t)||^2+\left|\left|\frac{\partial e_i(\cdot,t)}{\partial x}\right|\right|^2+\frac{1}{2}\frac{d\phi_k^2}{dt}\\
&\quad\leq\left\langle e_i(\cdot,t),\,\frac{\partial E_i(\cdot,t)}{\partial t}\right\rangle+\left\langle\left|\frac{\partial E_i(\cdot,t)}{\partial x}\right|,\left|\frac{\partial e_i(\cdot,t)}{\partial x}\right|\right\rangle\\
&\qquad\times\left\{1+\int_0^t\int_0^1\left[\left(1+\frac{1}{\epsilon_1}\right)\left(\frac{\partial\tilde{u}_1^h}{\partial x}\right)^2+\left(1+\frac{1}{\epsilon_2}\right)\left(\frac{\partial\tilde{u}_2^h}{\partial x}\right)^2\right]dxd\tau\right.\\
&\qquad\quad\left.+\int_0^t\int_0^1\left[(1+\epsilon_1)\left(\frac{\partial E_1}{\partial x}\right)^2+(1+\epsilon_2)\left(\frac{\partial E_2}{\partial x}\right)^2\right]dxd\tau\right\}\\
&\qquad+\left\langle\left|\frac{\partial\tilde{u}_i^h}{\partial x}\right|,\left|\frac{\partial e_i}{\partial x}\right|\right\rangle\left\{\int_0^t\int_0^1\left[\left(\frac{\partial E_1}{\partial x}\right)^2+\left(\frac{\partial E_2}{\partial x}\right)^2\right]dxd\tau\right.\\
&\qquad\qquad+2\left|\sup_{x,t}\left(\frac{\partial\tilde{u}_1^h}{\partial x}\right)\right|\int_0^t\int_0^1\left|\frac{\partial E_1}{\partial x}\right|dxd\tau\\
&\qquad\qquad\left.+2\left|\sup_{x,t}\left(\frac{\partial\tilde{u}_2^h}{\partial x}\right)\right|\int_0^t\int_0^1\left|\frac{\partial E_2}{\partial x}\right|dxd\tau\right\}.
\end{aligned}\tag{4.4}$$

Integrating (4.4) with respect to t, we have

$$\begin{aligned}
&\frac{1}{2}\,||e_i(\cdot,T)||^2+\int_0^T\left|\left|\frac{\partial e_i(\cdot,t)}{\partial x}\right|\right|^2dt+\frac{1}{2}\phi_k^2(T)\\
&\quad\leq\frac{1}{2}\,||e_i(\cdot,0)||^2+\int_0^T\int_0^1 e_i\frac{\partial E_i}{\partial t}dxdt+\int_0^T\int_0^1\left|\frac{\partial E_i}{\partial x}\frac{\partial e_i}{\partial x}\right|dx\\
&\qquad\times\left\{1+\int_0^t\int_0^1\left[\left(1+\frac{1}{\epsilon_1}\right)\left(\frac{\partial\tilde{u}_1^h}{\partial x}\right)^2+\left(1+\frac{1}{\epsilon_2}\right)\left(\frac{\partial\tilde{u}_2^h}{\partial x}\right)^2\right]dxd\tau\right.\\
&\qquad\quad\left.+\int_0^t\int_0^1\left[(1+\epsilon_1)\left(\frac{\partial E_1}{\partial x}\right)^2+(1+\epsilon_2)\left(\frac{\partial E_2}{\partial x}\right)^2\right]dxd\tau\right\}dt\\
&\qquad+\int_0^T\int_0^1\left|\frac{\partial\tilde{u}_i^h}{\partial x}\frac{\partial e_i}{\partial x}\right|dx\left\{\int_0^t\int_0^1\left[\left(\frac{\partial E_1}{\partial x}\right)^2+\left(\frac{\partial E_2}{\partial x}\right)^2\right]dxd\tau\right.\\
&\qquad\qquad+2\left|\sup_{x,t}\left(\frac{\partial\tilde{u}_1^h}{\partial x}\right)\right|\int_0^t\int_0^1\left|\frac{\partial E_1}{\partial x}\right|dxd\tau\\
&\qquad\qquad\left.+2\left|\sup_{x,t}\left(\frac{\partial\tilde{u}_2^h}{\partial x}\right)\right|\int_0^t\int_0^1\left|\frac{\partial E_2}{\partial x}\right|dxd\tau\right\}dt.
\end{aligned}\tag{4.5}$$

We can estimate the following factor in the last integral:

$$\int_0^t \int_0^1 \left[\left(\frac{\partial E_1}{\partial x} \right)^2 + \left(\frac{\partial E_2}{\partial x} \right)^2 \right] dx d\tau + 2 \left| \sup_{x,t} \left(\frac{\partial \tilde{u}_1^h}{\partial x} \right) \right| \int_0^t \int_0^1 \left| \frac{\partial E_1}{\partial x} \right| dx d\tau$$
$$+ 2 \left| \sup_{x,t} \left(\frac{\partial \tilde{u}_2^h}{\partial x} \right) \right| \int_0^t \int_0^1 \left| \frac{\partial E_2}{\partial x} \right| dx d\tau \leq \left|\left|\left| \frac{\partial E_1}{\partial x} \right|\right|\right|^2 + \left|\left|\left| \frac{\partial E_2}{\partial x} \right|\right|\right|^2$$
$$+ 2 \left| \sup_{x,t} \left(\frac{\partial \tilde{u}_1^h}{\partial x} \right) \right| \left[\!\left] \frac{\partial E_1}{\partial x} \right[\!\right] + 2 \left| \sup_{x,t} \left(\frac{\partial \tilde{u}_2^h}{\partial x} \right) \right| \left[\!\left] \frac{\partial E_2}{\partial x} \right[\!\right].$$

Denoting the right-hand side of this inequality by I, the last term of (3.5) becomes

$$I \int_0^T \left| \sup_x \frac{\partial \tilde{u}_i^h}{\partial x} \right| \int_0^1 \left| \frac{\partial e_i}{\partial x} \right| dx dt \leq I \sqrt{\int_0^T \left| \sup_x \frac{\partial \tilde{u}_i^h}{\partial x} \right|^2 dt} \sqrt{\int_0^T \left(\int_0^1 \left| \frac{\partial e_i}{\partial x} \right| dx \right)^2 dt}$$
$$\leq I \sqrt{\int_0^T \left| \sup_x \frac{\partial \tilde{u}_i^h}{\partial x} \right|^2 dt} \left|\left|\left| \frac{\partial e_i}{\partial x} \right|\right|\right|$$
$$\leq \frac{\epsilon_3}{2} I^2 \int_0^T \left| \sup_x \frac{\partial \tilde{u}_i^h}{\partial x} \right|^2 dt + \frac{1}{2\epsilon_3} \left|\left|\left| \frac{\partial e_i}{\partial x} \right|\right|\right|^2.$$

