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A Restricted Multinomial Hybrid Selection Procedure

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Analysts using simulation models often must assess a large number of alternatives in order to determine which are most effective. If effectiveness corresponds to the likelihood of yielding the best outcome, this becomes a multinomial selection problem. Unfortunately, existing procedures were developed primarily for evaluating small sets of alternatives, so parameters required to implement them may not be readily available or the sampling costs may be prohibitive when a large number of alternatives are present. We propose a truncated, sequential multinomial subset selection procedure that restricts the maximum subset size. Numerical comparisons show that our procedure can be much more efficient than the leading unrestricted procedure. Our procedure requires only one calculated parameter rather than four. We provide extensive tables for cases involving large numbers of alternatives.

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1. INTRODUCTION

Decision making under uncertainty is prevalent in our lives. A decision-making process is a procedure—either formal or informal—that examines several alternatives and makes assessments about their merits. Examples abound. Commuters have the choice of several routes to take to their place of work and may wish to identify those that are consistently the fastest. Pharmaceutical companies may have several potential new treatment programs, from which they will choose the most promising one(s) for further research and development. Military planners usually have more than one strategy or plan that can be used in a specific combat situation and may need to assess which one has the greatest probability of achieving the desired results.

If the consequences of making a poor decision are minimal, informal decision-making processes are commonly used. If the consequences are moderate to severe, it is best to support the decision with quantitative assessments of the alternatives rather than rely on anecdotal evidence or expert opinion. The ANalysis Of VAriance (ANOVA)

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approach is often used for studying multiple systems, but the inferences resulting from the ANOVA approach are limited and can be extremely unsatisfying. In the ANOVA approach, the null hypothesis is that all alternatives are identical. Rejecting the null hypothesis only allows us to state that they are different—we cannot formally state which alternatives are better or worse. Conversely, failing to reject the null hypothesis may simply indicate an insufficient sample size (i.e., lack of statistical power). In contrast, statistical Ranking and Selection (R&S) procedures attempt to address questions of interest to decision makers directly rather than indirectly.

R&S procedures can be classified according to one of three different formulations, each of which provides some type of guarantee on the minimum probability of correct selection ($P(CS)$).

—*Indifference-zone formulation*: These procedures answer questions such as "which alternative is best?" by selecting a single alternative from the many choices. They require the decision maker to prespecify a minimum distance (difference or ratio) between parameters of the best and second-best alternative that is of practical interest to detect. The guarantee on the $P(CS)$ is achieved whenever the true distance meets or exceeds this value. Put in other words, this value defines the region in which the $P(CS)$ guarantees must be met.
—*Subset selection formulation*: These procedures answer questions such as "which alternatives are good?" by selecting a nonempty subset, of random size, that contains the best alternative with some specified probability of correct selection.
—*Multiple comparison formulation*: These procedures answer questions such as "how good are the alternatives?" by using simultaneous confidence intervals to estimate the magnitudes of the differences among the alternatives' means, with high probability of correctly doing so. In some cases, Multiple Comparison with the Best (MCB) methods can augment indifference-zone or subset selection approaches (e.g., Nelson and Matejcik [1995]; Hsu [1996]).

Most of the R&S procedures define the *best* alternative as that having the largest expected response μ. In this article, we define the best alternative more broadly as that which has the largest probability of yielding the desired response. Here, the desired response could be the largest value, the smallest value, the smallest deviation from a target value, the smallest variance, and so forth. Procedures that adopt this definition of best are known as *multinomial selection procedures* and result in selecting the system most likely to be the best [Kim and Nelson 2006].

A subset selection procedure outputs a subset that has a random size. If the alternatives are truly very close to each other—or if, simply by chance, the sampling makes the choice of the best alternative more difficult—the size of the subset can be quite close to the original number of alternatives. In this case, restricting the size of the selected subset so that it will not exceed some prespecified value is beneficial. This is called a *restricted subset selection problem* [Gupta and Panchapakesan 1979].

A few multinomial selection procedures are available in the literature:

(1) *Indifference-zone procedures*: These procedures were born with the *single-stage* procedure $\mathcal{M}_{\text{BEM}}$ [Bechhofer et al. 1959]. Bechhofer et al. [1968] proposed an open-ended, *sequential* sampling procedure ($\mathcal{M}_{\text{BKS}}$) for selecting the single multinomial event that has the largest probability. The main difference between single-stage and open-ended procedures is that the former will always sample a fixed number of observations, whereas the latter will sample observations until a rule is triggered. Bechhofer and Goldsman [1986] later added two new stopping rules to procedure $\mathcal{M}_{\text{BKS}}$; their procedure $\mathcal{M}_{\text{BG}}$ limits the total sampling required and reduces the

observed overprotection on the probability of correct selection when compared to $\mathcal{M}_{\text{BKS}}$. The $\mathcal{M}_{\text{AVC}}$ (*All Vector Comparisons*) procedure was proposed by Miller et al. [1998] as an improvement to the $\mathcal{M}_{\text{BEM}}$ procedure when there are measured values for each alternative on each trial (i.e., the measurements are quantitative), resulting in 34% to 44% savings in their experimental settings.

(2) *Multinomial subset selection procedures*: Gupta and Nagel [1967] were the first to address the multinomial subset selection problem with the procedure $\mathcal{M}_{\text{GN}}$. Later, Chen and Hsu [1991] added several stopping and truncation rules to $\mathcal{M}_{\text{GN}}$, with the goal of reducing the expected size of the selected subset.

Parameters needed for using these procedures must often be determined numerically rather than analytically. The papers mentioned previously (and others in the literature) tend to present these parameters for situations where only a small set of alternatives (typically, 10 or fewer) exist. Our work is motivated by the need to run restricted multinomial subset selection procedures in a simulation environment, where there can be a large number of alternatives (say, 20 to 100).

In this article, we adapt the indifference-zone procedure $\mathcal{M}_{\text{BG}}$ of Bechhofer and Goldsman [1986] to the restricted multinomial subset selection problem. Chen [1988] suggested that in future research, the sampling and stopping rules of the $\mathcal{M}_{\text{BKS}}$ indifference-zone procedure could also be utilized in a subset selection approach. However, to the best of our knowledge, this article contains the first such adaptation. It is clear that our proposal is a combination of the strong features of indifference-zone and subset-selection procedures—that is, it is a hybrid procedure.

In Section 2, we provide mathematical definitions of the selection goal, along with other notation. Section 3 contains detailed descriptions of the original procedure of Bechhofer and Goldsman [1986] on which our proposal is based, as well as the procedure of Chen and Hsu [1991]. We describe our procedure in Section 4 and present numerical results in Section 5 for selection experiments involving small and large numbers of alternatives. Our conclusions appear in Section 6.

2. THE SELECTION NOTATION AND GOAL

Consider a selection problem involving k systems $\pi_1, \pi_2, \ldots, \pi_k$. These can be considered cells in a multinomial distribution, and throughout this article, we use the terms *system* and *cell* interchangeably. Let p_i $(i = 1, \ldots, k)$ denote the unknown probability associated with observing cell π_i $(0 \leq p_i \leq 1)$, where the p_i satisfy $\sum_{i=1}^{k} p_i = 1$. We denote the ordered values of the unknown cell probabilities by $p_{[1]}, p_{[2]}, \ldots, p_{[k]}$, where $p_{[1]} \leq p_{[2]} \leq \cdots \leq p_{[k]}$, and the corresponding cells by $\pi_{[1]}, \pi_{[2]}, \ldots, \pi_{[k]}$. The values of p_i and $p_{[j]}$ $(i, j = 1, \ldots, k)$, and also the pairing of π_i with $\pi_{[j]}$, are assumed to be completely unknown. If a single observation is taken from this multinomial distribution, and the outcome falls in cell π_i, we define this a *success* for π_i. Let y_{im} denote the total number of successes for π_i after m observations are taken. Clearly, $\sum_{i=1}^{k} y_{im} = m$. Let $y_{[1]m}, y_{[2]m}, \ldots, y_{[k]m}$ be the ordered values of y_{im}, and let $\widehat{\pi}_{[i]}$ be the cell associated with $y_{[i]m}$.

Let P^* denote the minimum desired $P(CS)$ $(1/k < P^* < 1)$, let t denote the maximum allowable size of the selected subset $(1 \leq t < k)$, and let θ^* represent the smallest ratio between the largest and second-largest probabilities that the experimenter is interested in detecting. The region $[p_{[k]}/\theta^*, p_{[k]}]$ is called the *indifference zone*, and all systems with probabilities in this region are considered to be equally good for practical purposes. P^*, t, and θ^* are all specified by the experimenter before sampling begins.

The goal of the experimenter is the selection of a subset with bounded size $t < k$ that contains $\pi_{[k]}$ whenever the ratio between the largest and second largest probabilities is greater than or equal to a chosen value θ^*. If this goal is achieved, it is said that a correct selection (CS) has been made. In other words, if $p_{[k]} \geq \theta^* p_{[k-1]}$, then the experimenter would like to select a subset that contains $\pi_{[k]}$; if $p_{[k]} < \theta^* p_{[k-i]}$, for some i, $1 \leq i \leq k-1$, then the experimenter is indifferent between a subset that contains $\pi_{[k]}$ and another that contains $\pi_{[k-j]}$ for some $0 < j \leq i$.

If $t = 1$, this simplifies to the standard indifference-zone formulation, and a correct selection is said to be made only if $\pi_{[k]}$ is selected. If $p_{[k]} \geq \theta^* p_{[k-1]}$, then the experimenter would like to select $\pi_{[k]}$; if $p_{[k]} < \theta^* p_{[k-i]}$, then the experimenter is indifferent between alternatives $\pi_{[k]}$ and $\pi_{[k-i]}$.

For either case ($t > 1$ or $t = 1$), the selection procedure should achieve or exceed a specified minimum probability of correct selection. Put mathematically, the selection procedure should provide the following probability guarantee:

$$P(CS) \geq P^* \text{ whenever } p_{[k]} \geq \theta^* p_{[k-1]}. \tag{1}$$

Finally, let n be the maximum number of observations allowed to be taken, N be the number of observations used by the algorithm, and S be the number of systems in the output subset. Let $E[N]$ and $E[S]$ denote, respectively, the expected value of N and S.

3. EXISTING PROCEDURES

We now describe, in detail, the two procedures that are the current best in the literature for multinomial selection. The first, $\mathcal{M}_{\mathrm{BG}}$, uses an indifference-zone approach to select a single multinomial cell. The second, $\mathcal{M}_{\mathrm{CH}}$, is a multinomial subset selection procedure.

3.1. The $\mathcal{M}_{\mathrm{BG}}$ Procedure

The $\mathcal{M}_{\mathrm{BG}}$ algorithm [Bechhofer and Goldsman 1986] added two new stopping rules, called *truncation* and *curtailment*, to the $\mathcal{M}_{\mathrm{BKS}}$ algorithm [Bechhofer et al. 1968], which is a sequential procedure for selecting *the single* multinomial event that has the largest probability among the alternatives under comparison. An algorithmic description of the procedure follows.

