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Rice, Donald B.

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AN EXPERIMENTAL METHOD FOR PROFIT MAXIMIZATION

DONALD B. RICE*

SUMMARY

This paper develops a procedure for locating profit maximizing operating conditions when the input variables are subject to control by the experimenter and observations on physical output may be taken subject to experimental error at any combination of these inputs. This procedure uses the Box–Wilson technique, a well-known statistical search procedure, together with the price of the output and the cost of the inputs so as to search for maximum profit points rather than maximum output points.

The result is an experimental search method which maximizes profit when the functional form of the physical response surface (production function) is unknown. In this case, of course, standard mathematical programming models are of no use in the absence of the required functional forms. This functional form, though unknown, is determined by the existing investment in capital equipment.

These procedures, like the Box–Wilson technique, are more likely to be applicable to flow process reactor systems in the chemical and oil industries, but are certainly not limited to such applications.

INTRODUCTION

There has been an interesting acceptance of the Box–Wilson technique among engineers concerned with the application of statistical methods to production problems. In this paper it has been attempted to provide a procedure, based on the Box–Wilson technique, which is applicable to profit maximization rather than to output maximization. This procedure carries with it the same advantages and inherent difficulties as the Box–Wilson technique, but nevertheless can be a useful tool for production management in some cases where mathematical programming methods fail for lack of functional forms needed for the model.

THE BOX–WILSON TECHNIQUE

The Box–Wilson technique (Hunter, 1958 and 1959; Duncan, 1959; Davies, 1956; Box & Wilson, 1951) is an experimental procedure for searching for an optimum response when observations can be taken directly on that response. Complete factorial or fractional replicate experimental designs (Bradley, 1958; Hunter, 1958 and 1959) are used to generate observations on the response. These observations are then used to fit a first-order linear regression model to the response surface, e.g.

$$y = b_o + b_1 x_1 + b_2 x_2 + \dots + b_n x_n \tag{1}$$

where y is the response to be optimized and $x_1, \dots, x_n$ are the n input variables whose levels are under control. Least squares procedures are used to estimate $b_o, b_1, \dots, b_n$. Define

$$x = t + d \, . \, b, \quad d \geqq 0 \tag{2}$$

* Defense Management Program, U.S. Naval Postgraduate School, Monterey, California 93940, U.S.A.
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where x denotes a point in the space of the inputs, t is the center point of the experimental design, d is a scaler and $b = (b_1, b_2, \ldots, b_n)$. Equation (2) then defines the estimated path of steepest ascent on the response surface.

Observations are taken along this path at positive values of d, the path parameter, until an optimum response on that path is located. This sub-optimal point is used as the center point of another experimental design and the procedure repeated until a point is located where no b_i, $i = 1, \ldots, n$, is statistically significant (nonzero).

The object of the Box–Wilson technique is to reduce experimentation to a minimum, and it is therefore particularly useful when experimentation is costly. The technique makes no explicit treatment of experimental cost but holds this cost to a minimum by reducing the number of observations required to locate an optimum.

Some Economic Considerations

The Box–Wilson technique is a method for locating the conditions which give a maximum response. It is most directly useful when the response is some measure of output. The Box–Wilson technique can be used to locate maximum profit conditions, for example, only if observations can be taken on the level of profit as a response at various levels of the inputs. This requires either that profit can be measured directly or that measures can be taken on some monotone function of profit. Such measurements are unlikely to be available for many production processes.

The conditions for maximum profit may be found from

$$\max\left\{R(y) - \min C(x_1, \ldots, x_n) \atop \text{subject to } y = f(x_1, \ldots, x_n)\right\} \tag{3}$$

where $y \geqq 0$, $x_i \geqq 0$, $i = 1, \ldots, n$. The function R is revenue, C is the cost function, f is the physical response (production function), y is the level of physical output, and x_i is the level of the ith input. The necessary conditions for a solution are given by

$$\frac{\mathrm{d}R}{\mathrm{d}y} = \frac{\mathrm{d}C}{\mathrm{d}y} = \frac{C_1(x)}{f_1(x)} = \ldots = \frac{C_n(x)}{f_n(x)} \tag{4}$$

(Baumol, 1961) where $C_i(x)$ and $f_i(x)$ denote the partial derivatives of C and f with respect to x_i. Conditions (4) will be exploited in the next section.

