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P-stable high-order super-implicit and Obrechkoff methods for periodic initial value problems

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Abstract

This paper discusses the numerical solution of periodic initial value problems. Two classes of methods are discussed, super-implicit and Obrechkoff. The advantage of Obrechkoff methods is that they are high-order one-step methods and thus will not require additional starting values. On the other hand they will require higher derivatives of the right-hand side. In cases when the right-hand side is very complex, we may prefer super-implicit methods. We develop a super-implicit P-stable method of order 12 and Obrechkoff method of order 18.
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Keywords: Obrechkoff methods; Super-implicit; Initial value problems

1. Introduction

In this paper we discuss the numerical solution of a special class (for which y' is missing) of second-order initial value problems (IVPs),

$$y''(x) = f(x, y(x)), \qquad y(0) = y_0, \qquad y'(0) = y'_0. \tag{1}$$

There is a vast literature for the numerical solution of these problems as well as for the general second-order IVPs,

$$y''(x) = f(x, y(x), y'(x)), \qquad y(0) = y_0, \qquad y'(0) = y'_0. \tag{2}$$

See for example the excellent book by Lambert [1]. One class of methods is due to Obrechkoff [2].[1] These methods for the solution of *first-order* IVPs are given by (see e.g. Lambert [1], pp. 199–204, or Lambert and Mitchell, [3])

$$\sum_{j=0}^{k} \alpha_j y_{n+j} = \sum_{i=1}^{\ell} h^i \sum_{j=0}^{k} \beta_{ij} y_{n+j}^{(i)}, \qquad \alpha_k = 1. \tag{3}$$

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[1] The Bulgarian mathematician Academician Nikola Obrechkoff (1896–1963, born in Varna) did pioneering work in such diverse fields as analysis, algebra, number theory, numerical analysis, summation of divergent series, probability and statistics.

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According to Lambert and Mitchell [3], the error constant decreases more rapidly with increasing ℓ rather than the step k. It is difficult to satisfy the zero stability for large k. The weak stability interval appears to be small. The advantage of Obrechkoff methods is the fact that these are one-step high-order methods and as such do not require additional starting values. A list of Obrechkoff methods for $\ell = 1, 2, \ldots, 5 - k$, $k = 1, 2, 3, 4$ is given in Lambert and Mitchell [3]. For example, for $k = 1$ and $\ell = 3$ we get an implicit method of order 6 with an error constant

$$C_7 = -\frac{1}{100\,800}$$

and the method is

$$y_{n+1} - y_n = \frac{h}{2}\left(y'_{n+1} + y'_n\right) - \frac{h^2}{10}\left(y''_{n+1} - y''_n\right) + \frac{h^3}{120}\left(y'''_{n+1} + y'''_n\right). \tag{4}$$

Obrechkoff methods for the solution of *second-order* IVPs (1) can be found in Ananthakrishnaiah [4], and Simos [5]. See also more recent work by Sakas and Simos [6–8], Simos [9,10] and Neta [11]. In Rai and Ananthakrishnaiah [12], Obrechkoff methods for general second-order differential equations (2) are developed.

The other class, called super-implicit, was developed recently by Fukushima [13] for the first-order IVPs and for the special second-order IVPs (1). The methods are called super-implicit because they require the knowledge of functions not only at past and present but also at future time steps. Fukushima developed Cowell and Adams type super-implicit methods of arbitrary degree and auxiliary formulas to be used in the starting and ending procedures. The resulting methods work as a one-step method integrating a large time interval. Symmetric Cowell type methods of order up to 12 are given. The integration error grows linearly with respect to time as in symmetric multistep methods.

The general form of such methods for the second-order IVPs (1) is given by

$$y_{n+1} + \sum_{j=1}^{k} \alpha_j y_{n+1-j} = h^2 \sum_{j=0}^{\ell} \beta_j f_{n+1+m-j}, \qquad \alpha_k \beta_0 \beta_\ell \neq 0. \tag{5}$$

The first step is evaluating y_1 using the initial conditions and some future values

$$y_1 = y_0 + h y'_0 + h^2 \sum_{j=0}^{\ell} b_j^{(0)} f_j. \tag{6}$$

Next, obtain the additional values $y_2, \ldots, y_{m-1}$, using

$$y_{n+1} = 2y_n - y_{n-1} + h^2 \sum_{j=0}^{\ell} b_j^{(n)} f_j, \quad n = 1, \ldots, m. \tag{7}$$