Procedure $\mathcal{M}_{\mathrm{BG}}$

Inputs from the decision maker:

P^*: The desired minimum probability of correct selection.

k: The number of cells in the multinomial distribution.

θ^*: The desired indifference-zone value.

Other inputs:

n: The maximum sample size (a function of P^*, k, and θ^*).

Sampling rule: Draw observations from the multinomial distribution one at a time. Continue sampling until one of the following three stopping criteria is met.

Stopping rule 1: Calculate

$$z_m \leftarrow \sum_{i=1}^{k-1} \left(1/\theta^*\right)^{(y_{[k]m} - y_{[i]m})}.$$

If $z_m \leq (1 - P^*)/P^*$, then stop.

Stopping rule 2: If $y_{[k]m} - y_{[k-1]m} \geq n - m$, then stop.

Stopping rule 3: If $m = n$, then stop.

Return $\widehat{\pi}_{[k]}$ as the best. In case of ties, randomly select one of the π_i's associated with $y_{[k]m}$ as the best.

Here, stopping rule 1 is the original stopping rule used in the open-ended procedure $\mathcal{M}_{\text{BKS}}$. Stopping rules 2 and 3 are, respectively, the curtailment and truncation stopping rules. Rule 3 was added because it was observed that stopping rule 1 overprotects the probability of correct selection. With this truncation rule, it is possible to decrease the expected number of observations taken while achieving (1). The curtailment stopping rule 2 was added because if it holds, then the current $\widehat{\pi}_{[k-1]}$ will, at best, be tied with the current $\widehat{\pi}_{[k]}$ after an additional $n-m$ observations are taken. This is true because using rule 2 along with rule 3 allows the $P(CS)$ to remain unchanged from using just rule 3. Note that rule 3 is redundant since rule 2 will apply when $m = n$, but we have retained it for consistency with the original algorithm in Bechhofer and Goldsman [1986].

The Least Favorable Configuration (LFC) is the configuration of the p_i, $1 \le i \le k$, that minimizes the probability of correct selection. The slippage configuration, typically written as $p_{[i]} \equiv p$ for $i = 1, \ldots, k-1$ and $p_{[k]} = \theta^* p$, was stated (but not proven) to be the LFC for procedure $\mathcal{M}_{\text{BG}}$ [Bechhofer and Goldsman 1986]. Note that because $\sum_{i=1}^{k} p_i = 1$, the slippage configuration can be completely defined as

$$p_{[i]} = \begin{cases} \dfrac{1}{k-1+\theta^*} & \text{if } i = 1, \ldots, k-1; \quad \text{and} \\[2ex] \dfrac{\theta^*}{k-1+\theta^*} & \text{if } i = k. \end{cases} \tag{2}$$

3.2. The $\mathcal{M}_{\text{CH}}$ Procedure

Chen and Hsu [1991] proposed an algorithm we call $\mathcal{M}_{\text{CH}}$ for multinomial subset selection, which combines stopping rules proposed by Bechhofer and Chen [1991], Panchapakesan [1971], and Ramey and Alam [1979] with a truncation rule. Note that $\mathcal{M}_{\text{CH}}$ does not allow the experimenter to restrict the maximum subset size.

Procedure $\mathcal{M}_{\text{CH}}$

Inputs from the decision maker:

P^*: The desired minimum probability of correct selection.

k: The number of cells in the multinomial distribution.

Other inputs:

n: The maximum sample size (tabled choices depend on P^* and k).

D, M, r: Other parameters for the procedure (functions of P^*, k, and n).

Sampling rule: Draw observations from the multinomial distribution one at a time. Continue sampling until one of the following four stopping criteria is met.

Stopping rule 1: Stop if $y_{[k]m} - y_{[k-1]m} \ge n - m + D$. The selected subset contains the single cell $\widehat{\pi}_{[k]}$.

Stopping rule 2: Stop if $y_{[k]m} = M$. The selected subset contains the single cell $\widehat{\pi}_{[k]}$.

Stopping rule 3: Stop if $y_{[k]m} - y_{[k-1]m} = r$. The selected subset contains the single cell $\widehat{\pi}_{[k]}$.

Stopping rule 4: Stop if $m = n$. The selected subset contains all π_i that satisfy $y_{in} \ge y_{[k]n} - D$.

The LFC for $\mathcal{M}_{\text{CH}}$ is given by the form

$$(p_{[1]}, p_{[2]}, \ldots, p_{[k]}) = (0, 0, \ldots, 0, u, p, p, \ldots, p)$$

for some $0 \leq u \leq p < 1$. The determination of u is made numerically; usually it is equal to either 0 or $1/k$, but for some procedure parameters, it falls strictly between these values.

Chen and Hsu [1991] wrote a FORTRAN program for finding the procedure parameters for k up to 50 that, unfortunately, is no longer available. In their paper, they provide tables only for $k \leq 10$.

4. A NEW RESTRICTED SUBSET SELECTION PROCEDURE

Our procedure adapts the open-ended sequential stopping rule of the $\mathcal{M}_{\text{BKS}}$ procedure (also used in $\mathcal{M}_{\text{BG}}$) so that it can be used for subset selection, not just for selecting the single best multinomial cell. Like $\mathcal{M}_{\text{BG}}$ and $\mathcal{M}_{\text{CH}}$, our procedure is truncated, which is beneficial to those running experiments who may be faced with restrictions on the maximum sample size due to real-world constraints on time or budget. Unlike $\mathcal{M}_{\text{BG}}$, our procedure $\mathcal{M}_{\text{NEW}}$ allows the decision maker to select more than a single alternative; this can substantially reduce the sampling requirements and may be particularly helpful if several alternatives have similarly good performance and secondary criteria are of interest. Unlike $\mathcal{M}_{\text{CH}}$, the $\mathcal{M}_{\text{NEW}}$ procedure allows the decision maker to restrict the bounded size of the selected subset. A description follows.

Procedure $\mathcal{M}_{\text{NEW}}$

Inputs from the decision maker:

P^*: The desired minimum probability of correct selection.

k: The number of cells in the multinomial distribution.

θ^*: The desired indifference-zone value.

t: The maximum subset size to be selected.

Other inputs:

n: The maximum sample size (a function of P^*, k, θ^*, and t).

Sampling rule: Draw observations from the multinomial distribution one at a time. Continue sampling until one of two stopping criteria is met.

Stopping rule 1 (sequential rule): Define

$$P^s(CS) \leftarrow 1 - \prod_{c=1}^{s} \left(\frac{z_{cm}}{1 + z_{cm}} \right), \; s = 1, 2, \ldots, t,$$

where

$$z_{cm} \leftarrow \sum_{i=1}^{k-c} \left(\frac{1}{\theta^*} \right)^{(y_{[k-c+1]m} - y_{[i]m})}.$$

For the smallest s for which $P^s(CS) \geq P^*$, stop and randomize the order of those $\widehat{\pi}_{[i]}$ that have equal counts. Return the subset $\left\{ \widehat{\pi}_{[k-s+1]}, \widehat{\pi}_{[k-s+2]}, \ldots, \widehat{\pi}_{[k]} \right\}$.

Stopping rule 2 (curtailment and truncation rule): If stopping rule 1 does not apply and if $y_{[k-t+1]m} - y_{[k-t]m} \geq n - m$, stop and randomize the order of those $\widehat{\pi}_{[i]}$ that have equal counts. Return the subset $\left\{ \widehat{\pi}_{[k-t+1]}, \widehat{\pi}_{[k-t+2]}, \ldots, \widehat{\pi}_{[k]} \right\}$.

Note that stopping rule 2 will always return a subset of size t, whereas stopping rule 1 will return a subset of size s, where $1 \leq s \leq t$. Note further that stopping rule 2 corresponds to curtailment if $m < n$ and truncation if $m = n$.

4.1. Theoretical Properties

In this section, we consider some properties of $\mathcal{M}_{\text{BG}}$ and $\mathcal{M}_{\text{NEW}}$ under the slippage configuration. Consider again the $\mathcal{M}_{\text{BG}}$ stopping rule 1, which stops sampling if

$$z_m \equiv \sum_{i=1}^{k-1}(1/\theta^*)^{(y_{[k]m}-y_{[i]m})} \leq \frac{1-P^*}{P^*}. \tag{3}$$

For a given z_m, note that

$$z_m \leq \frac{1-P^*}{P^*} \iff P^* \leq \frac{1}{1+z_m}.$$

Define

$$P_m = \frac{1}{1+z_m}; \tag{4}$$

this is the maximum value of P^* for which procedure $\mathcal{M}_{\text{BG}}$ would terminate at m, given the outcome vector $\overrightarrow{y_m} = (y_{1m}, \ y_{2m}, \ldots, y_{km})^T$. Correspondingly, under the slippage configuration, $1 - P_m$ is the probability of making an incorrect selection (i.e., $\Pr(\pi_{[k]} \neq \widehat{\pi}_{[k]})$ at stage m, given the outcome vector $\overrightarrow{y_m}$, if we stop and select the cell associated with $y_{[k]}$ as best.

We generalize the statistics z_m and P_m of Equations (3) and (4) as follows:

$$z_{cm} = \sum_{i=1}^{k-c}\left(1/\theta^*\right)^{(y_{[k-c+1]m}-y_{[i]m})} \tag{5}$$

$$P_{cm} = \frac{1}{1+z_{cm}}. \tag{6}$$

Under the slippage configuration, P_{cm} is the conditional probability that $\pi_{[k]}$ is associated with $\widehat{\pi}_{[k-c+1]}$, conditioned on $\pi_{[k]} \notin \{\widehat{\pi}_{[k-c+2]}, \ldots, \widehat{\pi}_{[k]}\}$ for the given m and $\overrightarrow{y_m}$. When $c = 1$, (5) simplifies to (3) and (6) simplifies to (4). The main idea behind $\mathcal{M}_{\text{NEW}}$ is as follows: the probability that the cell which truly has the largest $p_{[k]}$ is contained in the subset $\{\widehat{\pi}_{[k-s+1]}, \ \widehat{\pi}_{[k-s+2]}, \ldots, \ \widehat{\pi}_{[k]}\}$ is the complement of the probability that none of the subset components are associated with $\pi_{[k]}$. This fact leads to

$$\begin{aligned}
&\Pr\left(\pi_{[k]} \notin \left\{\widehat{\pi}_{[k-s+1]}, \ldots, \widehat{\pi}_{[k]}\right\}\right) \\
&\quad = \Pr\left(\pi_{[k]} \neq \widehat{\pi}_{[k-s+1]} \mid \pi_{[k]} \notin \left\{\widehat{\pi}_{[k-s+2]}, \ldots, \widehat{\pi}_{[k]}\right\}\right) \\
&\qquad \times \Pr\left(\pi_{[k]} \notin \left\{\widehat{\pi}_{[k-s+2]}, \ldots, \widehat{\pi}_{[k]}\right\}\right) \\
&\quad = (1 - P_{sm}) \times \Pr\left(\pi_{[k]} \notin \left\{\widehat{\pi}_{[k-s+2]}, \ldots, \widehat{\pi}_{[k]}\right\}\right).
\end{aligned} \tag{7}$$

Using successive expansions of $\Pr(\pi_{[k]} \notin \{\widehat{\pi}_{[k-s+2]}, \ldots, \widehat{\pi}_{[k]}\})$ leads to

$$\Pr\left(\pi_{[k]} \notin \left\{\widehat{\pi}_{[k-s+1]}, \ldots, \widehat{\pi}_{[k]}\right\}\right) = \prod_{c=1}^{s}(1 - P_{cm}) = \prod_{c=1}^{s}\left(\frac{z_{cm}}{1+z_{cm}}\right),$$

and hence, if sampling stops at stage m with a subset of size $s \leq t$,

$$P(CS)_m = 1 - \Pr\left(\pi_{[\mathrm{k}]} \notin \left\{\widehat{\pi}_{[\mathrm{k-s+1}]}, \ldots, \widehat{\pi}_{[\mathrm{k}]}\right\}\right) = 1 - \prod_{\mathrm{c=1}}^{\mathrm{s}}\left(\frac{\mathrm{z_{cm}}}{1+\mathrm{z_{cm}}}\right), \tag{8}$$

where $P(CS)_m$ is the probability of correct selection if the sampling stops at stage m.