Thus, (4.5) yields

$$\left|\left|\left| \frac{\partial e_i}{\partial x} \right|\right|\right|^2 \leq \frac{1}{2} ||e_i(\cdot, 0)||^2 + \frac{1}{2} \left(\epsilon_4 \, |||e_i|||^2 + \frac{1}{\epsilon_4} \left|\left|\left| \frac{\partial E_i}{\partial t} \right|\right|\right|^2 \right)$$
$$+ \frac{1}{2} \left(L + \sum_{m=1}^{2} (1 + \epsilon_m) \left|\left|\left| \frac{\partial E_m}{\partial x} \right|\right|\right|^2 \right) \left(\epsilon_5 \left|\left|\left| \frac{\partial E_i}{\partial x} \right|\right|\right|^2 + \frac{1}{\epsilon_5} \left|\left|\left| \frac{\partial e_i}{\partial x} \right|\right|\right|^2 \right)$$
$$+ \frac{\epsilon_3}{2} \left[\sum_{m=1}^{2} \left(\left|\left|\left| \frac{\partial E_m}{\partial x} \right|\right|\right|^2 + 2 \left| \sup_{x,t} \left(\frac{\partial \tilde{u}_m^h}{\partial x} \right) \right| \left[\!\left] \frac{\partial E_m}{\partial x} \right[\!\right] \right) \right]^2$$
$$\times \int_0^T \left| \sup_x \frac{\partial \tilde{u}_i^h}{\partial x} \right|^2 dt + \frac{1}{2\epsilon_3} \left|\left|\left| \frac{\partial e_i}{\partial x} \right|\right|\right|^2,$$

where

$$L = 1 + \int_0^t \int_0^1 \left[\left(1 + \frac{1}{\epsilon_1} \right) \left(\frac{\partial \tilde{u}_1^h}{\partial x} \right)^2 + \left(1 + \frac{1}{\epsilon_2} \right) \left(\frac{\partial \tilde{u}_2^h}{\partial x} \right)^2 \right] dx d\tau.$$

Since E_i is the interpolation error, we have from (3.4),

$$|||E_i||| \leq C_0 h^j |||u_i|||,$$
$$\left|\left|\left|\frac{\partial E_i}{\partial x}\right|\right|\right| \leq C_1 h^{j-1} |||u_i|||,$$
$$\left|\left|\left|\frac{\partial E_i}{\partial t}\right|\right|\right| \leq C_2 h^j \left|\left|\left|\frac{\partial u_i}{\partial t}\right|\right|\right|,$$
$$\left[\right]\frac{\partial E_i}{\partial x}\left[\right] \leq C_3 h^{j-1} []u_i[], \qquad i = 1, 2.$$

Therefore,

$$\begin{aligned}\left|\left|\left|\frac{\partial e_i}{\partial x}\right|\right|\right|^2 \leq\ & \frac{1}{2}||e_i(\cdot,0)||^2 + \frac{\epsilon_4}{2}|||e_i|||^2 + \frac{1}{2\epsilon_4}C_2^2 h^{2j}\left|\left|\left|\frac{\partial u_i}{\partial t}\right|\right|\right|^2 \\ & + \frac{1}{2}\left(L + \sum_{m=1}^{2}(1+\epsilon_m)C_1^2 h^{2(j-1)}|||u_m|||^2\right) \\ & \times \left(\epsilon_5 C_1^2 h^{2(j-1)}|||u_i|||^2 + \frac{1}{\epsilon_5}\left|\left|\left|\frac{\partial e_i}{\partial x}\right|\right|\right|^2\right) \\ & + \frac{\epsilon_3}{2}\left[\sum_{m=1}^{2}\left(C_1^2 h^{2j-2}|||u_m|||^2 + 2\left|\sup_{x,t}\left(\frac{\partial \tilde{u}_m^h}{\partial x}\right)\right| C_3 h^{j-1} []u_m[]\right)\right]^2 \\ & \times \int_0^T \left|\sup_x \frac{\partial \tilde{u}_i^h}{\partial x}\right|^2 dt + \frac{1}{2\epsilon_3}\left|\left|\left|\frac{\partial e_i}{\partial x}\right|\right|\right|^2.\end{aligned}$$

Now we collect terms with norms of the error e_i to obtain

$$\begin{aligned}& \left|\left|\left|\frac{\partial e_i}{\partial x}\right|\right|\right|^2 - \frac{\epsilon_4}{2}|||e_i|||^2 - \frac{1}{2}\left(L + \sum_{m=1}^{2}(1+\epsilon_m)C_1^2 h^{2(j-1)}|||u_m|||^2\right)\frac{1}{\epsilon_5}\left|\left|\left|\frac{\partial e_i}{\partial x}\right|\right|\right|^2 \\ & - \frac{1}{2\epsilon_3}\left|\left|\left|\frac{\partial e_i}{\partial x}\right|\right|\right|^2 \leq \frac{1}{2}||e_i(\cdot,0)||^2 + \frac{1}{2\epsilon_4}C_2^2 h^{2j}\left|\left|\left|\frac{\partial u_i}{\partial t}\right|\right|\right|^2 \\ & + \frac{1}{2}\left(L + \sum_{m=1}^{2}(1+\epsilon_m)C_1^2 h^{2(j-1)}|||u_m|||^2\right)\epsilon_5 C_1^2 h^{2(j-1)}|||u_i|||^2 \\ & + \frac{\epsilon_3}{2}\left[\sum_{m=1}^{2}\left(C_1^2 h^{2j-2}|||u_m|||^2 + 2\left|\sup_{x,t}\left(\frac{\partial \tilde{u}_m^h}{\partial x}\right)\right| C_3 h^{j-1} []u_m[]\right)\right]^2 \\ & \times \int_0^T \left|\sup_x \frac{\partial \tilde{u}_i^h}{\partial x}\right|^2 dt.\end{aligned}$$

Next we use the Poincaré inequality

$$\left|\left|\frac{\partial e_i}{\partial x}\right|\right| \geq C_p \,||e_i||$$

to show that

$$\left|\left|\left|\frac{\partial e_i}{\partial x}\right|\right|\right| \geq C_p \,|||e_i|||\,.$$

So, the last relation becomes

$$\begin{aligned}
&\left(1 - \frac{\epsilon_4}{2C_p^2} - \frac{L}{2\epsilon_5} - \frac{1}{2\epsilon_5}\sum_{m=1}^{2}(1+\epsilon_m)C_1^2 h^{2(j-1)}|||u_m|||^2 - \frac{1}{2\epsilon_3}\right)\left|\left|\left|\frac{\partial e_i}{\partial x}\right|\right|\right|^2 \\
&\quad \leq \frac{1}{2}\,||e_i(\cdot,0)||^2 + \frac{1}{2\epsilon_4}C_2^2 h^{2j}\left|\left|\left|\frac{\partial u_i}{\partial t}\right|\right|\right|^2 \\
&\qquad + \frac{1}{2}\epsilon_5 C_1^2 h^{2(j-1)}|||u_i|||^2\left(L + \sum_{m=1}^{2}(1+\epsilon_m)C_1^2 h^{2(j-1)}|||u_m|||^2\right) \\
&\qquad + \frac{\epsilon_3}{2}\left[\sum_{m=1}^{2}\left(C_1^2 h^{2j-2}\,|||u_m|||^2 + 2\left|\sup_{x,t}\left(\frac{\partial \tilde{u}_m^h}{\partial x}\right)\right| C_3 h^{j-1}\,[]\,u_m\,[]\right)\right]^2 \\
&\quad \times \int_0^T \left|\sup_x \frac{\partial \tilde{u}_i^h}{\partial x}\right|^2 dt.
\end{aligned}$$