The coefficients $b_j^{(n)}$ are given in Fukushima. Then the method is given by

$$y_{n+1} = 2y_n - y_{n-1} + h^2 \sum_{j=0}^{\ell} b_j^{(m,\ell)} f_{n+1+m-j}. \tag{8}$$

For example, the 12th-order Cowell type super-implicit is given by

$$\begin{aligned} y_{n+1} = 2y_n - y_{n-1} + h^2 \Bigg(& \frac{31\,494\,553}{39\,916\,800} f_n + \frac{9\,186\,203}{79\,833\,600}\left(f_{n-1} + f_{n+1}\right) \\ & - \frac{222\,331}{19\,958\,400}\left(f_{n-2} + f_{n+2}\right) + \frac{40\,489}{22\,809\,600}\left(f_{n-3} + f_{n+3}\right) \\ & - \frac{17\,453}{79\,833\,600}\left(f_{n-4} + f_{n+4}\right) + \frac{317}{22\,809\,600}\left(f_{n-5} + f_{n+5}\right)\Bigg). \end{aligned} \tag{9}$$

Thus we have to solve a system of nonlinear equations. In order to make the system smaller, one can subdivide the total interval of integration into subintervals. In any case we require special formulas to obtain the ending values.

Before we continue, we need several definitions. For the multistep method to solve the second-order IVP

$$\sum_{i=0}^{k} a_i y_{n+i} = h^2 \sum_{i=0}^{k} b_i f_{n+i} \tag{10}$$

we define the characteristic polynomials

$$\rho(z) = \sum_{i=0}^{k} a_i z^i \tag{11}$$

and

$$\sigma(z) = \sum_{i=0}^{k} b_i z^i. \tag{12}$$

The order of the method is defined to be p if for an adequately smooth arbitrary test function $\zeta(x)$,

$$\sum_{i=0}^{k} a_i \zeta(x+ih) - h^2 \sum_{i=0}^{k} b_i \zeta''(x+ih) = C_{p+2} h^{p+2} \zeta^{(p+2)}(x) + O(h^{p+3})$$

where C_{p+2} is the error constant. The method is assumed to satisfy the following:

1. $a_k = 1$, $|a_0| + |b_0| \neq 0$, $\sum_{i=0}^{k} |b_i| \neq 0$,
2. ρ and σ have no common factor (irreducibility)
3. $\rho(1) = \rho'(1) = 0$, $\rho''(1) = 2\sigma(1)$ (consistency)
4. The method is zero-stable.

The method is called symmetric if

$$a_i = a_{k-i}, \qquad b_i = b_{k-i} \quad \text{for } i = 0, 1, \ldots, k.$$

Definition (*Lambert and Watson, [14]*). The method described by the characteristic polynomials ρ, σ is said to have *interval of periodicity* $(0, H_0^2)$ if for all H^2 in the interval the roots of

$$V(z, H^2) = \rho(z) + H^2 \sigma(z) = 0, \;\; H = \omega h$$

satisfy

$$z_1 = e^{i\theta(H)}, \qquad z_2 = e^{-i\theta(H)},$$
$$|z_s| \leq 1, \quad s = 3, 4, \ldots, k,$$

where $\theta(H)$ is a real function.

Definition (*Lambert and Watson, [14]*). The method described by the characteristic polynomials ρ, σ is said to be *P-stable* if its interval of periodicity is $(0, \infty)$.

Lambert and Watson [14] proved that a method described by ρ, σ has a nonvanishing interval of periodicity only if it is symmetric and for P-stability the order cannot exceed 2. Fukushima [15] has proved that the condition is also sufficient. To be precise, we quote the result of Fukushima [15].

Theorem. *Consider an irreducible, convergent, symmetric multistep method. Define a function*

$$g(\theta) = -\frac{\rho(e^{i\theta})}{\sigma(e^{i\theta})}.$$

Then the method has a nonvanishing interval of periodicity if and only if

1. $g(\theta)$ *has no nonzero double roots in the interval* $[0, \pi]$, *or*
2. $g''(\theta)$ *is positive on all the nonzero double roots of* $g(\theta)$ *in the interval* $[0, \pi]$.