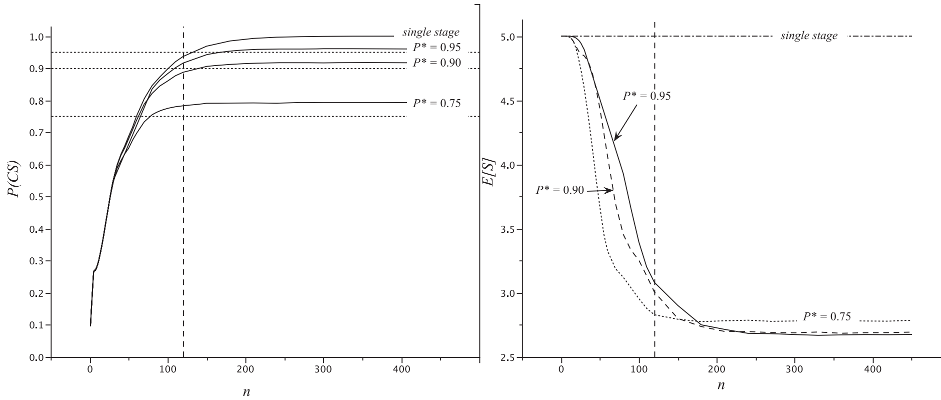

Fig. 1. Achieved $P(CS)$ and $E[S]$ for given P^* in stopping rule 1, $k = 100$, $t = 5$, slippage configuration.

Equation (8) provides the $P(CS)_m$ value if the sequential stopping rule is employed. Like $\mathcal{M}_{\mathrm{BG}}$, on which it is based, the sequential rule is almost always conservative. Except in rare instances, a value will exist that will allow the procedure to be truncated while still meeting the probability requirement when $t = 1$. If truncation is employed, then curtailment does not affect the $P(CS)$, nor does it affect the size of the selected subset. As in $\mathcal{M}_{\mathrm{BG}}$, when curtailment applies, it reduces the sampling required—that is, stopping rule 2 halts the procedure with $m < n$.

A key requirement of any selection procedure is a guarantee that $P(CS) \geq P^*$ for some $1/k < P^* < 1$ of interest. If the form of the LFC can be proven, then by finding the parameters that guarantee the desired $P(CS)$ in the LFC, all other configurations will have a $P(CS) \geq P^*$. Appropriate parameters can sometimes be found analytically but often require either computationally intensive exact expansions or numerical simulations. Proofs of LFCs tend to be much more complicated for multinomial selection problems than for other types of selection procedures because the y_i's are not independent; see, for example, Kesten and Morse [1959] and Gastaldi [2005]. At the time of this writing, a proof of the LFC for $\mathcal{M}_{\mathrm{BG}}$ is not yet available, although Bechhofer and Goldsman [1986] conjecture that the slippage configuration is the LFC. Similarly, we conjecture that the slippage configuration given by (2) is the LFC for procedure $\mathcal{M}_{\mathrm{NEW}}$. This is the motivation for designing our procedure's sample-size calculations based on the slippage configuration.

Finally, we illustrate an interesting property of our procedure in Figure 1. This provides the achieved $P(CS)$, as a function of n, when we fix the P^* in stopping rule 1 of $\mathcal{M}_{\mathrm{NEW}}$ at 0.75, 0.90, and 0.95. Such a plot could be used to determine the appropriate values of n to achieve the desired overall $P(CS)$, although we used a smarter search algorithm for finding n. The curves were constructed by performing 200,000 Monte Carlo replications for 1(1)30(5)60(10)120(30)450. In line with the theoretical properties, the general $P(CS)$ trends rise rapidly and then stabilize at values slightly higher than the nominal P^*. (Recall, however, that $P(CS)$ is not strictly monotonic in n because of the discrete nature of the multinomial sampling.) The curves all start out providing essentially the same performance guarantee but then diverge. This suggests that there are multiple ways to achieve a desired $P(CS)$ using procedure $\mathcal{M}_{\mathrm{NEW}}$. The tabled values of n that we provide correspond to using the overall P^* as the criterion for stopping in stopping rule 1 and lead to the lowest $E[S]$ values. However, one could get by with a lower n and achieve the same overall probability guarantee, $P(CS) \geq P^*$, by using a $P' > P^*$ as the criterion in stopping rule 1. In particular, we ran a variant of $\mathcal{M}_{\mathrm{NEW}}$

that never invokes stopping rule 1 and display the results in Figure 1 as $P' = 1.0$ (*single stage*). This last curve will never reach the value 1.0, but where it crosses a $P(CS)$ value of interest (such as 0.90), it provides values of n that correspond to a single-stage procedure that always returns a subset of size t. We prefer the $\mathcal{M}_{\mathrm{NEW}}$ procedure as reported, because when there are clear differences in the multinomial cell probabilities, the procedure can terminate early with a subset substantially smaller than t. Nonetheless, it is interesting to see how a single-stage procedure could be obtained. An intermediate procedure between the reported procedure and the single stage one can also be used. For example, the dashed vertical line in Figure 1 corresponds to $n = 120$. If $P' = 0.75$, the procedure will achieve $P(CS) = 0.783$ and $E[S] = 2.83$; if $P' = 0.90$, the procedure will achieve $P(CS) = 0.888$ and $E[S] = 3.01$; if $P' = 0.95$, the procedure will achieve $P(CS) = 0.916$ and $E[S] = 3.08$; and if a single stage procedure that always selects five cells is used, the achieved $P(CS) = 0.937$. In other words, by accepting a slight increase in $E[S]$, we can achieve a much higher $P(CS)$ by using $P' = 0.95$ instead of $P' = 0.75$.

5. RESULTS

We examine the performance of our procedure in Sections 5.1, 5.2, and 5.3; discuss tables for its use in Section 5.4; and provide some examples where it might be beneficial in Section 5.5.

5.1. Performance Comparison

Before proceeding, a more detailed discussion about *indifference-zone* and *subset selection* procedures is in order. Indifference-zone procedures require that the decision maker establishes a minimum distance between the parameters of the best and second-best alternative, whereas subset selection procedures do not need such establishment (as discussed in Section 1). Accordingly, indifference-zone procedures guarantee that $P(CS) \geq P^*$ for a more limited set of probability configurations than subset selection procedures. As these procedures are defined under distinct parameter spaces, a straightforward comparison of their performances is challenging. Nevertheless, if the decision maker is willing to establish an indifference zone, it is interesting to see how $\mathcal{M}_{\mathrm{NEW}}$ compares in relation to a classical subset selection procedure. This comparison has a practical motivation rather than a methodological one: we would like to see how efficient our procedure is when compared with $\mathcal{M}_{\mathrm{CH}}$ in case the decision maker accepts as output a subset instead of only a single alternative and he or she can establish an indifference-zone parameter.

We compare the performance of $\mathcal{M}_{\mathrm{NEW}}$ with the results of $\mathcal{M}_{\mathrm{CH}}$ for $k \in \{3, 4\}$, $P^* \in \{0.75, 0.90, 0.95\}$, $t \in \{1, 2\}$, and $\theta^* = 3.0$ in the slippage configuration $(p, p, \ldots, p, \theta^* p)$. We decided on procedure $\mathcal{M}_{\mathrm{CH}}$ because Bechhofer et al. [1995, p. 241] suggest that it is the best algorithm proposed so far in the literature for multinomial subset selection by recommending it when the required constants (D, M, r) are available. Table I shows the results of this comparison, where $P(CS)$ is the achieved probability of correct selection, $E[S]$ is the expected subset size, and $E[N]$ is the expected sample size. Our simulations were based in 20,000,000 Monte Carlo replications for $P^* = 0.90$ and 0.95, and 40,000,000 replications for $P^* = 0.75$, so that the standard errors of all estimates are quite small. The $P(CS)$ was calculated as the observed proportion of correct selections among the replications.