Now choose ϵ_3, ϵ_4 and ϵ_5 so that the coefficient of $\left|\left|\left|\frac{\partial e_i}{\partial x}\right|\right|\right|^2$ is positive, say C_4. The right-hand side depends on the grid size h and the known error at time $t = 0$, i.e.,

$$||e_i(\cdot,0)|| \leq C_5 h^j\,||u_{i0}||\,.$$

Thus,

$$\begin{aligned}
\left|\left|\left|\frac{\partial e_i}{\partial x}\right|\right|\right|^2 \leq h^{2j-2}\Bigg\{ &\frac{C_5^2}{2C_4}h^2\,||u_{i0}||^2 + \frac{C_2^2}{2C_4\epsilon_4}h^2\left|\left|\left|\frac{\partial u_i}{\partial t}\right|\right|\right|^2 \\
&+ \frac{\epsilon_5 C_1^2}{2C_4}|||u_i|||^2\left(L + \sum_{m=1}^{2}(1+\epsilon_m)C_1^2 h^{2(j-1)}|||u_m|||^2\right) \\
&+ \frac{\epsilon_3}{2C_4}\left[\sum_{m=1}^{2}\left(C_1^2 h^{j-1}\,|||u_m|||^2 + 2\left|\sup_{x,t}\left(\frac{\partial \tilde{u}_m^h}{\partial x}\right)\right| C_3[]u_m[]\right)\right]^2 \\
&\times \int_0^T \left|\sup_x \frac{\partial \tilde{u}_i^h}{\partial x}\right|^2 dt \Bigg\}.
\end{aligned}$$

Recall that

$$|||e_i|||_1^2 = |||e_i|||^2 + \left|\left|\left|\frac{\partial e_i}{\partial x}\right|\right|\right|^2 \leq \left(1 + \frac{1}{C_p^2}\right)\left|\left|\left|\frac{\partial e_i}{\partial x}\right|\right|\right|^2,$$

so taking into account the last estimate we obtain

$$
\begin{aligned}
|||e_i|||_1^2 \le h^{2j-2}C_6 \Bigg\{ & \frac{C_5^2}{2C_4}h^2\,||u_{i0}||^2 + \frac{C_2^2}{2C_4\epsilon_4}h^2 \left|\left|\left| \frac{\partial u_i}{\partial t} \right|\right|\right|^2 \\
& + \frac{\epsilon_5 C_1^2}{2C_4}|||u_i|||^2 \left(L + \sum_{m=1}^{2}(1+\epsilon_m)C_1^2 h^{2(j-1)}|||u_m|||^2 \right) \\
& + \frac{\epsilon_3}{2C_4}\left[\sum_{m=1}^{2} \left(C_1^2 h^{j-1}\,|||u_m|||^2 + 2\left| \sup_{x,t}\left(\frac{\partial \tilde{u}_m^h}{\partial x} \right) \right| C_3[]u_m[] \right) \right]^2 \\
& \times \int_0^T \left| \sup_x \frac{\partial \tilde{u}_i^h}{\partial x} \right|^2 dt \Bigg\},
\end{aligned}
$$

where $C_6 = \left(1 + \frac{1}{C_p^2}\right)$.

From this, using the triangle inequality

$$
|||u_i - u_i^h|||_1^2 = |||u_i - \bar{u}_i^h + \bar{u}_i^h - u_i^h|||_1^2 \le 2|||E_i|||_1^2 + 2|||e_i|||_1^2
$$

and estimating

$$
\begin{aligned}
|||E_i|||_1^2 &= |||E_i|||^2 + \left|\left|\left| \frac{\partial E_i}{\partial x} \right|\right|\right|^2 \le C_0^2 h^{2j}|||u_i|||^2 + C_1^2 h^{2j-2}|||u_i|||^2 \\
&= \left(C_0^2 h^2 + C_1^2 \right) h^{2j-2}|||u_i|||^2,
\end{aligned}
$$

we finally get

$$
\begin{aligned}
|||u_i - u_i^h|||_1 \le h^{j-1}\Bigg\{ & c_1 h^2\,||u_{i0}||^2 + c_2 h^2 \left|\left|\left| \frac{\partial u_i}{\partial t} \right|\right|\right|^2 \\
& + c_3|||u_i|||^2 + c_4 h^{2(j-1)} \sum_{m=1}^{2} |||u_m|||^2 \\
& + c_5 \left[\sum_{m=1}^{2} \left(c_6 h^{j-1}\,|||u_m|||^2 + c_7[]u_m[] \right) \right]^2 \Bigg\}^{1/2},
\end{aligned}
$$

where

$$
\begin{aligned}
c_1 &= \frac{C_5^2 C_6}{C_4}, \quad c_2 = \frac{C_2^2 C_6}{C_4\epsilon_4}, \quad c_3 = \frac{\epsilon_5 C_1^2 C_6}{C_4}L + 2C_0^2 h^2 + 2C_1^2, \\
c_4 &= (1 + \max\{\epsilon_1, \epsilon_2\})\,\frac{\epsilon_5 C_1^4 C_6}{C_4}, \quad c_5 = \frac{\epsilon_3 C_6}{C_4}\int_0^T \left| \sup_x \frac{\partial \tilde{u}_i^h}{\partial x} \right|^2 dt, \\
c_6 &= C_1^2, \quad c_7 = 2\left| \sup_{x,t}\left(\frac{\partial \tilde{u}_m^h}{\partial x} \right) \right| C_3.
\end{aligned}
$$

This completes the proof of Theorem 4.1. □

5 Numerical solution

For the numerical solution of (3.5) and (3.6), we let $\phi_1(x), \ldots, \phi_N(x)$ be a basis for S_h (where N is the dimension of S_h). Thus any $u_i^h \in S_h$ can be represented as follows:

$$u_i^h(x,t) = \sum_{j=1}^{N} u_{ij}(t)\phi_j(x). \tag{5.1}$$

Since (3.5) and (3.6) are valid for all $v_i^h \in S_h$, one can let $v_i^h = \phi_k$. Using (5.1), we are led to the following system for the vectors of weights $\mathbf{u_i}(t) = (u_{i1}(t), u_{i2}(t), \ldots, u_{iN}(t))$:

$$\mathbf{M}\dot{\mathbf{u}}_i + \mathbf{K}(u_1, u_2)\mathbf{u}_i = \mathbf{F}_i, \qquad i = 1,\ 2, \tag{5.2}$$

$$\mathbf{M}\mathbf{u}_i(0) = \mathbf{W}_i, \qquad i = 1,\ 2, \tag{5.3}$$

where

$$M_{jk} =< \phi_k, \phi_j >, \tag{5.4}$$

$$K(u_1, u_2)_{jk} =< (1 + \sigma_h(t))\phi_k', \phi_j' >, \tag{5.5}$$

$$F_{ik} =< \phi_k, f_i >, \qquad W_{ik} =< \phi_k, u_{i0} > . \tag{5.6}$$

Now we can evaluate $\sigma_h(t)$ as follows:

$$\begin{aligned}
\sigma_h(t) &= \int_0^t \int_0^1 \left[\left(\sum_{\ell=1}^{N} u_{1\ell}\phi_\ell' \right)^2 + \left(\sum_{\ell=1}^{N} u_{2\ell}\phi_\ell' \right)^2 \right] dx d\tau \\
&= \int_0^t \int_0^1 \left[\sum_{\ell=1}^{N}\sum_{m=1}^{N} u_{1\ell}u_{1m}\phi_\ell'\ \phi_m' + \sum_{\ell=1}^{N}\sum_{m=1}^{N} u_{2\ell}u_{2m}\phi_\ell'\phi_m' \right] dx d\tau \\
&= \sum_{\ell=1}^{N}\sum_{m=1}^{N} \int_0^t (u_{1\ell}u_{1m} + u_{2\ell}u_{2m}) \underbrace{\left[\int_0^1 \phi_\ell'\phi_m' dx \right]}_{=\tilde{K}_{\ell m}} d\tau \\
&= \sum_{\ell=1}^{N}\sum_{m=1}^{N} \tilde{K}_{\ell m} \int_0^t (u_{1\ell}u_{1m} + u_{2\ell}u_{2m})\, d\tau.
\end{aligned} \tag{5.7}$$

