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***Abstract*—We construct the wideband ambiguity function for signals represented by their fractional Fourier transforms. Because the reflected signal must be represented as a Doppler scaled version of the transmitted signal, this wideband form of ambiguity does not enjoy many of the same properties as the narrowband form (which is formed from a Doppler shifted version of the signal). We present the general result and also examine an approximation appropriate to wideband signals reflected from slowly moving targets.**

***Index Terms*—Ambiguity, fractional Fourier transform, wideband.**

I. Introduction

THE ambiguity function is a useful tool for describing the ability of a waveform to simultaneously estimate the range and range-rate (speed) of targets in active (correlation-based) radar and sonar systems. The narrowband limit of this function has been well studied in both the time and frequency domains and is fundamental to modern radar and sonar system design.

With the development of the fractional Fourier transform (FrFT), an important relationship between the FrFT representation of the narrowband ambiguity function and its transformation under the rotation operator has been observed [1]. The FrFT can be viewed as an arbitrary rotation of a signal in time-frequency space, and this relationship cleanly connects this transformed signal with the rotation of its narrowband ambiguity function. (A similar relationship exists for the Wigner distribution and has been found to be useful in signal processing applications—see, for example, [2], [5], and [6].) Because the ambiguity function can be interpreted as an imaging kernel, the link between the FrFT representation and ambiguity is also a link between this representation and inverse synthetic aperture imaging.

The simple connection between the FrFT and a rotation of the narrowband ambiguity function is valid only in the narrowband limit. *Wideband* ambiguity functions are used to describe waveforms whose bandwidths are proportionally large in comparison to their center frequencies. These waveforms have always been important in sonar systems and, in recent years, have become applicable to radar system analysis.

The question then is: What is the form of the wideband ambiguity function in the fractional domain, and is there a relationship between this transform and the rotation operator?

II. Radar/Sonar Data

We begin by establishing the relationship between the data collected by a correlation receiver and an echo wave scattered from a target so that the important role of the ambiguity function as an imaging kernel can be illustrated. For simplicity, we will use the scalar wave equation—extension to the vector wave equation is straightforward.

The wave field at time t and spatial position $\boldsymbol{x}$ obeys the inhomogeneous wave equation

$$\left(\nabla^2 - \tilde{c}^{-2}(t,\boldsymbol{x})\partial_t^2\right) u(t,\boldsymbol{x}) = -f(t,\boldsymbol{x}) \tag{1}$$

where $\tilde{c}(t,\boldsymbol{x})$ denotes the field propagation speed, and $f(t,\boldsymbol{x})$ is the source distribution. The Green's function for a source-free region of space satisfies $(\nabla^2 - c^{-2}\partial_t^2)g_0(t,\boldsymbol{x};t',\boldsymbol{x}') = -\delta(t-t')\delta(\boldsymbol{x}-\boldsymbol{x}')$, where c is the free-space propagation speed. Particular solutions of (1) are $u(t,\boldsymbol{x}) = \int g_0(t,\boldsymbol{x};t',\boldsymbol{x}')\, f(t',\boldsymbol{x}')\mathrm{d}t'\mathrm{d}\boldsymbol{x}'$.

We write the total field as the sum of an incident (transmitted field) and a field scattered from the target so that $u(t,\boldsymbol{x}) = u_{\mathrm{inc}}(t,\boldsymbol{x}) + u_{\mathrm{sc}}(t,\boldsymbol{x})$. The incident field is considered to be established by the source distribution of a point radiator (an "antenna") transmitting a signal $s_{\mathrm{inc}}(t)$ from location $\boldsymbol{x}$ and satisfies $(\nabla^2 - c^{-2}\partial_t^2)u_{\mathrm{inc}}(t,\boldsymbol{y}) = -s_{\mathrm{inc}}(t)\delta(\boldsymbol{y}-\boldsymbol{x})$ with solution $u_{\mathrm{inc}}(t,\boldsymbol{y}) = \int g_0(t,\boldsymbol{y};t',\boldsymbol{y}')\, s_{\mathrm{inc}}(t')\delta(\boldsymbol{y}'-\boldsymbol{x})\mathrm{d}t'\mathrm{d}\boldsymbol{y}'$.