Chen and Hsu [1991] consider values of $n = 10(5)30$ when $\theta^* = 3.0$. They remark that multiple combinations of the remaining parameters (D, M, r) are available that achieve the minimum probability requirements, but that the (D, M, r) combinations that yield the smallest $E[S]$ values may not be optimal for minimizing $E[N]$. To perform the comparisons, we first compute n for $\mathcal{M}_{\mathrm{NEW}}$ for $t = 1, 2$. For comparisons against $t = 1$,

Table I. Comparison between $\mathcal{M}_{\text{NEW}}$ and $\mathcal{M}_{\text{CH}}$

		$\mathcal{M}_{\text{NEW}^a}$				$\mathcal{M}_{\text{CH}^b}$				Rel. Eff. of $\mathcal{M}_{\text{NEW}^a}$ to $\mathcal{M}_{\text{CH}^b}$		
k	P^*	(t, n)	$P(CS)$	$E[S]$	$E[N]$	(D, M, r, n)	$P(CS)$	$E[S]$	$E[N]$	n	$E[S]$	$E[N]$
3	0.75	(1,5)	0.7573	1.000	3.24	(0,3,2,10)	0.7962	1.000	3.68	0.50	1.00	0.88
		(2,2)	0.8000	2.000	1.00					0.20	2.00	0.27
	0.90	(1,12)	0.9030	1.000	6.97	(0,5,3,15)	0.9004	1.000	6.76	0.80	1.00	1.03
		(2,5)	0.9109	2.000	2.85	(0,5,3,10)	0.9042	1.047	6.61	0.50	1.91	0.76
	0.95	(1,20)	0.9505	1.000	8.90	(0,7,4,20)	0.9505	1.000	9.77	1.00	1.00	0.91
		(2,8)	0.9578	2.000	4.93	(1,6,4,10)	0.9604	1.261	8.11	0.80	1.59	0.61
4	0.75	(1,9)	0.7519	1.000	4.91	(0,3,3,10)	0.7542	1.000	5.17	0.90	1.00	0.49
		(2,2)	0.7778	2.000	2.00					0.20	2.00	0.38
	0.90	(1,19)	0.9037	1.000	9.84	(0,6,4,25)	0.9075	1.000	11.00	0.76	1.00	0.89
		(2,9)	0.9067	1.762	6.02	(1,5,3,10)	0.9186	1.424	7.68	0.90	1.24	0.78
	0.95	(1,26)	0.9512	1.000	12.97	(0,8,5,30)	0.9527	1.000	14.81	0.87	1.00	0.88
		(2,14)	0.9523	1.911	8.22	(1,7,4,15)	0.9522	1.239	11.43	0.93	1.54	0.72
						(2,5,4,10)	0.9611	1.829	8.59	1.40	1.04	0.96

[a]Standard error of estimate ≤ 0.0001 for $P(CS)$ and $E[S]$, and ≤ 0.0002 for $E[N]$.
[b]Source: Chen and Hsu [1991, pp. 408–409].

we choose the (D, M, r) values from Chen and Hsu [1991] associated with the smallest n for which $E[S] = 1.000$. For comparisons against $t = 2$, we choose the (D, M, r) values associated with values n_{NEW} rounded to the nearest multiples of five, or the minimum n_{CH} reported in their table. By n_{NEW} and n_{CH}, we mean, respectively, the values of n for the procedures $\mathcal{M}_{\text{NEW}}$ and $\mathcal{M}_{\text{CH}}$.

Analysis of Table I shows that although neither procedure dominates, our procedure competes favorably with the current best procedure in the literature. When $t = 1$, $\mathcal{M}_{\text{NEW}}$ always has a smaller maximum sample size n. $\mathcal{M}_{\text{NEW}}$ also performs well with respect to $E[N]$. In one instance, it requires 3% additional sampling, on average; however, in the others, the relative efficiency in $E[N]$ ranges from 49% to 91%.

When $t = 2$, our procedure uses a larger n in only one case ($k = 4$, $P^* = 0.95$, $t = 2$), whereas it achieves as much as an 80% reduction in n in other situations. In terms of $E[N]$, when $t = 2$, $\mathcal{M}_{\text{NEW}}$ requires between 27% and 96% of the sampling required by $\mathcal{M}_{\text{CH}}$, on average. These reductions do come at the cost of a larger expected subset size, but $E[S]$ increases by less than 1.0 in all cases. The experimenter may be quite willing to end up with a slightly larger subset in exchange for substantial reductions in the maximum and expected sampling requirements, particularly if the time available to perform the experiment is limited.

Besides usually requiring both smaller n and $E[N]$, $\mathcal{M}_{\text{NEW}}$ has other advantages over $\mathcal{M}_{\text{CH}}$. First, after the decision maker establishes the indifference-zone value θ^*, for a given k and P^*, $\mathcal{M}_{\text{NEW}}$ needs only two parameters: t and n. Both of these are natural for a decision maker to interpret, and if the decision maker is able to specify either one, then a one-dimensional search can be used to determine the other. In contrast, for a given k and P^*, the procedure $\mathcal{M}_{\text{CH}}$ needs four parameters (D, M, r, and n). This makes the search space for suitable parameter combinations much larger. We remark that due to the discrete nature of the sampling, the values of the achieved $P(CS)$, $E[N]$, and $E[S]$ are not always monotonic in n, particularly for small problems. This fact complicates the search for both procedures—but especially the higher-dimensional search for $\mathcal{M}_{\text{CH}}$.

If the decision maker is not willing to establish an indifference zone, either because he or she does not feel comfortable in doing this or because the decision maker wants a $P(CS)$ guarantee when all systems have essentially the same probability of occurrence

Table II. Estimates for the Performance of $\mathcal{M}_{NEW}$ for Different Configurations When $P^* = 0.90$, $\theta^* = 3.0$, $k = 10$, and $t = 3$

Statistic	Configuration	n_{tied}	$t = 1$	$t = 2$	$t = 3$
$E[N]$	Slippage	—	24.016	20.856	18.081
	Random	—	16.365	12.147	8.950
	$\mathcal{M}_{CH}WC$	$t+1$	10.601	8.213	8.962
		t	4.000	5.813	7.262
		$t-1$	—	4.000	5.814
$E[S]$	Slippage	—	1.000	1.773	2.593
	Random	—	1.000	1.623	2.391
	$\mathcal{M}_{CH}WC$	$t+1$	1.000	1.885	2.655
		t	1.000	1.875	2.398
		$t-1$	—	1.000	1.875

(i.e, the equal-probability configuration with $\theta^* = 1.0$), then we recommend the use of procedure $\mathcal{M}_{CH}$ instead of ours.

5.2. Performance Under Alternative Configurations

As stated earlier, we conjecture that the slippage configuration is the LFC for $\mathcal{M}_{NEW}$. This means that our procedure may perform much better on practical problems than would be indicated by looking at the $E[N]$, $E[S]$, and $P(CS)$ values in Table I and all of the Tables in Appendix A. In addition to the slippage configuration, we examine two other types of configurations.

The first is a random configuration, which we generate as follows:

(1) Let w_i be independent $U(0, 1)$ numbers, $i = 1, \ldots, k-1$;
(2) Set $w_k = \theta^* \times \max(w_i)$;
(3) Set $p_i = w_i / \sum_{j=1}^{k} w_j$.

Second, we consider configurations similar to the worst case configuration for procedure $\mathcal{M}_{CH}$, which we call "$\mathcal{M}_{CH}WC$." Here, $p_{[i]} = 1/n_{tied}$ for $i = k - n_{tied} + 1, \ldots, k$ and $p_i = 0$ otherwise. In other words, there are exactly n_{tied} alternatives tied for the best. In these cases, a correct selection is made for $\mathcal{M}_{NEW}$ whenever the selected subset contains at least one of the n_{tied} best alternatives. Although these are difficult configurations for the selection goal of $\mathcal{M}_{CH}$, they are very favorable configurations for $\mathcal{M}_{NEW}$ in terms of the achieved $P(CS)$; they are also associated with lower $E[N]$ as long as n_{tied} is not much larger than t.

Table II illustrates the expected sample sizes and the expected subset sizes for these three types of configurations for a representative case where $k = 10$, $\theta^* = 3.0$, and $P^* = 0.90$. We provide results for $n_{tied} = t - 1$, $n_{tied} = t$, and $n_{tied} = t + 1$ in the $\mathcal{M}_{CH}WC$ configuration. All computations are based on 500,000 Monte Carlo replications. By replications, we mean different sample paths for the slippage and $\mathcal{M}_{CH}WC$ configurations, and different random w_i values for the random configurations.

The results in Table II show that $E[N]$ is much smaller for the random and $\mathcal{M}_{CH}WC$ configurations, stopping after 17% to 68% of the sampling (on average) that would be required under the conjectured LFC. The differences in expected subset size are smaller and less consistent; the reductions in $E[S]$ are largest when $\mathcal{M}_{CH}WC$ has fewer than t alternatives tied for best.

The cumulative distribution functions of N for the various alternatives are given in Figure 2 for $P^* = 0.90$, $k = 10$, and $t = 3$. These provide a more detailed look at the performance of the procedures. In addition to the configurations of Table II, we also furnish the results for the equal-probability case. $\mathcal{M}_{NEW}$ halts very quickly in the

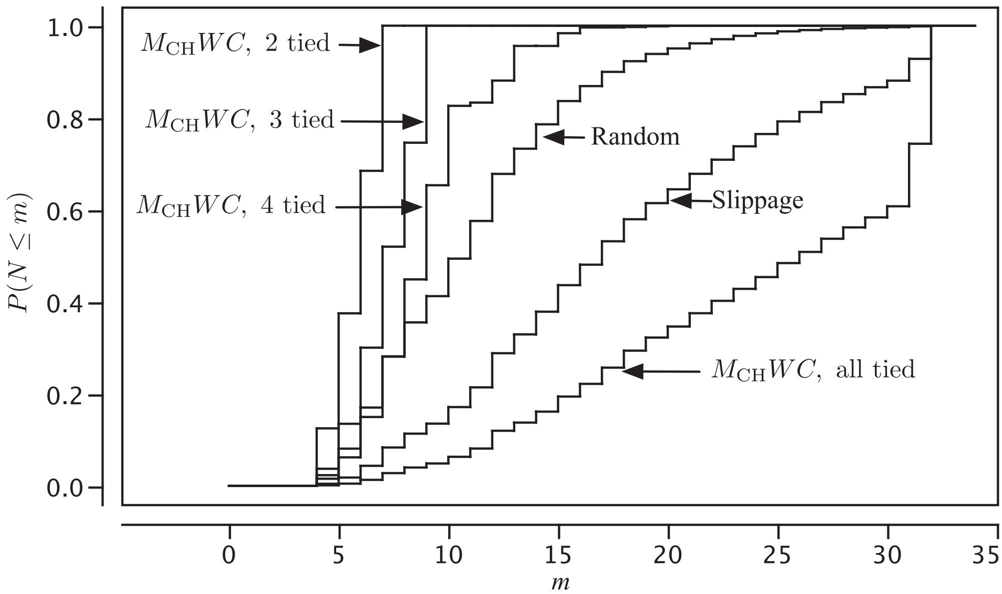

Fig. 2. Cumulative distribution functions of the sample size N for different configuration types, using $\mathcal{M}_{\rm NEW}$ when $P^* = 0.90$, $\theta^* = 3.0$, $k = 10$, and $t = 3$.

$\mathcal{M}_{\rm CH}WC$ when $n_{\rm tied}$ is small. For the slippage configuration and the equal-probability configuration, the distributions are multimodal with high probabilities of requiring all n samples. Similar behavior occurs for other combinations of probability configurations, P^*, k, and t.

5.3. Performance in Large-Scale Problems

Figure 3 illustrates the behavior of our procedure for $k = 10(10)100(100)500$, $P^* = 0.75$, $t = 5$, and $\theta^* = 5$. As the reader can see, $\mathcal{M}_{\rm NEW}$ can be used in very big problems. The trade-off of using our procedure in such big problems is the total number of observations to be taken—for example, to select a subset of bounded size of $t = 5$ with required probability of correct selection of $P^* = 0.75$ when the indifference-zone parameter is $\theta^* = 5$, it will be necessary to take as many as $n \times k = 916 \times 500 = 458,000$ observations for $k = 500$. Despite not being a theoretical limit, we believe that problems bigger than $k = 500$ will not be practical in the real world.