The time integral can be approximated by the trapezoidal rule using the equally spaced point t_p, where $t_0 = 0$, $t_n = t$ and $\Delta t = t_i - t_{i-1},\ i = 1, \ldots, n$:

$$\begin{aligned}
&\int_0^t (u_{1\ell}u_{1m} + u_{2\ell}u_{2m})\, d\tau \\
&\quad = \sum_{p=0}^{n} \Delta t \zeta_p \left(u_{1\ell}(t_p)u_{1m}(t_p) + u_{2\ell}(t_p)u_{2m}(t_p) \right) + O\left((\Delta t)^2 \right),
\end{aligned} \tag{5.8}$$

where $\zeta_p = 1/2$ for $p = 0, n$ and $\zeta_p = 1$ for $p = 1, \ldots, n-1$. Combining (5.8) and (5.7) with (5.5), we get

$$\begin{aligned}
&K(u_1, u_2)_{jk}\\
&= \left\langle \left(1 + \sum_{\ell=1}^{N} \sum_{m=1}^{N} \tilde{K}_{\ell m} \sum_{p=0}^{n} \Delta t \zeta_p \left(u_{1\ell}(t_p) u_{1m}(t_p) + u_{2\ell}(t_p) u_{2m}(t_p) \right) \right) \phi'_k, \phi'_j \right\rangle \\
&= \left(1 + \sum_{\ell=1}^{N} \sum_{m=1}^{N} \tilde{K}_{\ell m} \sum_{p=0}^{n} \Delta t \zeta_p \left(u_{1\ell}(t_p) u_{1m}(t_p) + u_{2\ell}(t_p) u_{2m}(t_p) \right) \right) \tilde{K}_{jk} \\
&= \left(1 + \sum_{p=0}^{n} \Delta t \zeta_p v(t_p) \right) \tilde{K}_{jk}, \qquad (5.9)
\end{aligned}$$

where $v(t) = \mathbf{u}_1^T(t) \tilde{\mathbf{K}} \mathbf{u}_1(t) + \mathbf{u}_2^T(t) \tilde{\mathbf{K}} \mathbf{u}_2(t)$. To solve the system (5.2) and (5.3), we use Taylor's series. Let

$$\mathbf{u}_i(t + \Delta t) = \mathbf{u}_i(t) + \Delta t \dot{\mathbf{u}}_i(t) + \frac{1}{2} (\Delta t)^2 \ddot{\mathbf{u}}_i(t) + O\left((\Delta t)^3 \right). \qquad (5.10)$$

Differentiating (5.2) with respect to t, we obtain

$$\mathbf{M}\, \ddot{\mathbf{u}}_i + \mathbf{K}(u_1, u_2) \dot{\mathbf{u}}_i + \dot{\mathbf{K}} \mathbf{u}_i = \dot{\mathbf{F}}_i, \qquad i = 1,\ 2, \qquad (5.11)$$

where the matrices $\mathbf{M}$ and $\mathbf{F}$ are defined by (5.4), (5.6) and

$$\begin{aligned}
\dot{K}_{kj} &= \left\langle \dot{\sigma}_h \phi'_j, \phi'_k \right\rangle \\
&= \left\langle \left[\int_0^1 \left(\sum_{\ell=1}^{N} u_{1\ell}(t) \phi'_\ell \right)^2 + \left(\sum_{\ell=1}^{N} u_{2\ell}(t) \phi'_\ell \right)^2 \right] \phi'_j, \phi'_k \right\rangle \\
&= \left[\sum_{\ell=1}^{N} \sum_{m=1}^{N} \left(u_{1\ell}(t) u_{1m}(t) + u_{2\ell}(t) u_{2m}(t) \right) \int_0^1 \phi'_\ell \phi'_m \right] \tilde{K}_{kj} \\
&= \left[\sum_{\ell=1}^{N} \sum_{m=1}^{N} \left(u_{1\ell}(t) u_{1m}(t) + u_{2\ell}(t) u_{2m}(t) \right) \tilde{K}_{m\ell} \right] \tilde{K}_{kj} \\
&= \left(\mathbf{u}_1^T(t) \tilde{\mathbf{K}} \mathbf{u}_1(t) + \mathbf{u}_2^T(t) \tilde{\mathbf{K}} \mathbf{u}_2(t) \right) \tilde{K}_{kj} = v(t) \tilde{K}_{kj}. \qquad (5.12)
\end{aligned}$$

Now multiplying (5.10) by $\mathbf{M}$ and using (5.2), (5.11) and (5.12), and after dropping terms of order higher than the second, we obtain

$$\begin{aligned}
\mathbf{M}\left(\frac{\mathbf{u}_i(t+\Delta t)-\mathbf{u}_i(t)}{\Delta t}\right) &= \mathbf{M}\dot{\mathbf{u}}_i(t)+\frac{1}{2}\Delta t\mathbf{M}\ddot{\mathbf{u}}_i(t)\\
&= [\mathbf{F}_i-\mathbf{K}(u_1,u_2)\mathbf{u}_i]+\frac{1}{2}\Delta t\left[\dot{\mathbf{F}}_i-\mathbf{K}(u_1,u_2)\dot{\mathbf{u}}_i-\dot{\mathbf{K}}\mathbf{u}_i\right]\\
&= [\mathbf{F}_i-\mathbf{K}(u_1,u_2)\mathbf{u}_i]\\
&\quad+\frac{1}{2}\Delta t\left[\dot{\mathbf{F}}_i-\mathbf{K}(u_1,u_2)\mathbf{M}^{-1}\left(\mathbf{F}_i-\mathbf{K}(u_1,u_2)\mathbf{u}_i\right)-\dot{\mathbf{K}}\mathbf{u}_i\right]\\
&= \left[\mathbf{F}_i+\frac{1}{2}\Delta t\dot{\mathbf{F}}_i-\frac{1}{2}\Delta t\mathbf{K}(u_1,u_2)\mathbf{M}^{-1}\mathbf{F}_i\right]\\
&\quad-\mathbf{K}(u_1,u_2)\left[\mathbf{u}_i-\frac{1}{2}\Delta t\mathbf{M}^{-1}\mathbf{K}(u_1,u_2)\mathbf{u}_i\right]-\frac{1}{2}\Delta t\dot{\mathbf{K}}\mathbf{u}_i.
\end{aligned} \tag{5.13}$$

Substituting for $\mathbf{K}$ and $\dot{\mathbf{K}}$ from (5.9) and (5.12), we get from (5.13),

$$\begin{aligned}
&\mathbf{M}\left(\frac{\mathbf{u}_i(t+\Delta t)-\mathbf{u}_i(t)}{\Delta t}\right)\\
&\quad=\left[\mathbf{F}_i+\frac{1}{2}\Delta t\dot{\mathbf{F}}_i-\frac{1}{2}\Delta t\left(1+\sum_{p=0}^{n}\Delta t\zeta_p v(t_p)\right)\tilde{\mathbf{K}}\mathbf{M}^{-1}\mathbf{F}_i\right]\\
&\qquad-\Delta t\left(1+\sum_{p=0}^{n}\Delta t\zeta_p v(t_p)\right)\\
&\qquad\times\tilde{\mathbf{K}}\left[\mathbf{u}_i-\frac{1}{2}\Delta t\,\mathbf{M}^{-1}\left(1+\sum_{p=0}^{n}\Delta t\zeta_p v(t_p)\right)\tilde{\mathbf{K}}\mathbf{u}_i\right]-\frac{1}{2}\Delta t\,v(t)\tilde{\mathbf{K}}\mathbf{u}_i.
\end{aligned} \tag{5.14}$$