However, higher-order P-stable methods were developed by introducing off-step points or higher derivatives of $f(x, y)$ (Obrechkoff).

2. Second-order IVPs

The numerical integration methods for (1) can be divided into two distinct classes: (a) problems for which the solution period is known (even approximately) in advance; (b) problems for which the period is not known (Ananthakrishnaiah, [4]). For the first class, see Gautschi [16] and Neta [17] and references therein. Here we consider the second class only, i.e. we are not assuming any knowledge of the solution period.

In this section we take the P-stable method of order 12 given by Wang et al. [18],

$$y_{n+1} - 2y_n + y_{n-1} = h^2 \left(\frac{229}{7788} \left(y''_{n+1} + y''_{n-1}\right) - \alpha_2 y''_n \right) - h^4 \left(\frac{1}{2360} \left(y^{(4)}_{n+1} + y^{(4)}_{n-1}\right) - \frac{711}{12\,980} y^{(4)}_n \right) + h^6 \left(\frac{127}{39\,251\,520} \left(y^{(6)}_{n+1} + y^{(6)}_{n-1}\right) + \frac{2923}{3\,925\,152} y^{(6)}_n \right) \tag{13}$$

and show how to get a super-implicit P-stable method equivalent to it. This method has a truncation error

$$\frac{45\,469}{1\,697\,361\,329\,664\,000} h^{14} \left(\omega^{12} y''(x) - y^{(14)} \right)$$

where α_2 is given by

$$\alpha_2 = -\frac{3665}{3894} + \frac{45\,469}{1\,697\,361\,329\,664\,000} (\omega h)^{12} + O\left((\omega h)^{14}\right). \tag{14}$$

In order to get a super-implicit method we can use the idea of Neta and Fukushima [19], i.e. replace the sixth- and fourth-order derivatives by second-order derivatives at neighboring points. Suppose we have a method

$$y_{n+1} - 2y_n + y_{n-1} + h^2 \left(\alpha_1 \left(y''_{n+1} + y''_{n-1}\right) - \alpha_2 y''_n\right) + h^4 \left(\beta_1 \left(y^{(4)}_{n+1} + y^{(4)}_{n-1}\right) + \beta_2 y^{(4)}_n\right) + h^6 \left(\gamma_1 \left(y^{(6)}_{n+1} + y^{(6)}_{n-1}\right) + \gamma_2 y^{(6)}_n\right) = O\left(h^{14}\right). \tag{15}$$

Replacing $h^4 \left(\beta_1 \left(y^{(4)}_{n+1} + y^{(4)}_{n-1}\right) + \beta_2 y^{(4)}_n\right)$ by

$$h^2 \left(A_4 y''_n + B_4 \left(y''_{n+1} + y''_{n-1}\right) + D_4 \left(y''_{n+2} + y''_{n-2}\right) + F_4 \left(y''_{n+3} + y''_{n-3}\right) + H_4 \left(y''_{n+4} + y''_{n-4}\right) + K_4 \left(y''_{n+5} + y''_{n-5}\right)\right) \tag{16}$$

and using MAPLE (Redfern, [20]) we found for $\alpha_1, \beta_1, \beta_2, \gamma_1, \gamma_2$ given in Wang et al. [18]

$$\begin{aligned} A_4 &= \frac{7629}{47\,200} & B_4 &= -\frac{864\,739}{9\,345\,600} \\ D_4 &= \frac{56\,009}{4\,088\,700} & F_4 &= -\frac{32\,667}{14\,537\,600} \\ H_4 &= \frac{6101}{21\,806\,400} & K_4 &= -\frac{2339}{130\,838\,400}. \end{aligned} \tag{17}$$

Similarly, replacing $h^6 \left(\gamma_1 \left(y^{(6)}_{n+1} + y^{(6)}_{n-1}\right) + \gamma_2 y^{(6)}_n\right)$ by

$$h^2 \left(A_6 y''_n + B_6 \left(y''_{n+1} + y''_{n-1}\right) + D_6 \left(y''_{n+2} + y''_{n-2}\right) + F_6 \left(y''_{n+3} + y''_{n-3}\right) + H_6 \left(y''_{n+4} + y''_{n-4}\right) + K_6 \left(y''_{n+5} + y''_{n-5}\right)\right) \tag{18}$$

and using MAPLE we found

$$\begin{aligned} A_6 &= -\frac{22\,243\,211}{2\,344\,091\,200} & B_6 &= \frac{32\,341\,679}{4\,710\,182\,400} \\ D_6 &= -\frac{3\,023\,063}{1\,177\,545\,600} & F_6 &= \frac{4\,446\,259}{9\,420\,364\,800} \\ H_6 &= -\frac{288\,089}{4\,710\,182\,400} & K_6 &= \frac{37\,487}{9\,420\,364\,800}. \end{aligned} \tag{19}$$