Another interesting phenomenon in Figure 3 is the reduction of $E[S]$ and the ratio $E[N]/n$ as the number of systems increases. This occurs because in the slippage configuration, the cell probability for any system other than the best is inversely related to the number of systems under evaluation. If a very large number of systems are tied for second best, it is unlikely that $t - 1$ or more will remain consistently close to the true best as sampling progresses. The sequential stopping rule is more likely to be invoked before the curtailment and truncation rule, and tends to be associated with reduced sampling requirements. It is also the only rule that can result in a subset size S smaller than t.

5.4. Tables

Our work was motivated by a need for readily accessible (or easy to compute) parameters for multinomial subset selection procedures involving a large number of alternatives. In Tables III through XII of Appendix A, we present the truncation

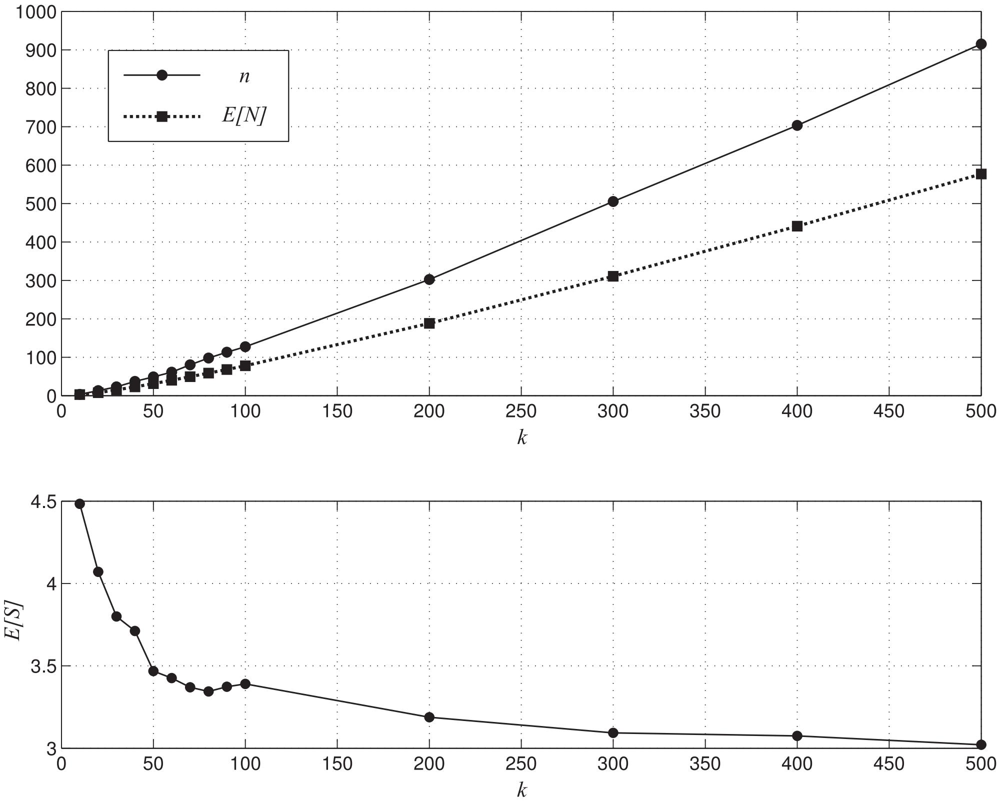

Fig. 3. n, $E[N]$, and $E[S]$ for $k = 10(10)100(100)500$ in the slippage configuration when $P^* = 0.75$, $t = 5$, and $\theta^* = 5$.

numbers n needed to implement our procedure. We provide tables for $k \in \{3, 4, \ldots, 9, 10, 20, \ldots, 100\}$, $P^* \in \{0.75, 0.90, 0.95\}$, and several values of t and θ^*. Although the n values are all that are needed to implement the procedure, we also report estimates of $P(CS)$, the expected subset size $E[S]$, and the expected sample size $E[N]$ under the associated slippage configurations. All values were calculated using 500,000 Monte Carlo replications.

The values of θ^* in Tables III through X are the same ones that Bechhofer and Goldsman [1986] use for $k = 2, 3, \ldots, 6$. For $k = 7, 8, 9, 10$, we use the values from their first paper [Bechhofer and Goldsman 1985]; for $k = 20, 30, \ldots, 100$, we use a different approach for specifying θ^* values. In these cases, we first specify an unnormalized difference δ_u between the best and second-best alternatives (e.g., $\delta_u = 0.05$). This, in turn, determines the value of θ^* in the slippage configuration as follows:

$$\theta^* = \frac{1 + (k-1)\delta_u}{1 - \delta_u}.$$

The probabilities in the slippage configuration are then determined from Equation (2). The numbers in Tables XI and XII represent differences of $\delta_u = 0.02$, 0.05, and 0.08. Note how efficient $\mathcal{M}_{\text{NEW}}$ can be: an analyst interested in a difference of at least $\delta_u = 0.08$ can go from a set of 100 alternatives to a subset of five or fewer, with at most 89 multinomial observations, while guaranteeing that the $P(CS) \geq 0.95$.

5.5. Examples

Example 1: *Pandemic modeling*. Disease transmission rates may vary for different demographic segments of a population, along with the severity of the symptoms. Several types of interventions may be possible, including school closings, treatment alternatives following diagnosis, and broad-based or targeted vaccinations. Public health officials in a particular city may be interested in using simulation models to explore the effects of various disease characteristics and potential interventions on the local population. Suppose that there are three different policies for school closings, five treatment alternatives, and four different ways in which vaccines can be distributed (including no vaccines). As the intervention can only be executed a single time, then selecting the best interventions based on long-run performance (expected value) makes no sense! It is more natural to approach this problem with a multinomial selection procedure instead of a selection-of-the-best procedure—one that aims to select the system (or subset) associated with the best performance measure expected value. Of the 60 alternatives that result from combining these interventions, officials are interested in a subset, of no more than three, that contains the intervention most likely to yield the best results. If they desire a $P(CS)$ guarantee of 0.90 whenever $\delta_u = 0.05$, then Table XII indicates that $n = 170$ is the maximum number of observations needed by $\mathcal{M}_{\text{NEW}}$.

Example 2: *Military tactic development*. Air-to-air combat tactics can be evaluated through simulation models. Vieira Jr. [2011] studied beyond-visual-range combat with the objective of identifying the optimal tactic. Several of the decision variables involved in an air-to-air combat are continuous (e.g., distance, altitude, and speed) and must be discretized when running the simulation. Even with a parsimonious discretization of these variables, a very large number of alternatives must be compared to each other. Clearly, the overriding consideration in air-to-air combat problem involves the risk factor. A nation is not interested in having a slightly greater average success than its enemy over a period of several years of combat. What really matters is to have the greatest probability of being the winner in each individual engagement (i.e., each unique trial). Again, using a selection-of-the-best procedure is not the appropriate solution, and the $\mathcal{M}_{\text{NEW}}$ procedure can be a useful screening procedure for this class of problems. The reason is that it is an efficient way to winnow the large number of alternatives down to a small number that can be examined in more depth. If there are 80 variants under investigation, the decision maker is interested in identifying at most eight tactics to study in more detail, and wants a $P(CS)$ guarantee of 0.95 when $\delta_u = 0.02$, then Table XIII shows that $\mathcal{M}_{\text{NEW}}$ needs at most $n = 702$ observations and, on average, will end after observing $E[N] = 376.52$ samples. The output set will have an average size of $E[S] = 6.274$.

6. CONCLUSIONS

In this article, we adapt a truncated sequential indifference-zone multinomial selection procedure, proposed by Bechhofer and Goldsman [1986], to the *restricted multinomial subset selection* problem. We compare the performance of our hybrid procedure, $\mathcal{M}_{\text{NEW}}$, to the procedure of Chen and Hsu [1991], which is considered the best unrestricted multinomial subset selection procedure currently available, and show that $\mathcal{M}_{\text{NEW}}$ performs favorably with respect to a variety of measures. $\mathcal{M}_{\text{NEW}}$ is easier to implement, because it has only one calculated parameter that the decision maker is not free to choose (n), whereas $\mathcal{M}_{\text{CH}}$ has four such parameters (D, M, r, and n). This makes it easier to use $\mathcal{M}_{\text{NEW}}$ than $\mathcal{M}_{\text{CH}}$, as well as making it easier to obtain suitable parameters for configurations not covered by existing tables. Under the slippage configuration, $\mathcal{M}_{\text{NEW}}$ typically requires less sampling than $\mathcal{M}_{\text{CH}}$, in terms of both the maximum and expected sample size, at the cost of a small increase in the expected size of the selected subset.

Our work was motivated in part by a need to run restricted multinomial selection procedures in a simulation environment where there are a large number of alternatives. Consequently, in addition to providing tables of the truncation numbers n needed to implement our procedure for small numbers of systems under evaluation ($k = 2, 3, \ldots, 10$), we also provide tables for large values of k ($k = 20, 30, \ldots, 100$).

APPENDIX

A. TABLES TO IMPLEMENT $\mathcal{M}_{\text{NEW}}$

Table III. The Value of n for $k = 3$ with the Estimated $P(CS)$, $E[S]$, and $E[N]$ under the Slippage Configuration

		$P^* = 0.75$				$P^* = 0.90$				$P^* = 0.95$			
θ^*	t	n	$P(CS)$	$E[S]$	$E[N]$	n	$P(CS)$	$E[S]$	$E[N]$	n	$P(CS)$	$E[S]$	$E[N]$
3.0	1	5	0.7578	1.000	3.24	12	0.9027	1.000	6.97	20	0.9506	1.000	8.90
	2	1	0.8003	2.000	1.00	5	0.9109	2.000	2.85	8	0.9578	2.000	4.93
2.8	1	6	0.7622	1.000	3.70	15	0.9053	1.000	7.77	22	0.9517	1.000	10.48
	2	1	0.7905	2.000	1.00	5	0.9075	1.784	3.30	9	0.9517	2.000	5.06
2.6	1	7	0.7536	1.000	3.94	16	0.9021	1.000	9.17	25	0.9514	1.000	12.26
	2	1	0.7828	2.000	1.00	5	0.9059	2.000	3.32	12	0.9509	1.847	5.80
2.4	1	8	0.7595	1.000	5.40	22	0.9017	1.000	10.43	31	0.9519	1.000	14.48
	2	1	0.7726	2.000	1.00	5	0.9038	2.000	4.15	11	0.9509	2.000	6.80
2.2	1	10	0.7501	1.000	6.00	25	0.9011	1.000	13.31	41	0.9508	1.000	17.58
	2	1	0.7615	2.000	1.00	8	0.9080	2.000	4.96	14	0.9504	2.000	8.22
2.0	1	14	0.7579	1.000	8.24	34	0.9015	1.000	17.18	52	0.9510	1.000	23.03
	2	2	0.7502	2.000	1.00	10	0.9038	2.000	5.98	19	0.9509	2.000	10.52
1.8	1	19	0.7569	1.000	11.61	50	0.9004	1.000	23.73	71	0.9502	1.000	32.63
	2	2	0.7918	2.000	2.00	14	0.9015	2.000	7.78	31	0.9507	2.000	14.10
1.6	1	32	0.7522	1.000	17.63	84	0.9007	1.000	37.32	125	0.9503	1.000	50.30
	2	2	0.7683	2.000	2.00	23	0.9017	2.000	11.99	46	0.9506	2.000	21.58
1.4	1	72	0.7511	1.000	34.12	169	0.9004	1.000	73.64	262	0.9504	1.000	99.08
	2	3	0.7514	2.000	2.34	47	0.9004	2.000	22.43	89	0.9501	2.000	41.87
1.2	1	279	0.7510	1.000	118.02	680	0.9001	1.000	253.94	960	0.9500	1.000	345.42
	2	12	0.7520	2.000	7.84	173	0.9001	2.000	73.96	362	0.9502	2.000	137.61