If we take $t=t_n$ as in (5.8) and denote $\mathbf{u}_i^n=\mathbf{u}_i(t_n)$, then (5.14) can be written as follows:

$$\begin{aligned}
\mathbf{M}\left(\frac{\mathbf{u}_i^{n+1}-\mathbf{u}_i^n}{\Delta t}\right)&=\left[\mathbf{F}_i^n+\frac{1}{2}\Delta t\dot{\mathbf{F}}_i^n-\frac{1}{2}\Delta t\left(1+\sum_{p=0}^{n}\Delta t\zeta_p v^p\right)\tilde{\mathbf{K}}\mathbf{M}^{-1}\mathbf{F}_i^n\right]\\
&\quad-\left(1+\sum_{p=0}^{n}\Delta t\zeta_p v^p\right)\tilde{\mathbf{K}}\\
&\quad\times\left[\mathbf{u}_i^n-\frac{1}{2}\Delta t\mathbf{M}^{-1}\left(1+\sum_{p=0}^{n}\Delta t\zeta_p v^p\right)\tilde{\mathbf{K}}\mathbf{u}_i^n\right]-\frac{1}{2}\Delta t\,v^n\tilde{\mathbf{K}}\mathbf{u}_i^n.
\end{aligned} \tag{5.15}$$

Now let us denote

$$\phi^n = 1 + \sum_{p=0}^{n} \Delta t \zeta_p v^p. \tag{5.16}$$

Using notation (5.16) from (5.15), we have

$$\begin{aligned}\mathbf{M}\left(\frac{\mathbf{u}_i^{n+1}-\mathbf{u}_i^n}{\Delta t}\right) &= -\phi^n\tilde{\mathbf{K}}\left[\mathbf{u}_i^n - \frac{1}{2}\Delta t\,\mathbf{M}^{-1}\phi^n\tilde{\mathbf{K}}\mathbf{u}_i^n\right] - \frac{1}{2}\Delta t\, v^n\tilde{\mathbf{K}}\mathbf{u}_i^n\\ &\quad + \left[\mathbf{F}_i^n + \frac{1}{2}(\Delta t)\dot{\mathbf{F}}_i^n - \frac{1}{2}\Delta t\,\phi^n\tilde{\mathbf{K}}\mathbf{M}^{-1}\mathbf{F}_i^n\right], \qquad i = 1,\ 2.\end{aligned} \tag{5.17}$$

Note that in (5.17), ϕ^n and v^n depend on both $\mathbf{u}_1$ and $\mathbf{u}_2$. We can update both ϕ^n and v^n after we solve the two systems or we can update them after solving each system.

In our first numerical experiment we have chosen the right-hand side so that the exact solution is given by

$$u_1(x,t) = x(1-x)\sin(x+t)$$

and

$$u_2(x,t) = x(1-x)\cos(x+t)\,.$$

In this case the right-hand side is

$$\begin{aligned}f_1(x,t) \;&=\; x(1-x)\cos(x+t) - \left(1+\frac{11}{30}t\right)(-2\sin(x+t) + 2(1-x)\cos(x+t)\\ &\quad - 2x\cos(x+t) - x(1-x)\sin(x+t))\end{aligned}$$

and

$$\begin{aligned}f_2(x,t) \;&=\; -x(1-x)\sin(x+t) - \left(1+\frac{11}{30}t\right)(-2\cos(x+t) - 2(1-x)\sin(x+t)\\ &\quad + 2x\sin(x+t) - x(1-x)\cos(x+t))\,.\end{aligned}$$

The parameters used are $N = 100$ which dictates $h = 0.01$. In the next two figures we plotted the numerical solution (marked with $*$) and the exact solution at $t = 0.5$ (Figs. 1 and 2) and $t = 1.0$ (Figs. 3 and 4). It is clear that the two solutions are almost identical (Table 1).

Note that the energy norm of the error decreases linearly with h as expected by Theorem 4.1.

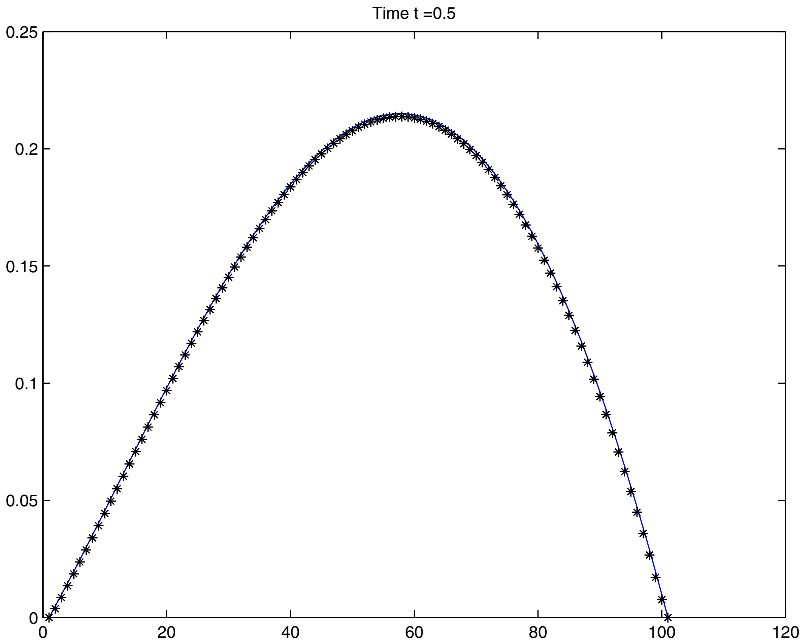


Fig. 1 The solution at $t = 0.5$. The exact solution u_1 is a *solid line* and the numerical solution is marked by $*$

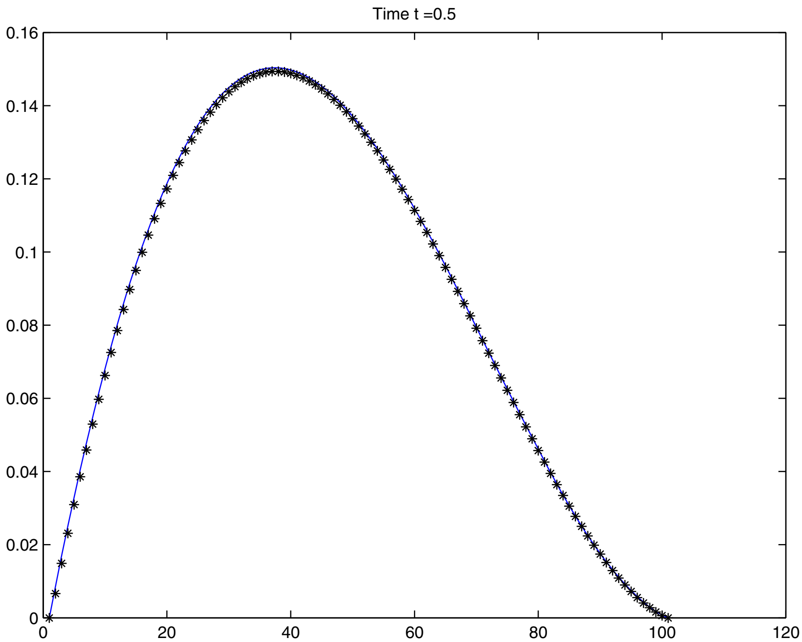


Fig. 2 The solution at $t = 0.5$. The exact solution u_2 is a *solid line* and the numerical solution is marked by $*$