The method is then

$$\begin{aligned} y_{n+1} &= 2y_n - y_{n-1} + h^2 \left(N_0 y''_n + N_1 \left(y''_{n+1} + y''_{n-1}\right) + N_2 \left(y''_{n+2} + y''_{n-2}\right)\right. \\ &\quad \left. + N_3 \left(y''_{n+3} + y''_{n-3}\right) + N_4 \left(y''_{n+4} + y''_{n-4}\right) + N_5 \left(y''_{n+5} + y''_{n-5}\right)\right) + O\left(h^{14}\right) \end{aligned} \tag{20}$$

where

$$\begin{aligned} N_0 &= -\frac{358\,413\,373}{2\,355\,091\,200} + \alpha_2 & N_1 &= \frac{9\,186\,203}{79\,833\,600} \\ N_2 &= -\frac{222\,331}{19\,958\,400} & N_3 &= \frac{40\,489}{22\,809\,600} \\ N_4 &= -\frac{17\,453}{79\,833\,600} & N_5 &= \frac{317}{22\,809\,600}. \end{aligned} \tag{21}$$

Upon choosing $\alpha_2 = \frac{3665}{3894} + O\left(h^{12}\right)$, as given in Wang et al. [18], we get a 12th-order method with local truncation error $-\frac{6\,803\,477}{2\,615\,348\,736\,000} h^{14} \left(y^{(14)} - \frac{454\,690}{4\,415\,456\,573}\omega^{12} y''(x)\right)$. This method is actually very similar to (9) with the only exception being the coefficient of f_n. The error constant is larger than that of (13). This phenomenon was discovered by Neta and Fukushima [19].

The problem is that we are requiring five points in the future; on the other hand, we do *not* need a special formula for the first derivative but do need special formulas for starting and ending points of the integration interval.

We will now try to get another method by using more points on the left. Clearly, to satisfy the first two conditions of consistency, $\rho(1) = \rho'(1) = 0$, and to satisfy the zero stability, we need

$$\rho(\zeta) = (\zeta - 1)^2(\zeta - a)(\zeta - b) \tag{22}$$

with a and b bounded by 1 but not equal to 1. Adding the condition of symmetry, we find that $a = i$, and $b = -i$, i.e.

$$\rho(\zeta) = \zeta^4 - 2\zeta^3 + 2\zeta^2 - 2\zeta + 1. \tag{23}$$

Consider the method (see, for example, the P-stable four-step method in Wang, [21])

$$\begin{aligned} y_{n+2} - 2y_{n+1} + 2y_n - 2y_{n-1} + y_{n-2} &= -h^2 \left(\alpha_1 \left(y''_{n+2} + y''_{n-2}\right) + \alpha_2 \left(y''_{n+1} + y''_{n-1}\right) + \alpha_3 y''_n\right) \\ &\quad - h^4 \left(\beta_1 \left(y^{(4)}_{n+2} + y^{(4)}_{n-2}\right) + \beta_2 \left(y^{(4)}_{n+1} + y^{(4)}_{n-1}\right) + \beta_3 y^{(4)}_n\right) \\ &\quad - h^6 \left(\gamma_1 \left(y^{(6)}_{n+2} + y^{(6)}_{n-2}\right) + \gamma_2 \left(y^{(6)}_{n+1} + y^{(6)}_{n-1}\right) + \gamma_3 y^{(6)}_n\right). \end{aligned} \tag{24}$$