The Profit Maximizing Procedure

A procedure will now be developed for locating the levels of the inputs, x_i, $i = 1, \ldots, n$, which give a maximum level of profit. Assume that the physical response surface is continuous, but that its functional form is unknown. This procedure, like the Box–Wilson technique, will use evolutionary methods of searching for the optimum point. Over the range of levels of y to be examined a price per unit of y will be taken as given, denoted p, and a cost per unit of input x_i, denoted c_i, $i = 1, \ldots, n$.

These assumptions result in a revenue function

$$R(y) = p \cdot y \tag{5}$$

and a cost function

$$C(x) = c_0 + c_1 x_1 + c_2 x_2 + \ldots + c_n x_n. \tag{6}$$

Let equation (1) be a first-order regression model used to approximate the response function f over the region of the experimental design just as for an application of the Box–Wilson procedure. Note that b_i, $i = 1, ..., n$, is an estimate of $f_i(x)$. Then conditions (4) become

$$p = \frac{c_1}{b_1} = \dots = \frac{c_n}{b_n} \tag{7}$$

or

$$(b_i p - c_i) = 0, i = 1, ..., n. \tag{8}$$

Equation (8) gives the conditions for profit maximization which will be searched for experimentally. The economic interpretation of (8) is the standard microeconomic solution for multiple inputs, viz. each input is employed up to that level where its marginal contribution to revenue is equal to its cost.

The expression $(b_i p - c_i)$, obtained from the known price of y and cost of x_i and the coefficient of x_i in the regression model, will be called the marginal profitability of input x_i, denoted m_i. Now define

$$x = t + d.m, \quad d \geqq 0 \tag{9}$$

where x, t and d are defined as before and $m = (m_1, m_2, ..., m_n)$. The path denoted by (9) is the estimated path of steepest ascent on the profit surface. A search is conducted among positive values of d until a maximum level of profit is located along path (9). The point where this sub-optimization occurs is then used as the center point of a new experimental design and the procedure iterated.

The foregoing assumed that no direct monotone measure of profit exists and the constancy of p and c_i, $i = 1, ..., n$, is unlikely to hold over all levels of y and corresponding points x. Therefore the change in profit, denoted ΔP, is used to locate the maximum profit point along a given path (9).

Let the actual level of profit at t be P_o. Using m_i as the appropriate coefficients a prediction equation for profit is given by

$$\begin{aligned} P_p(x) &= m_1(x_1 - t_1) + \dots + m_n(x_n - t_n) + P_o \\ &= \sum_{i=1}^{n} (b_i p - c_i)(x_i - t_i) + P_o \end{aligned} \tag{10}$$

where $P_p(x)$ is predicted profit at x and $t = (t_1, ..., t_n)$ is the center point of the experimental design used to estimate b_i, $i = 1, ..., n$. Then the predicted change in profit, denoted ΔP_p, resulting from a move from t to some x is given by

$$\Delta P_p(x) = \sum_{i=1}^{n} (b_i p - c_i)(x_i - t_i). \tag{11}$$

This prediction equation is not adequate, because it rapidly becomes inaccurate as the length of $(x - t)$ becomes large. Under the assumptions of constant price and costs over the range of levels of interest, an equation which gives the actual change in profit, denoted $\Delta P_a(x)$, can be written,

$$\Delta P_a(x) = p(y_x - y_t) - \sum_{i=1}^{n} c_i(x_i - t_i) \tag{12}$$

where y_x is the level of output at x and y_t the level of output at t. Equation (12) is the function to be maximized over positive values of d in (9) to find the point of maximum profit on a given path. Equation (11) will be useful for other purposes in later sections.

In "real" problems the level of the physical input, x_i, may be measured in pounds, barrels per day, etc., and the cost numbers, c_i, must be computed to comply with the measure used for x_i. Output y may be measured in similar units, or even as per cent conversion on a chemical reaction, and the price, p, must, likewise, be appropriately calculated.

Problems in Application

The above procedure is generally applicable to production processes with multiple inputs where observations on the level of physical output may be taken at any desired level of the inputs. In practice, the procedure must be extended to account for cost of experimentation and the form of the response surface.

If the cost of experimentation is negligible, and the response surface is unimodal, the above procedure is iterated until conditions (8) are found, i.e. until the hypothesis $m_i = 0$ cannot be rejected for any i. If the standard assumptions that measurement errors are normally distributed is made for the regression model, then the distribution of m_i is determined from the distribution of b_i, since m_i is a linear function of b_i.