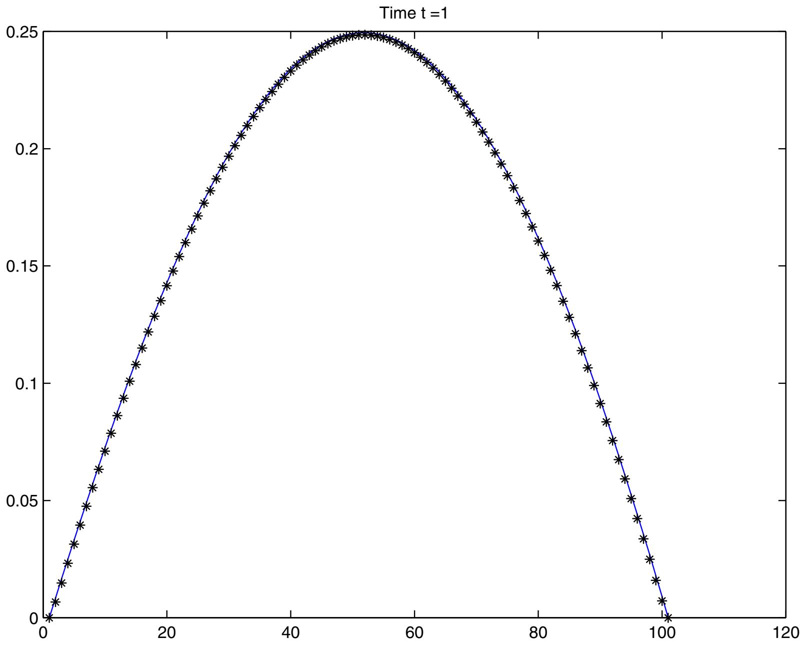


Fig. 3 The solution at $t = 1.0$. The exact solution u_1 is a *solid line* and the numerical solution is marked by $*$

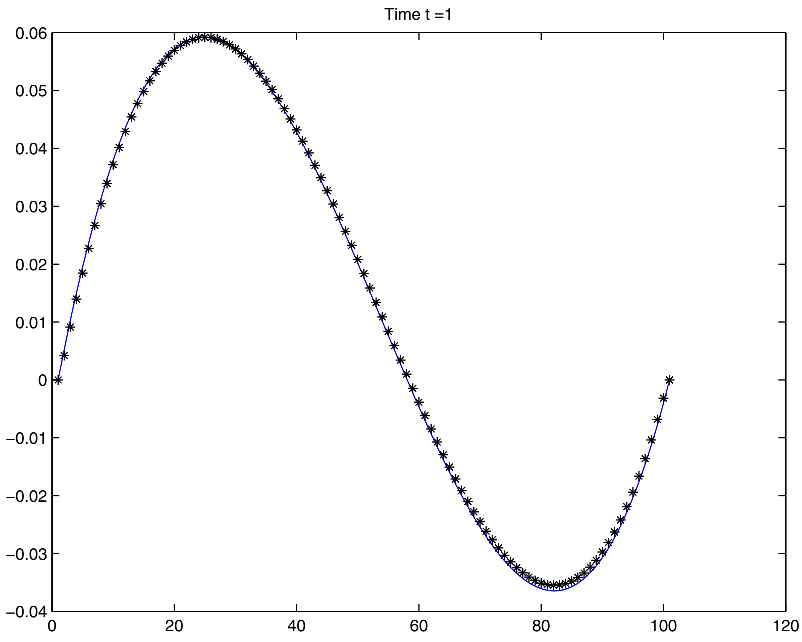


Fig. 4 The solution at $t = 1.0$. The exact solution u_2 is a *solid line* and the numerical solution is marked by $*$

Table 1 The empirical rate of convergence in the energy norm when integrating up to $t = .5$ and $t = 1$

$t = .5$			$t = 1$		
h	Energy norm	Rate	h	Energy norm	Rate
.2	.023124	.934145	.2	.030767	.941436
.04	.005142	.983124	.04	.006762	.988143
.02	.0026012	.992146	.02	.003409	.991434
.01	.0013077		.01	.001715	

In our second experiment we have taken a zero right-hand side and initial data given by

$$u_{10}(x) = u_1(x,0) = x(1-x)\sin(8\pi x), \quad u_{20}(x) = u_2(x,0) = x(1-x)\cos(4\pi x)\,.$$

In this case, we know (Theorem 2.2) that the solution will decay in time. The parameters N, h are as before. In Fig. 5 we plotted the initial data and in Fig. 6 we show the numerical solution at four different times. In both figures the top subplot is for u_1 and the bottom subplot is for u_2. It is clear that the numerical solution is approaching zero for all x. We have also plotted the maximum norm of the partial derivatives $\frac{\partial u_1}{\partial x}$ and $\frac{\partial u_2}{\partial x}$ versus the exponential $e^{-t/2}$. Figure 7 shows that the maximum norm of $\frac{\partial u_1}{\partial x}$ (top) and $\frac{\partial u_2}{\partial x}$ (bottom) decays exponentially. Therefore the numerical approximation of the x-derivative of the solutions obtained in our numerical experiment fully agrees with the theoretical results given in Theorem 2.2. Note that the line for the derivative is very close to the origin which means that the constant in Theorem 2.2 is very small.

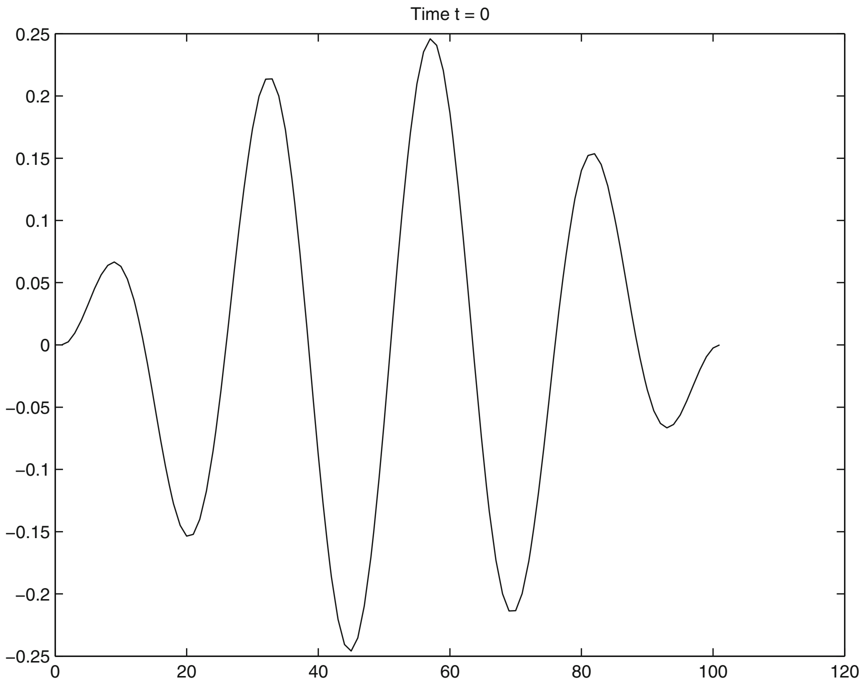
To test the numerical data for a longer time run, we have run the first example up to $t = 5$. The exact solution is plotted with the numerical solution at $t = 5$ in Figs. 8 and 9. Note the agreement between the finite element and exact solution at the end of the run.

We have experimented with several other initial solutions, and in all cases we observed an agreement with the exact solution.