Table IV. The Value of n for $k = 4$ with the Estimated $P(CS)$, $E[S]$, and $E[N]$ under the Slippage Configuration

		$P^* = 0.75$				$P^* = 0.90$				$P^* = 0.95$			
θ^*	t	n	$P(CS)$	$E[S]$	$E[N]$	n	$P(CS)$	$E[S]$	$E[N]$	n	$P(CS)$	$E[S]$	$E[N]$
	1	9	0.7531	1.000	4.90	19	0.9031	1.000	9.84	26	0.9514	1.000	12.98
3.0	2	4	0.7501	1.668	2.00	9	0.9066	1.763	6.02	14	0.9525	1.912	8.22
	3	1	0.8336	3.000	1.00	3	0.9173	3.000	2.67	6	0.9523	3.000	4.50
	1	9	0.7514	1.000	5.99	21	0.9008	1.000	11.16	30	0.9507	1.000	14.74
2.8	2	2	0.7620	2.000	2.00	10	0.9042	1.922	6.29	16	0.9507	1.829	9.40
	3	1	0.8280	3.000	1.00	3	0.9081	3.000	2.68	7	0.9550	3.000	5.15
	1	11	0.7552	1.000	7.06	26	0.9029	1.000	13.21	36	0.9503	1.000	17.19
2.6	2	4	0.7709	2.000	2.99	13	0.9042	2.000	7.05	19	0.9503	1.894	10.77
	3	1	0.8213	3.000	1.00	3	0.9089	3.000	3.00	9	0.9515	3.000	5.45
	1	15	0.7572	1.000	8.28	31	0.9029	1.000	15.94	44	0.9513	1.000	20.65
2.4	2	5	0.7707	2.000	3.32	14	0.9022	1.937	8.48	23	0.9510	1.962	12.54
	3	1	0.8155	3.000	1.00	5	0.9031	3.000	3.38	9	0.9515	3.000	6.45

Continued

Table IV. Continued

		$P^* = 0.75$				$P^* = 0.90$				$P^* = 0.95$			
θ^*	t	n	$P(CS)$	$E[S]$	$E[N]$	n	$P(CS)$	$E[S]$	$E[N]$	n	$P(CS)$	$E[S]$	$E[N]$
	1	17	0.7502	1.000	10.43	39	0.9003	1.000	19.81	56	0.9509	1.000	25.77
2.2	2	5	0.7568	1.903	3.65	17	0.9009	1.940	10.60	30	0.9509	1.946	15.62
	3	1	0.8070	3.000	1.00	4	0.9007	3.000	3.58	11	0.9514	3.000	7.74
	1	24	0.7549	1.000	13.78	54	0.9007	1.000	25.77	75	0.9508	1.000	33.94
2.0	2	8	0.7531	2.000	4.22	22	0.9004	1.973	13.41	38	0.9508	1.982	20.12
	3	1	0.8003	3.000	1.00	7	0.9070	3.000	5.03	17	0.9509	3.000	9.50
	1	34	0.7509	1.000	19.25	75	0.9010	1.000	36.90	106	0.9505	1.000	48.19
1.8	2	9	0.7528	2.000	6.04	33	0.9009	2.000	18.71	52	0.9501	1.990	28.25
	3	1	0.7910	3.000	1.00	9	0.9018	3.000	6.18	22	0.9507	3.000	12.97
	1	57	0.7523	1.000	31.07	126	0.9006	1.000	58.74	182	0.9503	1.000	76.30
1.6	2	14	0.7515	2.000	9.19	55	0.9003	2.000	29.10	91	0.9501	2.000	43.94
	3	1	0.7814	3.000	1.00	16	0.9023	3.000	9.14	36	0.9502	3.000	19.58
	1	123	0.7513	1.000	62.17	274	0.9006	1.000	117.37	381	0.9503	1.000	153.15
1.4	2	32	0.7503	2.000	17.87	118	0.9006	2.000	56.72	195	0.9503	2.000	86.79
	3	1	0.7736	3.000	1.00	30	0.9005	3.000	16.56	75	0.9500	3.000	37.25
	1	485	0.7502	1.000	218.66	1052	0.9002	1.000	414.03	1443	0.9501	1.000	539.28
1.2	2	118	0.7518	2.000	60.65	452	0.9001	2.000	195.48	753	0.9507	2.000	298.97
	3	1	0.7624	3.000	1.00	113	0.9009	3.000	52.25	283	0.9500	3.000	121.48

Table V. The Value of n for $k = 5$ with the Estimated $P(CS)$, $E[S]$, and $E[N]$ under the Slippage Configuration

		$P^* = 0.75$				$P^* = 0.90$				$P^* = 0.95$			
θ^*	t	n	$P(CS)$	$E[S]$	$E[N]$	n	$P(CS)$	$E[S]$	$E[N]$	n	$P(CS)$	$E[S]$	$E[N]$
	1	12	0.7618	1.000	7.44	24	0.9040	1.000	13.10	34	0.9512	1.000	16.49
3.0	2	5	0.7537	2.000	3.50	14	0.9000	1.902	8.05	20	0.9501	1.847	11.41
	3	2	0.7954	3.000	2.00	7	0.9021	2.742	5.13	12	0.9500	2.783	7.71
	1	13	0.7520	1.000	8.36	28	0.9023	1.000	15.02	39	0.9514	1.000	19.20
2.8	2	6	0.7684	1.918	4.02	16	0.9012	1.852	9.34	24	0.9509	1.888	13.02
	3	2	0.7831	3.000	2.00	8	0.9053	2.920	5.52	13	0.9500	2.769	8.89
	1	17	0.7557	1.000	9.81	34	0.9016	1.000	17.43	46	0.9512	1.000	22.64
2.6	2	7	0.7514	1.924	4.23	18	0.9003	1.887	10.96	29	0.9509	1.907	15.27
	3	2	0.7700	3.000	2.00	9	0.9028	3.000	6.19	17	0.9516	2.880	10.25
	1	20	0.7543	1.000	11.94	41	0.9008	1.000	21.09	58	0.9512	1.000	27.10
2.4	2	8	0.7577	1.910	5.80	22	0.9004	1.921	13.01	35	0.9507	1.920	18.17
	3	2	0.7553	3.000	2.00	12	0.9037	2.907	7.43	19	0.9502	2.938	11.89
	1	26	0.7557	1.000	15.18	52	0.9005	1.000	26.69	74	0.9509	1.000	33.95
2.2	2	10	0.7514	1.979	6.70	29	0.9009	1.932	16.17	43	0.9503	1.918	22.76
	3	3	0.7799	3.000	2.77	13	0.9021	2.958	9.16	25	0.9506	2.944	14.68
	1	34	0.7507	1.000	19.76	71	0.9006	1.000	35.18	98	0.9502	1.000	45.05
2.0	2	14	0.7501	2.000	8.39	37	0.9003	1.953	21.07	59	0.9512	1.960	29.68
	3	3	0.7587	3.000	2.78	18	0.9006	2.993	11.41	33	0.9501	2.971	18.90
	1	50	0.7514	1.000	28.34	104	0.9018	1.000	50.37	141	0.9502	1.000	64.42
1.8	2	19	0.7510	1.997	11.95	55	0.9010	1.988	29.62	84	0.9503	1.979	41.93
	3	6	0.7546	3.000	3.80	27	0.9008	3.000	15.74	48	0.9506	2.991	26.26
	1	85	0.7505	1.000	45.58	174	0.9001	1.000	81.19	241	0.9501	1.000	103.60
1.6	2	32	0.7505	2.000	19.08	91	0.9001	2.000	47.03	138	0.9506	1.999	66.81
	3	7	0.7525	3.000	5.50	44	0.9002	3.000	24.38	78	0.9504	3.000	41.30

Table V. Continued

θ^*	t	$P^* = 0.75$ n	$P(CS)$	$E[S]$	$E[N]$	$P^* = 0.90$ n	$P(CS)$	$E[S]$	$E[N]$	$P^* = 0.95$ n	$P(CS)$	$E[S]$	$E[N]$
	1	182	0.7501	1.000	92.42	369	0.9001	1.000	163.99	513	0.9502	1.000	209.85
1.4	2	68	0.7504	2.000	37.92	192	0.9002	2.000	93.74	289	0.9504	2.000	133.40
	3	15	0.7506	3.000	9.38	89	0.9005	3.000	47.77	166	0.9501	3.000	80.81
	1	738	0.7503	1.000	331.35	1462	0.9004	1.000	584.90	1974	0.9500	1.000	747.07
1.2	2	269	0.7506	2.000	131.67	769	0.9004	2.000	328.83	1136	0.9502	2.000	468.89
	3	56	0.7511	3.000	30.74	363	0.9002	3.000	162.47	655	0.9502	3.000	278.82

Table VI. The Value of n for $k = 6$ with the Estimated $P(CS)$, $E[S]$, and $E[N]$ under the Slippage Configuration