Unimodal surfaces, however, need not be everywhere concave. They will, in general, be concave in the region around the mode (peak). The constancy of output price and input costs insures that the profit maximizing solution must occur in the concave portion of the surface, since this is the region of decreasing marginal physical productivity. A comparison of equations (11) and (12) can be used to get some information on the shape of the response surface in the region being searched. Let x be determined by (9) and let x be outside of the region of the experimental design. Then if $\Delta P_a(x) > \Delta P_p(x)$, the profit surface is convex in the region near the path defined by (9). If $\Delta P_a(x) < \Delta P_p(x)$, the surface is concave. Since the solution must occur in a concave portion of the surface, another iteration should be run, even if conditions (8) are satisfied, if the experimentation has been conducted in a convex portion of the surface.

If costs are negligible, and the surface is multimodal, the procedures for the unimodal case are applicable, but they must be applied several times, from a variety of starting points spread over the space of the inputs, so as to locate solutions corresponding to all of the modes. There will, of course, be a locally optimal solution for each mode. Let $x_1, \ldots, x_m$ be the m locally optimal points corresponding to the m modes of the response surface. Then, using the form of equation (12), calculate

$$\Delta P_a(x_j) = p(y_{x_j} - y_{x_1}) - \sum_{i=1}^{n} c_i(x_{ij} - x_{i1}), j = 2, \ldots, m. \tag{13}$$

Then the optimal point is that x_j for which

$$\max_j \{\Delta P_a(x_j) : \Delta P_a > 0\} \tag{14}$$

occurs. If $\Delta P_a(x_j) \leqq 0, j = 2, \ldots, m$, then x_1 is the optimal point.

When experimental cost is not negligible, the returns from searching must be balanced against the cost of searching. Ideally, another experimental design would not be constructed, and observations taken along the resulting path of steepest ascent, if the cost of doing so exceeded the present value of the increase in future profit streams to be found by moving along that path. Determining for certain what this profit increase will be, before the observations are taken and the coefficients of the regression model computed, and without further experimentation along the appropriate path, is an obviously impossible task.

There is, however, a stopping rule, directly applicable to strictly concave response surfaces, which is extended to unimodal response surfaces. If the surface is concave throughout the region of interest, disregarding for the moment the effect of experimental error succeeding iterations of the procedure will result in successively smaller increases in profit.

Let K be the average cost per iteration of the procedure for both the observations in the experimental design used to estimate the regression coefficients and the observations taken for subsequent moves along the estimated path of steepest ascent on the profit surface. Let the increase in profit per time period resulting from the most recent iteration be denoted $\Delta P_a^\star$. Then a decision rule for stopping, which on the average is conservative (i.e. errs on the side of taking too many observations) is to make the final iteration that one for which the present value of its profit increase is less than K. That is, another iteration should not be run if the last one did not pay for itself.

Mathematically the rule is

$$\int_0^T (\Delta P_a^\star)\, e^{-rt}\, dt \leqq K \tag{15}$$

where K is the average cost of experimentation for one iteration, T is the number of time periods in the time horizon relevant for the decision, and r is the discount rate per time period. If T is finite (15) becomes

$$(\Delta P_a^\star)\left(\frac{1-e^{-rT}}{r}\right) \leqq K, \tag{16}$$

and if the time horizon is assumed to be infinite (15) becomes

$$(\Delta P_a^\star)\frac{1}{r} \leqq K. \tag{17}$$

The above stopping rule is not strictly applicable to general unimodal surfaces. It will be recalled, however, that the profit maximizing solution must occur in the concave portion of the response surface. Since all unimodal surfaces are concave around the mode, this stopping rule can be extended by a rule of thumb to unimodal surfaces. If the comparison of profit calculations using equations (11) and (12) shows that the most recent search has been conducted over a convex portion of the profit surface, another iteration should be run, even if equation (16) or (17) is satisfied. The reason for this is, of course, that in a convex region profit is increasing at an increasing rate. If (16) or (17) is satisfied, and the comparison from (11) and (12) shows the region to be concave, the search procedure should stop on a unimodal surface.

Multimodal surfaces present much greater difficulties where experimentation is costly. The above rule can be used to stop the search on any given model, but the rule does not extend to calculations based on a new starting point for the search procedure. If enough starting points are selected, and the procedure applied from each of them to locate solutions on each mode using the above stopping rule in each case, the result is clearly to incur more cost of experimentation than is profitable. There is a need for more research on the question of optimal sampling plans for searching multimodal surfaces.