We have more parameters, since we are allowing the use of five points. In order for the method to be P-stable, we apply it to the test equation

$$y''(x) = -\omega^2 y(x), \tag{25}$$

to get

$$\mathrm{e}^{\mathrm{i}\omega x} \left(A(h\omega) + B(h\omega)\cos(h\omega) + C(h\omega)\cos(2h\omega)\right) = 0 \tag{26}$$

where

$$\begin{aligned} A(h\omega) &= 2 - h^2\omega^2\alpha_3 + h^4\omega^4\beta_3 - h^6\omega^6\gamma_3, \\ B(h\omega) &= -4 - 2h^2\omega^2\alpha_2 + 2h^4\omega^4\beta_2 - 2h^6\omega^6\gamma_2, \\ C(h\omega) &= 2 - 2h^2\omega^2\alpha_1 + 2h^4\omega^4\beta_1 - 2h^6\omega^6\gamma_1. \end{aligned} \tag{27}$$

Following Wang et al. [18], we solve (26) for α_3 to get

$$\begin{aligned} \alpha_3 = {} & \frac{2}{h^2\omega^2} + h^2\omega^2\beta_3 - h^4\omega^4\gamma_3 + \left(\frac{-4}{h^2\omega^2} - 2\alpha_2 + 2h^2\omega^2\beta_2 - 2h^4\omega^4\gamma_2\right)\cos(h\omega) \\ & + \left(\frac{2}{h^2\omega^2} - 2\alpha_1 + 2h^2\omega^2\beta_1 - 2h^4\omega^4\gamma_1\right)\cos(2h\omega). \end{aligned} \tag{28}$$

Now substitute (28), after expanding the cosine into Taylor series, into (24) and expand into Taylor series to get a system of equations for the coefficients of h^{2m}, $m = 2, 3, \ldots, 9$:

$$7 + 24\alpha_1 + 6\alpha_2 + 12\beta_1 + 12\beta_2 + 6\beta_3 = 0 \tag{29}$$

$$31 + 240\alpha_1 + 15\alpha_2 + 720\beta_1 + 180\beta_2 + 360\gamma_1 + 360\gamma_2 + 180\gamma_3 = 0 \tag{30}$$

$$127 + 1792\alpha_1 + 28\alpha_2 + 13\,440\beta_1 + 840\beta_2 + 40\,320\gamma_1 + 10\,080\gamma_2 = 0 \tag{31}$$

$$73 + \frac{11\,520}{7}\alpha_1 + \frac{45}{7}\alpha_2 + 23\,040\beta_1 + 360\beta_2 + 172\,800\gamma_1 + 10\,800\gamma_2 = 0 \tag{32}$$

$$2047 + 67\,584\alpha_1 + 66\alpha_2 + 1\,520\,640\beta_1 + 5940\beta_2 + 21\,288\,960\gamma_1 + 332\,640\gamma_2 = 0 \tag{33}$$

$$8191 + 372\,736\alpha_1 + 91\alpha_2 + 12\,300\,288\beta_1 + 12\,012\beta_2 + 276\,756\,480\gamma_1 + 1\,081\,080\gamma_2 = 0 \tag{34}$$

$$4681 + \frac{1\,966\,080}{7}\alpha_1 + \frac{120}{7}\alpha_2 + 12\,779\,520\beta_1 + 3120\beta_2 + 421\,724\,160\gamma_1 + 411\,840\gamma_2 = 0 \tag{35}$$

$$13\,1071 + 10\,027\,008\alpha_1 + 153\alpha_2 + 601\,620\,480\beta_1 + 36\,720\beta_2 + 27\,373\,731\,840\gamma_1 + 6\,683\,040\gamma_2 = 0 \tag{36}$$

Solving for the coefficients using MAPLE, we have

$$\begin{aligned} \alpha_1 &= -\frac{55\,321\,909\,809\,919}{2\,132\,415\,136\,051\,200} & \alpha_2 &= -\frac{518\,228\,348\,369}{520\,609\,164\,075} \\ \beta_1 &= \frac{43\,680\,311\,221}{142\,161\,009\,070\,080} & \beta_2 &= -\frac{92\,737\,040\,519}{1\,665\,949\,325\,040} \\ \beta_3 &= \frac{9\,222\,970\,982\,471}{213\,241\,513\,605\,120} & \gamma_1 &= -\frac{384\,479\,909\,371}{223\,903\,589\,285\,376\,000} \\ \gamma_2 &= -\frac{1\,724\,668\,910\,507}{1\,749\,246\,791\,292\,000} & \gamma_3 &= \frac{194\,077\,077\,322\,127}{111\,951\,794\,642\,688\,000}. \end{aligned} \tag{37}$$