Remark At each time step one can advance $\mathbf{u}_1$ and then advance $\mathbf{u}_2$ using either the previous $\mathbf{u}_1$ or the most current one. As a result, the matrices are smaller and banded.

5.1 Comparison with the finite difference method

The authors of [19] have developed a finite difference scheme to solve the system (3.1). In the finite difference method, the system is nonlinear and Newton's method was used at each time step. This required computing and storing the Jacobian. Therefore the system becomes dense. On the other hand, in the finite element method we did not have to solve a nonlinear system. Therefore the system is banded and the bandwidth is dictated by the degree of the elements used. This fact had already been discussed by Neta and Igwe [21].



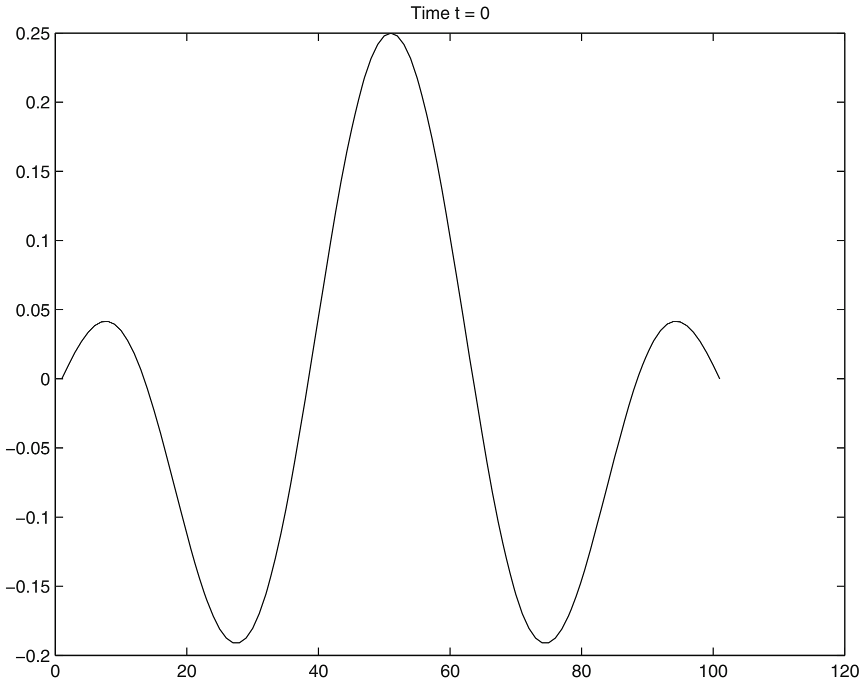


Fig. 5 The **initial data** $u_{10}(x) = x(1-x)\sin(8\pi x)$ (*top*) and $u_{20}(x) = x(1-x)\cos(4\pi x)$ (*bottom*) for Example 2

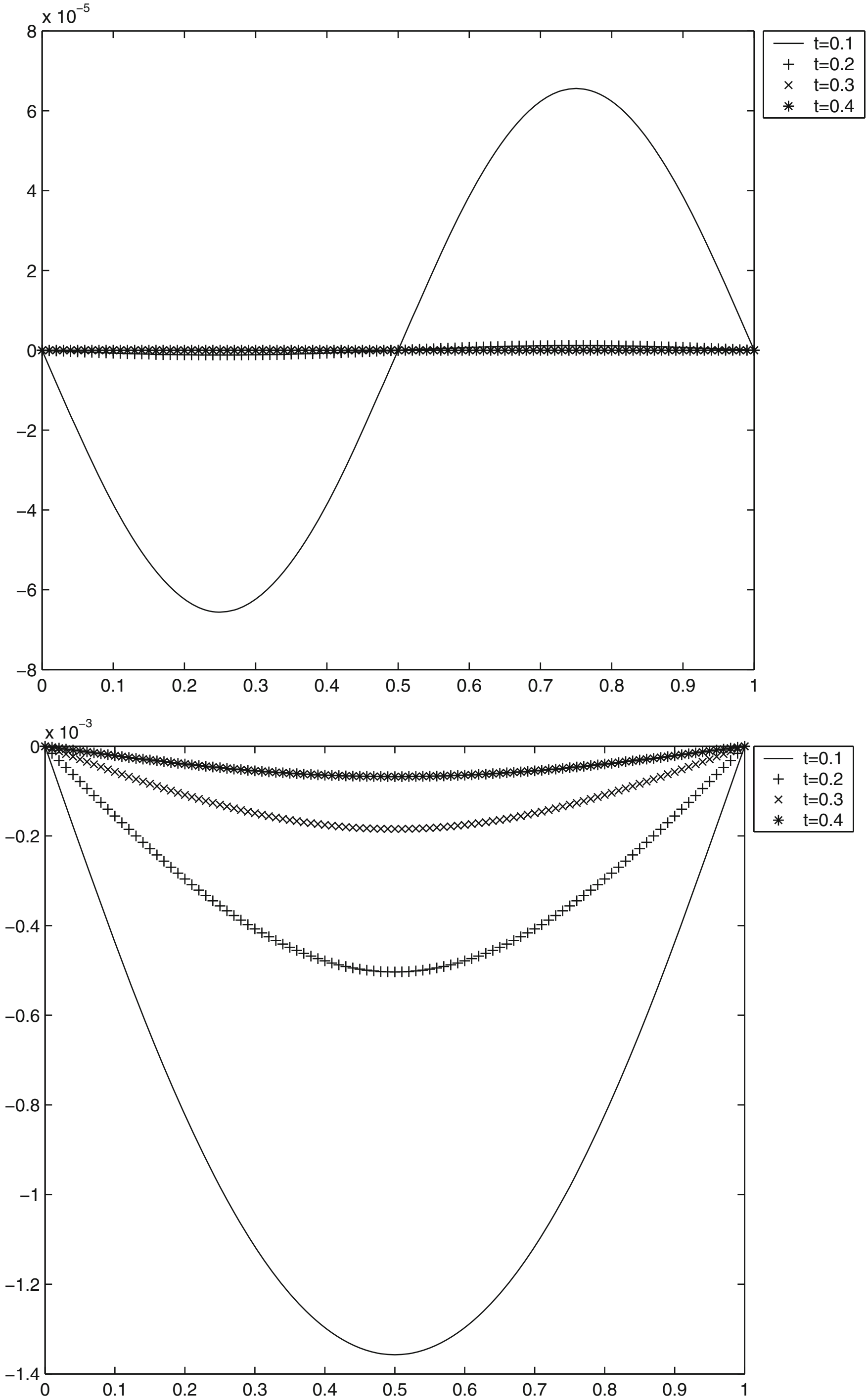


Fig. 6 The numerical solution at $t = 0.1, 0.2, 0.3, 0.4$ for u_1 (*top*) and u_2 (*bottom*)