A Note on Search Methodology

Three methodological questions confront the user of an evolutionary search procedure: 1. The selection of the starting point, 2. The method of taking observations along the search path, 3. The use of end game strategies. No definitive answers can be given to any of these problems, but the type of difficulties which they present to the user of such a search procedure can be indicated. At all times the user should take advantage of any information he has about the process under study, whether experimental or theoretical.

The selection of a starting point clearly affects the amount of search that must be conducted to locate the optimum. For many surface shapes, the closer the start to the optimum, the less search is needed to reach it. For processes already in operation the choice of starting point will likely be made at the existing operating conditions, since rapid and sharp changes in the level of output may be quite costly. For any process, whatever is known about the process should be used to locate a "near optimum" starting point to the extent possible.

Very little can be found in the literature on the Box–Wilson technique about the procedures to be followed in searching for the optimum on a given path. In any event, the solution can at best be an approximate one, since, in the presence of measurement errors, observations must be spaced some distance apart to ensure that even repeated observations can show whether the point results in a higher or lower level of output than nearby points. The length of this minimum spacing distance must depend on the precision with which the response and the level of the controlled input variables can be measured. Given this minimum distance, such techniques as Fibonacci or golden section search (Wilde, 1964) may be used on paths describing a unimodal curve if an upper limit can be placed on the value of d in equation (9) above. Alternatively the user might decide to increment d in amounts equivalent to the minimum spacing distance until a peak is crossed on the path. This latter method does not require limits on the variables or unimodal paths, but it should be noted that it will stop at the first ridge or peak crossed by the search path and not go on to peaks that may be located farther along the path. A third alternative is to take enough observations spread along the path to provide a sketch of the image of the response surface along the path. Then any parts of the path where the image promises high response values could be searched by a Fibonacci or golden section technique.

End game strategies are attempts to check that the optimum has, in fact, been located. When experimentation is costly the use of such strategies results in more cost of experimentation, which was not accounted for in the stopping rule calculations above. Most authors in this area recommend that an experimenter always try to fit a nonlinear function in the neighborhood of any apparent optimum to make sure that many directions are searched and that one is not stuck on a ridge rather than a peak. See Hunter (1958 and 1959) and Wilde (1964) for discussions of higher order models and their applications.

On décrit une méthode pour établir les modes opératoires susceptibles de donner un profit maximum lorsque les variables d'entrée sont sous le contrôle de l'expérimentateur, et quand on peut faire des observations sur la production (sauf erreur expérimentale) avec n'importe quelle combinaison des variables d'entrée. Ce procédé emploie la technique Box–Wilson, bien connue en statistique, en combinaison avec le prix de production et le prix des entrées, de manière à établir les points du profit maximum plutôt que les points de production maximum.

Il en résulte une méthode déterminative expérimentale qui indique le profit maximum lorsque la forme fonctionnelle de la surface de réponse est inconnue. Evidemment, en l'absence de la forme fonctionnelle, les modèles standard de programmage mathématique ne servent à rien. Cette forme fonctionnelle, bien qu'inconnue, est déterminée par le capital d'entreprise actuel.

Ces procédés, tout comme la technique Box–Wilson, sont plus susceptibles d'être employés à la chaîne de production des industries chimique et du pétrole, mais ils ne sont aucunement restreints à ces applications particulières.

Beschreibung einer Methode zur Bestimmung von Arbeitsverfahren, die ein Profitmaximum ergeben wenn die Energievariablen unter der Kontrolle des Forschers stehen, und wenn Beobachtungen über die Leistung, unter Berücksichtigung des Versuchsfehlers, bei irgend einer Kombination solcher Variablen angestellt werden können. Man bedient sich der Box–Wilson Methode, ein bekanntes statistisches Suchverfahren, in Verbindung mit dem Leistungspreis sowie den Eingabekosten, um derart nach Maximalprofitstellen anstatt Maximalleistungsstellen zu suchen.

Hiermit ergibt sich eine experimentelle Suchmethode, die ein Profitmaximum anzeigt wenn die Funktion der physischen Anspruchskurve (Produktionsfunktion) unbekannt ist. In diesem Falle wären natürlich die üblichen mathematischen Programm-Modelle zwecklos, da die benötigten Funktionen nicht zur Verfügung stehen. Die unbekannten Funktionen werden durch die vorhandene Kapitalanlage bestimmt.

Diese Verfahren, wie auch die Box–Wilson Methode, eignen sich eher zur Anwendung bei dem Fertigungsfluss der Erdöl- und chemischen Industrie, sind aber durchaus nicht auf solche Anwendungen beschränkt.

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