The local truncation error is

$$-\frac{14\,729\,175\,706\,111}{1\,299\,067\,775\,131\,517\,297\,786\,880\,000}h^{20}\left(y^{(20)}(x) + \omega^{18}y''(x)\right). \tag{38}$$

Thus this Obrechkoff method is P-stable of order 18 if we choose $\alpha_3 = \frac{15\,190\,029\,559\,381}{355\,402\,522\,675\,200} + O\left(h^{18}\right)$. We now obtain a super-implicit method equivalent to it. Following the same steps as earlier, we replace the fourth- and sixth-order derivatives by a combination of second-order derivatives at neighboring points. The work was done with MAPLE, and we get

$$\begin{aligned} y_{n+2} - 2y_{n+1} + 2y_n - 2y_{n-1} + y_{n-2} = {} & -h^2\left(N_0 y''_n + N_1\left(y''_{n+1} + y''_{n-1}\right) + N_2\left(y''_{n+2} + y''_{n-2}\right)\right. \\ & \left. + N_3\left(y''_{n+3} + y''_{n-3}\right) + N_4\left(y''_{n+4} + y''_{n-4}\right)\right) \end{aligned} \tag{39}$$

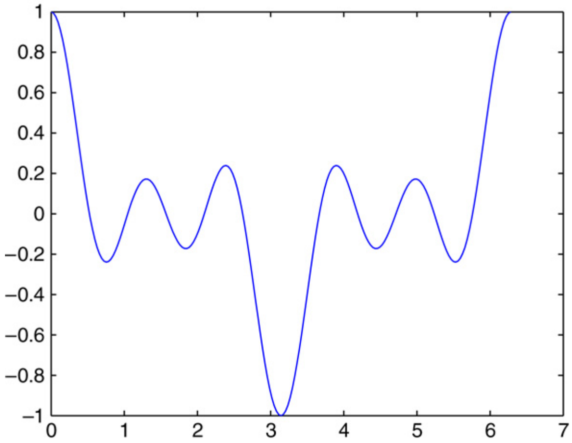

Fig. 1. Exact solution (43) over one period.

where

$$\begin{aligned} &N_0 = \alpha_3 - \frac{2\,369\,868\,964\,432\,087}{9\,595\,868\,112\,230\,400} \qquad N_1 = -\frac{362\,771}{453\,600} \\ &N_2 = -\frac{47\,057}{453\,600} \qquad N_3 = \frac{2707}{453\,600} \\ &N_4 = -\frac{641}{1\,814\,400}. \end{aligned} \tag{40}$$

Unfortunately this is only of order 10 with an error constant of $\frac{-4139}{79\,833\,600}$. In order to increase its order, one must use five points in the future. Thus taking the two additional values on the left did not alleviate this problem.

3. Numerical experiments

In our first experiment, we have used the 12th-order P-stable method due to Wang et al. [18] and our 18th-order P-stable method given by (24) and (37) to solve the following initial value problem:

$$y''(x) + \omega^2 y(x) = 8\left(\cos(x) + \frac{2}{3}\cos(3x)\right) \tag{41}$$

subject to

$$y(0) = 1, \qquad y'(0) = 0 \tag{42}$$

where $\omega = 5$. The exact solution is

$$y(x) = \frac{1}{3}\left(\cos(x) + \cos(3x) + \cos(5x)\right) \tag{43}$$

whose complex oscillatory pattern can be seen in Fig. 1. Both methods showed great results using $h = \pi/8$ and integrating up to $x = 10\pi$. See the plot of the numerical solutions and the exact solution in Fig. 2.

In our second example, we solved the almost periodic problem studied by Stiefel and Bettis [22],

$$\begin{aligned} &z'' + z = 0.001\mathrm{e}^{\mathrm{i}x} \\ &z(0) = 1 \\ &z'(0) = 0.9995\mathrm{i} \end{aligned} \tag{44}$$

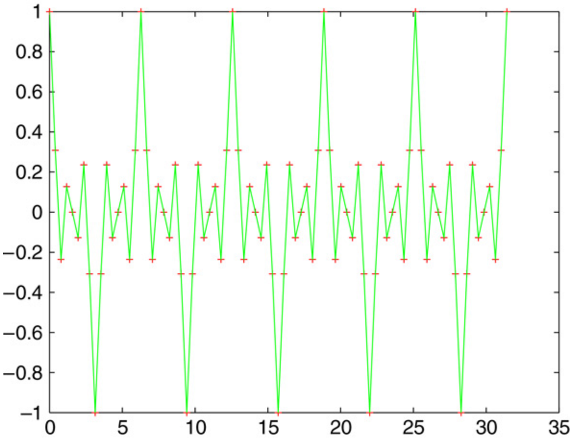

Fig. 2. Exact solution (43) and numerical solution over five periods.