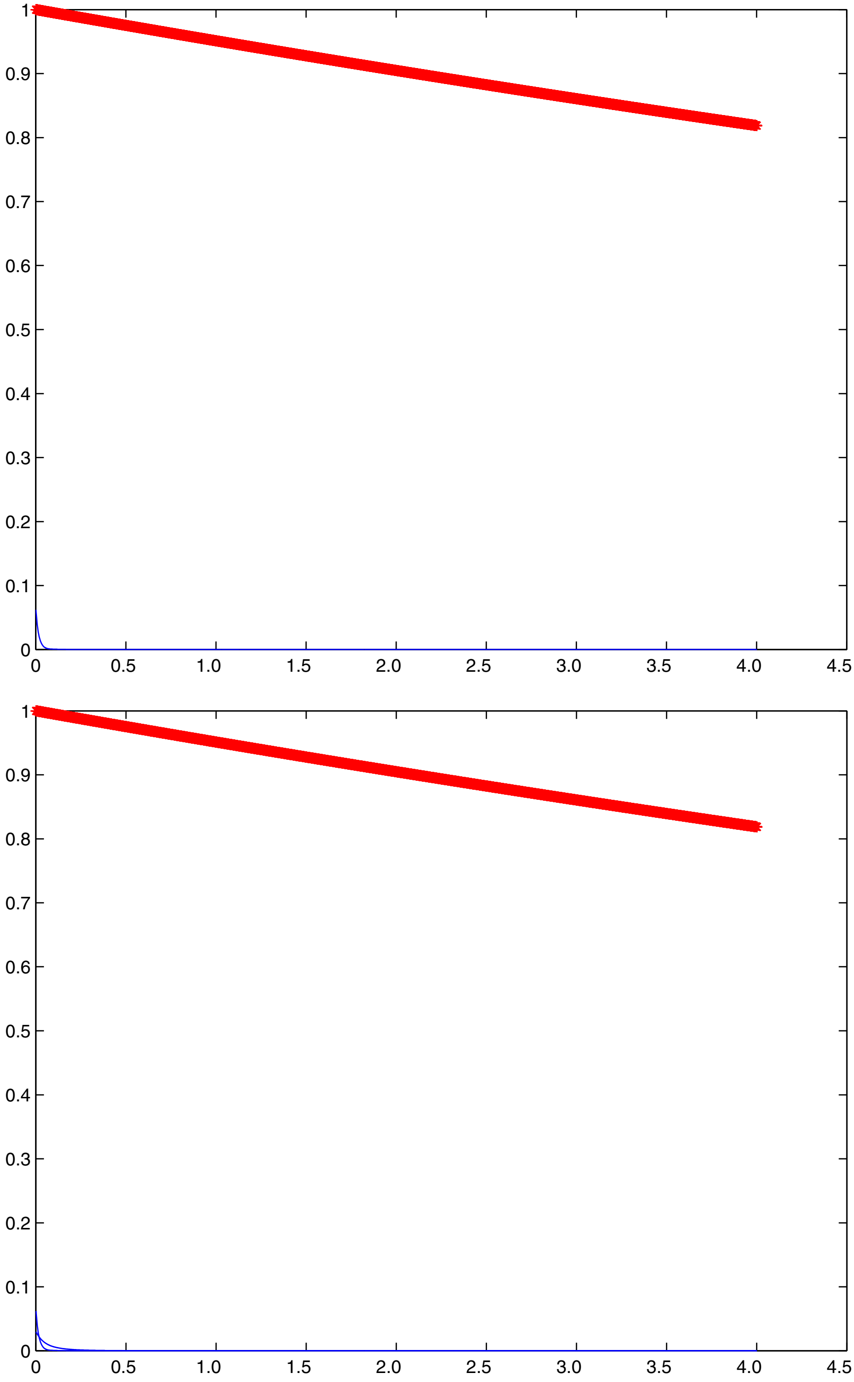


Fig. 7 The maximum norm of the numerical solution for $\frac{\partial u_1}{\partial x}$ (*top*) and $\frac{\partial u_2}{\partial x}$ (*bottom*) (Example 2) and $e^{-t/2}$. A *solid line* is used for $\frac{\partial u_1}{\partial x}$ and $\frac{\partial u_2}{\partial x}$ and a line marked with $*$ for the exponential

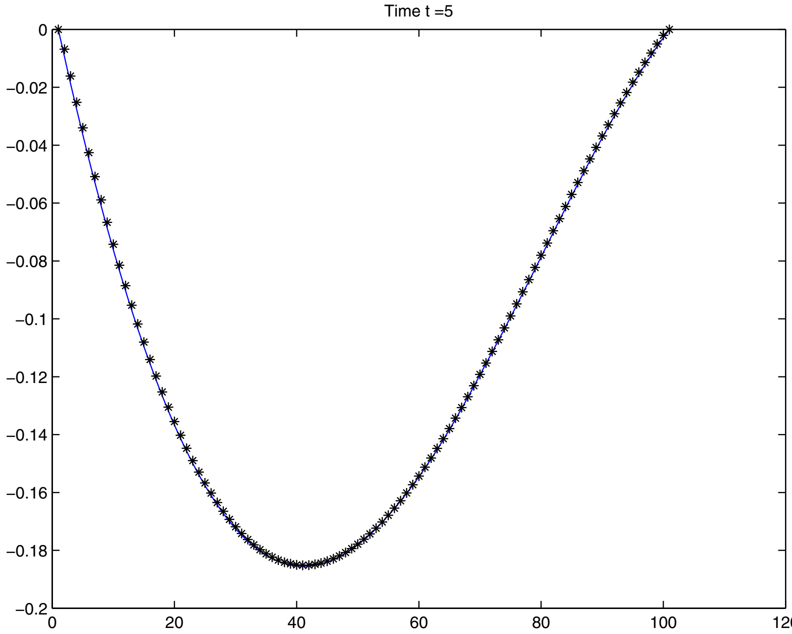


Fig. 8 The solution at $t = 5.0$. The exact solution u_1 is a *solid line* and the numerical solution is marked by $*$

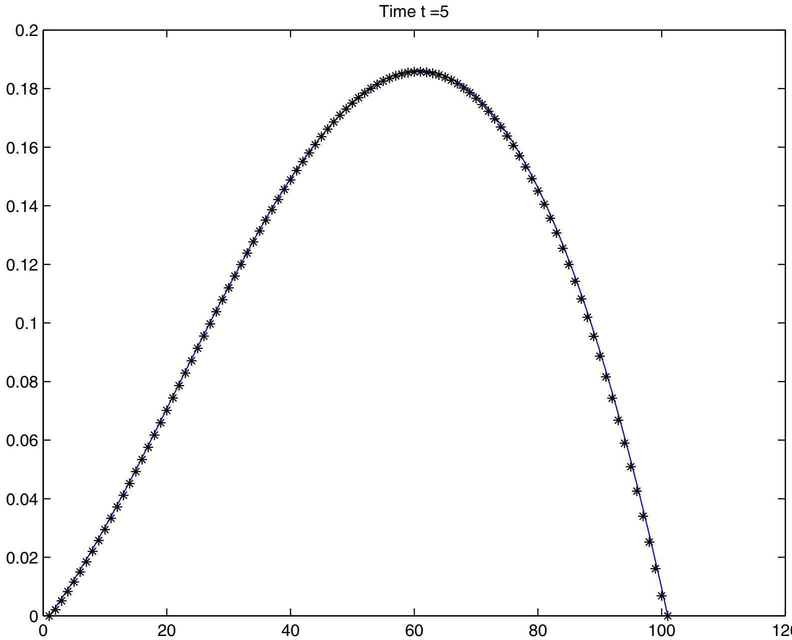


Fig. 9 The solution at $t = 5.0$. The exact solution u_2 is a *solid line* and the numerical solution is marked by $*$

We have recorded the CPU time it took our PC to solve the second example for $N = 25$ and $t = 1$. The finite difference solution took more than twice the time it took to solve the same problem using finite (linear) elements. The accuracy is the same.

6 Conclusions

In this paper we have used the finite element method to solve a system of two nonlinear integro-differential equations. We have shown that it gives the same accuracy as the finite difference scheme without the need to compute the Jacobian at each time step. We have also established the rate of convergence of the finite element method in an appropriate energy norm.

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