θ^*	t	$P^* = 0.75$ n	$P(CS)$	$E[S]$	$E[N]$	$P^* = 0.90$ n	$P(CS)$	$E[S]$	$E[N]$	$P^* = 0.95$ n	$P(CS)$	$E[S]$	$E[N]$
	1	16	0.7554	1.000	9.30	30	0.9017	1.000	16.03	41	0.9505	1.000	20.32
3.0	2	10	0.7541	1.864	4.90	20	0.9014	1.760	10.89	27	0.9502	1.848	14.61
	3	3	0.7771	3.000	2.78	12	0.9025	2.838	7.57	19	0.9519	2.722	11.04
	1	18	0.7517	1.000	10.73	35	0.9012	1.000	18.52	48	0.9503	1.000	23.54
2.8	2	9	0.7527	1.813	6.31	22	0.9004	1.835	12.48	32	0.9507	1.829	16.91
	3	3	0.7623	3.000	2.79	14	0.9012	2.799	8.30	22	0.9511	2.826	12.43
	1	21	0.7514	1.000	12.92	42	0.9012	1.000	21.96	57	0.9509	1.000	27.93
2.6	2	10	0.7522	1.907	6.92	25	0.9013	1.889	14.56	38	0.9516	1.871	19.99
	3	5	0.7601	2.897	3.71	15	0.9017	2.810	9.92	25	0.9501	2.844	14.68
	1	26	0.7520	1.000	15.76	52	0.9005	1.000	26.76	72	0.9506	1.000	33.57
2.4	2	13	0.7559	1.958	8.35	31	0.9004	1.876	17.67	45	0.9502	1.923	23.83
	3	6	0.7500	2.954	3.97	18	0.9001	2.921	11.45	31	0.9514	2.917	17.50
	1	34	0.7511	1.000	19.76	66	0.9009	1.000	33.77	91	0.9505	1.000	42.54
2.2	2	16	0.7514	1.943	10.33	40	0.9005	1.911	22.16	59	0.9508	1.914	30.10
	3	7	0.7503	2.906	4.65	24	0.9011	2.943	14.20	39	0.9509	2.912	21.61
	1	46	0.7535	1.000	26.42	89	0.9005	1.000	44.80	122	0.9502	1.000	56.59
2.0	2	22	0.7527	1.982	13.26	53	0.9012	1.949	29.13	77	0.9500	1.943	39.55
	3	8	0.7504	3.000	5.97	30	0.9002	2.959	18.61	50	0.9504	2.953	28.08
	1	67	0.7510	1.000	37.81	132	0.9007	1.000	64.44	178	0.9506	1.000	81.51
1.8	2	31	0.7519	1.984	19.00	78	0.9005	1.976	41.13	113	0.9504	1.977	56.37
	3	12	0.7506	2.975	8.33	44	0.9004	2.988	25.96	72	0.9501	2.978	39.78
	1	114	0.7501	1.000	61.35	224	0.9013	1.000	104.83	300	0.9503	1.000	131.95
1.6	2	52	0.7509	1.999	30.20	128	0.9010	1.998	65.93	186	0.9502	1.995	90.54
	3	20	0.7545	3.000	13.03	72	0.9005	3.000	41.03	121	0.9508	2.996	63.20
	1	248	0.7507	1.000	125.49	479	0.9005	1.000	213.25	640	0.9504	1.000	268.42
1.4	2	109	0.7501	2.000	60.64	272	0.9001	2.000	132.90	399	0.9503	2.000	182.76
	3	41	0.7509	3.000	25.07	158	0.9008	3.000	81.72	258	0.9500	3.000	125.97
	1	991	0.7504	1.000	452.95	1872	0.9004	1.000	766.86	2486	0.9501	1.000	964.41
1.2	2	443	0.7514	2.000	215.61	1045	0.9002	2.000	470.76	1543	0.9504	2.000	648.16
	3	162	0.7509	3.000	85.93	617	0.9002	3.000	284.91	989	0.9502	3.000	440.32

Table VII. The Value of n for $k = 7$ with the Estimated $P(CS)$, $E[S]$, and $E[N]$ under the Slippage Configuration

		$P^* = 0.75$				$P^* = 0.90$				$P^* = 0.95$			
θ^*	t	n	$P(CS)$	$E[S]$	$E[N]$	n	$P(CS)$	$E[S]$	$E[N]$	n	$P(CS)$	$E[S]$	$E[N]$
3.0	1	19	0.7548	1.000	11.63	36	0.9022	1.000	19.47	49	0.9505	1.000	24.06
	2	11	0.7557	1.796	7.22	24	0.9004	1.801	13.65	34	0.9518	1.800	18.09
	3	6	0.7534	2.911	4.09	17	0.9019	2.689	10.06	25	0.9512	2.750	14.14
2.4	1	33	0.7520	1.000	19.45	63	0.9006	1.000	32.35	85	0.9505	1.000	40.37
	2	19	0.7558	1.906	11.16	40	0.9007	1.870	22.41	56	0.9500	1.897	29.74
	3	9	0.7586	2.913	6.41	27	0.9014	2.875	15.95	40	0.9505	2.869	22.90
2.0	1	57	0.7501	1.000	33.09	109	0.9006	1.000	54.89	148	0.9505	1.000	68.43
	2	31	0.7528	1.972	18.41	69	0.9011	1.939	37.37	98	0.9506	1.936	49.80
	3	16	0.7537	2.974	9.87	44	0.9009	2.929	26.05	69	0.9502	2.931	37.58
1.6	1	145	0.7509	1.000	78.21	271	0.9001	1.000	128.99	364	0.9503	1.000	160.99
	2	73	0.7506	1.997	42.22	167	0.9009	1.993	85.85	236	0.9504	1.990	114.99
	3	36	0.7512	3.000	22.12	105	0.9003	2.995	58.49	164	0.9505	2.989	85.47

Table VIII. The Value of n for $k = 8$ with the Estimated $P(CS)$, $E[S]$, and $E[N]$ under the Slippage Configuration

		$P^* = 0.75$				$P^* = 0.90$				$P^* = 0.95$			
θ^*	t	n	$P(CS)$	$E[S]$	$E[N]$	n	$P(CS)$	$E[S]$	$E[N]$	n	$P(CS)$	$E[S]$	$E[N]$
3.0	1	23	0.7531	1.000	13.71	42	0.9013	1.000	22.61	57	0.9504	1.000	28.02
	2	15	0.7525	1.814	8.60	31	0.9013	1.771	16.45	41	0.9519	1.795	21.55
	3	11	0.7511	2.665	5.31	21	0.9001	2.617	12.78	31	0.9514	2.657	17.46
2.4	1	40	0.7540	1.000	23.68	74	0.9005	1.000	38.24	99	0.9503	1.000	47.24
	2	23	0.7515	1.880	14.45	50	0.9009	1.861	27.39	69	0.9501	1.880	35.74
	3	13	0.7516	2.898	9.01	35	0.9006	2.818	20.44	51	0.9503	2.822	28.41
2.0	1	70	0.7513	1.000	40.28	130	0.9010	1.000	65.30	173	0.9507	1.000	80.62
	2	40	0.7505	1.958	23.81	85	0.9014	1.928	45.85	121	0.9510	1.929	60.42
	3	22	0.7503	2.958	14.25	58	0.9005	2.908	33.64	87	0.9503	2.907	47.14
1.6	1	176	0.7501	1.000	95.64	321	0.9002	1.000	154.39	428	0.9506	1.000	191.01
	2	96	0.7518	1.994	55.41	205	0.9002	1.990	106.51	288	0.9504	1.986	140.39
	3	52	0.7505	2.998	32.16	141	0.9016	2.991	77.06	209	0.9502	2.982	108.61

Table IX. The Value of n for $k = 9$ with the Estimated $P(CS)$, $E[S]$, and $E[N]$ under the Slippage Configuration

		$P^* = 0.75$				$P^* = 0.90$				$P^* = 0.95$			
θ^*	t	n	$P(CS)$	$E[S]$	$E[N]$	n	$P(CS)$	$E[S]$	$E[N]$	n	$P(CS)$	$E[S]$	$E[N]$
3.0	1	26	0.7509	1.000	16.21	48	0.9015	1.000	26.07	65	0.9508	1.000	31.90
	2	17	0.7532	1.828	10.63	35	0.9007	1.769	19.40	47	0.9502	1.782	24.91
	3	10	0.7526	2.745	7.14	27	0.9020	2.666	15.31	37	0.9501	2.632	20.62
2.4	1	47	0.7509	1.000	27.63	85	0.9004	1.000	44.17	113	0.9504	1.000	54.24
	2	29	0.7523	1.865	17.74	59	0.9006	1.847	32.43	81	0.9505	1.871	41.91
	3	18	0.7540	2.834	11.70	44	0.9020	2.800	25.09	62	0.9504	2.804	34.11
2.0	1	82	0.7503	1.000	47.55	149	0.9001	1.000	75.74	198	0.9501	1.000	93.09
	2	50	0.7523	1.948	29.54	102	0.9007	1.919	54.72	141	0.9511	1.922	71.03
	3	30	0.7504	2.940	18.83	73	0.9003	2.889	41.63	107	0.9507	2.888	57.16
1.6	1	208	0.7500	1.000	113.54	374	0.9002	1.000	180.31	492	0.9504	1.000	221.27
	2	118	0.7501	1.990	68.83	247	0.9006	1.986	127.99	341	0.9504	1.982	166.55
	3	71	0.7507	2.994	43.19	174	0.9006	2.984	95.73	255	0.9502	2.975	132.46

Table X. The Value of n for $k = 10$ with the Estimated $P(CS)$, $E[S]$, and $E[N]$ under the Slippage Configuration

		$P^* = 0.75$				$P^* = 0.90$				$P^* = 0.95$			
θ^*	t	n	$P(CS)$	$E[S]$	$E[N]$	n	$P(CS)$	$E[S]$	$E[N]$	n	$P(CS)$	$E[S]$	$E[N]$
	1	31	0.7540	1.000	18.59	55	0.9019	1.000	29.43	73	0.9503	1.000	35.95
3.0	2	22	0.7549	1.800	12.49	41	0.9008	1.747	22.29	55	0.9511	1.769	28.58
	3	16	0.7516	2.699	8.82	32	0.9004	2.593	18.09	44	0.9509	2.639	23.89
	1	54	0.7526	1.000	32.01	97	0.9004	1.000	50.38	129	0.9510	1.000	61.47
2.4	2	34	0.7501	1.860	20.99	68	0.9001	1.842	37.54	92	0.9501	1.856	48.10
	3	23	0.7526	2.835	14.54	52	0.9008	2.768	29.72	75	0.9511	2.773	40.12
	1	96	0.7512	1.000	55.18	169	0.9003	1.000	86.47	224	0.9501	1.000	105.75
2.0	2	59	0.7512	1.941	35.29	119	0.9007	1.911	63.88	161	0.9501	1.914	81.81
	3	38	0.7511	2.919	23.78	89	0.9007	2.874	49.84	126	0.9500	2.870	67.20
	1	242	0.7509	1.000	132.13	433	0.9005	1.000	207.23	561	0.9500	1.000	252.84
1.6	2	145	0.7510	1.987	83.38	292	0.9009	1.983	150.33	396	0.9507	1.979	193.54
	3	90	0.7505	2.988	54.92	210	0.9000	2.977	115.48	300	0.9505	2.968	156.92

Table XI. The Value of n for $k = 20, 30, 40$, with the Estimated $P(CS)$, $E[S]$ and $E[N]$ under the Slippage Configuration