Table 1
Comparing the P-stable 10th-order due to Simos with the 12th-order due to Wang and with our 18th-order for the almost periodic problem

Step size	Simos	Wang	Neta
$\pi/4$	−6.788 (−12)	4.071 (−14)	−3.891 (−18)
$\pi/5$	−4.385 (−13)	2.677 (−15)	−6.339 (−20)
$\pi/6$	−4.707 (−14)	2.931 (−16)	−2.199 (−21)
$\pi/9$	−3.418 (−16)	1.800 (−18)	−1.324 (−24)
$\pi/12$	−9.100 (−18)	6.709 (−20)	−7.138 (−27)

whose theoretical solution is

$$\begin{aligned} z(x) &= u(x) + \mathrm{i}v(x) \\ u(x) &= \cos x + 0.0005x \sin x \\ v(x) &= \sin x - 0.0005x \cos x. \end{aligned} \tag{45}$$

The point $z(x)$ spirals slowly outwards, so that at time x its distance from the origin is

$$d(x) = \sqrt{1 + (0.0005x)^2}. \tag{46}$$

We have solved the problem for $0 \le x \le 40\pi$ using $h = \pi/4, \pi/5, \pi/6, \pi/9, \pi/12$. In Table 1 we present the results showing the error in the distance from the origin using our 18th-order method along with the result of Simos' 10th-order [5] and Wang's 12th-order [21].

In our next example, we solve the nonlinear Duffing equation

$$y''(x) + y(x) + y^3(x) = B\cos(\Omega x) \tag{47}$$

where $\Omega = 1.01$ and $B = .002$. We use the following as the exact solution:

$$\begin{aligned} y(x) = {} & 0.200179477536618 52\cos(\Omega x) + 0.246946143255583824 \times 10^{-3}\cos(3\Omega x) \\ & + 0.304014985249 \times 10^{-6}\cos(5\Omega x) + 0.374349084378 \times 10^{-9}\cos(7\Omega x) \\ & + 0.460964452 \times 10^{-12}\cos(9\Omega x) + 0.5676 \times 10^{-15}\cos(11\Omega x). \end{aligned} \tag{48}$$

We have summarized the results at the end of several periods (up to 50) using a step-size $h = \pi/8$ in Table 2. Notice that our method yields smaller absolute errors.

In our last example, we ran our 18th-order method using $h = \pi/12$ and compared the results to the 12th-order super-implicit method in Neta and Fukushima [19] for solving the nonlinear Duffing equation. We list the absolute errors in Table 3.

Table 2
Comparing the P-stable 12th-order due to Wang with our 18th-order for the nonlinear Duffing equation using $h = \pi/8$

Time	Wang	Neta
2π	1.34 (−13)	2.82 (−15)
4π	2.81 (−13)	2.31 (−15)
6π	4.06 (−13)	1.77 (−15)
8π	5.04 (−13)	1.25 (−15)
10π	5.68 (−13)	8.27 (−16)
20π		9.76 (−16)
40π		7.09 (−17)
60π		3.83 (−16)
80π		1.05 (−15)
100π		1.43 (−15)

Table 3
Comparing our 18th-order with our previous super-implicit 12th-order (Neta and Fukushima, [19]) for the nonlinear Duffing equation using $h = \pi/12$

Time	Neta	Super-implicit
2π	8.33 (−17)	2.53 (−07)
4π	1.94 (−16)	1.01 (−06)
8π	2.08 (−15)	3.95 (−06)
10π	5.16 (−15)	6.05 (−06)

4. Conclusions

In this paper we developed P-stable super-implicit and Obrechkoff methods. The advantage of Obrechkoff methods is that they are high-order one-step methods and thus will not require additional starting values. On the other hand, they will require higher derivatives of the right-hand side. In cases when the right-hand side is very complex, we may prefer super-implicit methods. We developed a super-implicit P-stable method of order 12 and an Obrechkoff method of order 18.

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