				$P^* = 0.75$				$P^* = 0.90$				$P^* = 0.95$			
k	δ_u	θ^*	t	n	$P(CS)$	$E[S]$	$E[N]$	n	$P(CS)$	$E[S]$	$E[N]$	n	$P(CS)$	$E[S]$	$E[N]$
			1	1313	0.7515	1.000	697.96	2114	0.9006	1.000	1011.45	2675	0.9502	1.000	1200.59
	0.02	1.408	2	914	0.7500	1.990	508.59	1563	0.9000	1.987	802.22	2025	0.9503	1.988	984.85
			3	688	0.7501	2.984	399.72	1282	0.9001	2.975	682.53	1697	0.9501	2.974	862.64
			1	219	0.7503	1.000	128.55	360	0.9001	1.000	187.63	460	0.9503	1.000	223.38
20	0.05	2.053	2	162	0.7511	1.873	95.84	282	0.9005	1.858	151.54	365	0.9502	1.865	185.93
			3	127	0.7503	2.804	76.75	239	0.9013	2.755	130.69	313	0.9501	2.751	164.26
			1	93	0.7515	1.000	56.19	153	0.9001	1.000	82.28	196	0.9502	1.000	98.36
	0.08	2.739	2	70	0.7518	1.767	42.76	123	0.9009	1.745	67.41	161	0.9508	1.730	82.63
			3	57	0.7511	2.610	34.88	105	0.9004	2.556	58.71	141	0.9501	2.529	73.80
			1	1008	0.7506	1.000	566.82	1572	0.9000	1.000	797.62	1970	0.9501	1.000	937.60
	0.02	1.612	3	617	0.7509	2.904	364.32	1073	0.9005	2.881	580.80	1378	0.9505	2.878	714.41
			4	521	0.7509	3.884	311.75	938	0.9004	3.839	522.30	1251	0.9513	3.826	656.83
			1	182	0.7515	1.000	109.68	293	0.9006	1.000	155.88	365	0.9500	1.000	183.18
30	0.05	2.579	3	123	0.7505	2.586	73.89	213	0.9008	2.528	117.23	273	0.9504	2.523	143.40
			4	106	0.7504	3.489	64.04	193	0.9007	3.362	106.78	251	0.9503	3.340	133.36
			1	79	0.7512	1.000	49.52	128	0.9001	1.000	70.56	164	0.9507	1.000	82.75
	0.08	3.609	3	58	0.7521	2.385	34.57	98	0.9008	2.288	54.27	127	0.9502	2.277	66.38
			4	50	0.7502	3.174	30.44	89	0.9006	3.015	49.92	116	0.9502	2.996	61.90
			2	668	0.7503	1.881	390.55	1067	0.9004	1.874	572.14	1344	0.9502	1.878	682.50
	0.02	1.816	4	493	0.7511	3.743	294.11	853	0.9005	3.678	471.16	1101	0.9503	3.661	580.83
			6	381	0.7505	5.704	234.38	705	0.9005	5.570	405.29	944	0.9502	5.525	515.81
			2	134	0.7511	1.675	80.93	218	0.9009	1.639	118.70	269	0.9504	1.667	140.67
40	0.05	3.105	4	103	0.7505	3.237	62.94	181	0.9003	3.106	100.09	232	0.9501	3.060	122.55
			6	83	0.7503	4.978	51.54	154	0.9009	4.740	87.66	207	0.9501	4.648	110.88
			2	60	0.7505	1.594	37.69	99	0.9000	1.545	55.51	126	0.9508	1.529	65.93
	0.08	4.478	4	48	0.7511	2.886	30.48	83	0.9014	2.775	47.63	111	0.9506	2.691	58.29
			6	42	0.7509	4.571	25.49	72	0.9000	4.156	42.45	100	0.9502	4.102	53.18

Table XII. The Value of n for $k = 50, 60, 70$, with the Estimated $P(CS)$, $E[S]$ and $E[N]$ under the Slippage Configuration

				$P^* = 0.75$				$P^* = 0.90$				$P^* = 0.95$			
k	δ_u	θ^*	t	n	$P(CS)$	$E[S]$	$E[N]$	n	$P(CS)$	$E[S]$	$E[N]$	n	$P(CS)$	$E[S]$	$E[N]$
50	0.02	2.020	2	601	0.7500	1.826	356.61	946	0.9003	1.811	513.78	1187	0.9502	1.819	607.45
			5	417	0.7505	4.546	252.15	719	0.9000	4.414	404.59	940	0.9502	4.380	500.20
			7	336	0.7509	6.494	209.94	623	0.9004	6.277	359.66	817	0.9505	6.201	454.76
	0.05	3.632	2	125	0.7504	1.609	76.47	197	0.9011	1.596	109.79	253	0.9502	1.580	129.84
			5	93	0.7505	3.854	56.47	165	0.9012	3.594	90.13	206	0.9503	3.557	109.62
			7	81	0.7513	5.708	47.87	145	0.9011	5.204	81.34	190	0.9500	5.022	101.59
	0.08	5.348	2	56	0.7511	1.557	36.30	94	0.9006	1.471	52.47	116	0.9505	1.508	61.55
			5	45	0.7523	3.493	27.76	77	0.9001	3.314	43.33	98	0.9509	3.047	53.19
			7	38	0.7506	5.388	23.29	69	0.9004	4.760	39.21	92	0.9502	4.217	49.93
60	0.02	2.225	3	492	0.7501	2.598	293.46	787	0.9001	2.551	431.97	999	0.9505	2.548	515.85
			6	364	0.7502	5.311	223.23	635	0.9003	5.116	360.61	836	0.9506	5.049	445.61
			9	287	0.7513	8.230	179.19	534	0.9001	7.884	312.69	716	0.9500	7.748	398.04
	0.05	4.158	3	108	0.7502	2.197	65.68	170	0.9008	2.146	95.81	219	0.9501	2.081	114.31
			6	89	0.7515	4.515	51.89	147	0.9002	4.105	82.54	192	0.9502	4.005	101.05
			9	69	0.7512	6.886	43.09	132	0.9005	6.410	73.58	166	0.9501	6.122	91.99
	0.08	6.217	3	48	0.7504	2.118	31.49	84	0.9005	1.924	46.68	104	0.9503	1.908	55.43
			6	43	0.7508	4.400	25.09	74	0.9000	3.656	40.69	89	0.9507	3.409	49.75
			9	35	0.7517	6.305	21.78	64	0.9011	5.425	36.95	80	0.9506	5.216	45.96
70	0.02	2.429	3	472	0.7520	2.512	279.73	745	0.9005	2.457	406.11	922	0.9503	2.459	480.76
			7	334	0.7500	6.051	203.56	585	0.9000	5.769	328.98	751	0.9501	5.676	406.34
			10	271	0.7513	8.948	168.63	504	0.9010	8.507	291.53	667	0.9500	8.315	369.17
	0.05	4.684	3	103	0.7517	2.150	63.34	166	0.9016	2.052	92.01	207	0.9504	1.992	108.99
			7	78	0.7506	4.961	48.74	135	0.9002	4.537	77.31	183	0.9511	4.368	95.23
			10	68	0.7501	7.212	42.43	124	0.9003	6.576	70.66	167	0.9501	6.581	87.87
	0.08	7.087	3	49	0.7509	2.034	30.90	80	0.9004	1.850	45.16	102	0.9509	1.812	53.59
			7	37	0.7505	4.649	23.67	67	0.9002	3.900	39.16	88	0.9518	3.869	47.17
			10	35	0.7503	5.936	22.07	62	0.9012	5.301	36.85	80	0.9504	5.798	43.53

Table XIII. The Value of n for $k = 80, 90, 100$, with the Estimated $P(CS)$, $E[S]$ and $E[N]$ under the Slippage Configuration

				$P^* = 0.75$				$P^* = 0.90$				$P^* = 0.95$			
k	δ_u	θ^*	t	n	$P(CS)$	$E[S]$	$E[N]$	n	$P(CS)$	$E[S]$	$E[N]$	n	$P(CS)$	$E[S]$	$E[N]$
80	0.02	2.633	4	414	0.7515	3.232	245.68	663	0.9007	3.124	362.92	820	0.9503	3.107	432.04
			8	307	0.7511	6.777	188.26	544	0.9000	6.403	305.35	702	0.9504	6.274	376.52
			12	242	0.7506	10.660	151.04	453	0.9003	10.015	265.34	617	0.9500	9.761	338.14
	0.05	5.211	4	94	0.7506	2.661	57.79	151	0.9008	2.530	84.58	187	0.9502	2.449	100.32
			8	74	0.7505	5.364	47.10	125	0.9010	4.872	73.57	171	0.9509	4.804	90.09
			12	56	0.7502	9.089	37.79	107	0.9002	7.675	65.75	154	0.9502	7.705	82.25
	0.08	7.957	4	47	0.7502	2.730	27.71	73	0.9000	2.199	42.25	95	0.9503	2.259	49.64
			8	35	0.7525	4.414	23.66	64	0.9012	3.962	38.47	85	0.9508	4.402	44.28
			12	33	0.7505	7.496	20.08	53	0.9013	6.747	34.11	79	0.9502	6.539	40.98
90	0.02	2.837	4	396	0.7502	3.138	237.37	628	0.9003	3.024	347.21	783	0.9502	2.998	411.60
			9	288	0.7506	7.510	176.16	510	0.9004	7.021	286.57	659	0.9503	6.824	353.50
			13	232	0.7508	11.284	145.32	437	0.9003	10.576	253.19	599	0.9505	10.279	322.21
	0.05	5.737	4	90	0.7502	2.620	56.56	147	0.9002	2.425	82.67	180	0.9504	2.405	96.94
			9	71	0.7501	5.886	45.13	120	0.9004	5.373	70.65	157	0.9500	5.131	85.98
			13	55	0.7511	10.155	36.40	106	0.9002	8.490	63.15	152	0.9502	7.622	80.49

Continued

Table XIII. Continued

	0.08	8.826	4	46	0.7505	2.659	27.37	68	0.9001	2.186	41.15	93	0.9500	2.180	48.48
			9	34	0.7516	4.519	23.73	62	0.9007	4.496	36.95	81	0.9500	4.626	42.81
			13	31	0.7517	8.287	19.68	53	0.9001	8.084	32.32	73	0.9506	6.246	40.65
100	0.02	3.041	5	361	0.7518	3.834	216.17	575	0.9006	3.642	319.72	724	0.9501	3.590	380.83
			10	279	0.7504	8.153	167.27	486	0.9009	7.603	271.38	630	0.9507	7.351	334.67
			15	210	0.7502	12.924	134.18	404	0.9004	12.076	236.25	550	0.9503	11.602	301.86
	0.05	6.263	5	77	0.7501	3.144	52.23	138	0.9006	2.846	78.00	170	0.9506	2.787	91.53
			10	67	0.7501	6.975	42.08	117	0.9002	6.094	67.13	151	0.9501	5.258	83.19
			15	57	0.7523	11.807	34.84	109	0.9002	10.059	59.29	135	0.9500	8.363	76.99
	0.08	9.696	5	41	0.7509	2.909	26.25	65	0.9021	2.438	39.84	89	0.9502	2.443	46.40
			10	33	0.7502	5.123	23.36	54	0.9009	5.426	34.52	79	0.9507	4.627	42.43
			15	27	0.7513	10.795	17.76	53	0.9014	9.635	30.92	64	0.9501	6.116	40.19

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