The scattered field is to be measured by sensors located in a target-free space. We assume these sensors to be co-located with the transmitter (i.e., a "monostatic" configuration). Substitution of $u(t,\boldsymbol{x}) = u_{\mathrm{inc}}(t,\boldsymbol{x}) + u_{\mathrm{sc}}(t,\boldsymbol{x})$ into (1) yields $(\nabla^2 - c^{-2}\partial_t^2)u_{\mathrm{sc}}(t,\boldsymbol{x}) = -V(t,\boldsymbol{x})\partial_t^2 u(t,\boldsymbol{x})$, where $V(t,\boldsymbol{x}) = c^{-2} - \tilde{c}^{-2}(t,\boldsymbol{x})$ is related to the target scattering density. The term $V(t,\boldsymbol{x})\partial_t^2 u(t,\boldsymbol{x})$ acts as a source distribution on the target, and so this equation has a solution given by the integral equation $u_{\mathrm{sc}}(t,\boldsymbol{x}) = \int g_0(t,\boldsymbol{x};t',\boldsymbol{y})\, V(t',\boldsymbol{y})\partial_{t'}^2 u(t',\boldsymbol{y})\mathrm{d}t'\mathrm{d}\boldsymbol{y}$. In the weak-scatterer approximation, we simplify this last result by substituting $u(t',\boldsymbol{y}) \approx u_{\mathrm{inc}}(t',\boldsymbol{y})$ to obtain

$$u_{\mathrm{sc}}(t,\boldsymbol{x}) = \int g_0(t,\boldsymbol{x};t',\boldsymbol{y})V(t',\boldsymbol{y})\partial_{t'}^2 u_{\mathrm{inc}}(t',\boldsymbol{y})\mathrm{d}t'\mathrm{d}\boldsymbol{y}.$$

The free-space Green's function is given by

$$g_0(t,\boldsymbol{x};t',\boldsymbol{x}') = \frac{\delta\left(t-t'-|\boldsymbol{x}-\boldsymbol{x}'|/c\right)}{4\pi|\boldsymbol{x}-\boldsymbol{x}'|} \tag{2}$$

which represents the field at position $\boldsymbol{x}$ and time t due to an incremental scattering event occurring at position $\boldsymbol{x}'$ and time t' when the wave propagation speed is c.

A. Fractional Fourier Transform

For our analysis, we will rewrite (2) in terms of the *fractional Fourier kernel*: The FrFT is defined by [4]

$$F_\alpha(\xi) \equiv \{\mathcal{F}^\alpha f\}(\xi) = \int K_\alpha(\xi,t)f(t)\mathrm{d}t.$$

The kernel K_α has various forms—for simplicity, we use

$$K_\alpha(\xi,t) = \sqrt{\frac{1+\mathrm{i}\cot\alpha}{2\pi}}\exp\left[-\frac{\mathrm{i}}{2}(\xi^2+t^2)\cot\alpha + \mathrm{i}\xi t\csc\alpha\right].$$

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It is straightforward to show that K_α obeys $K_{-\alpha}(\xi,t)=\overline{K}_\alpha(\xi,t)$; $\int K_\alpha(\xi,t')K_\psi(t',t)\mathrm{d}t' = K_{\alpha+\psi}(\xi,t)$; and $\int K_\alpha(t',\xi)\overline{K}_\alpha(t',t)\mathrm{d}t' = \delta(\xi-t)$. In addition, when α is a multiple of $\pi/2$, the FrFT reduces to the ordinary Fourier transform (or the inverse).

Evidently, (2) can therefore be represented as

$$g_0(t,\boldsymbol{x};t',\boldsymbol{x}') = \frac{\delta\left(t-t'-|\boldsymbol{x}-\boldsymbol{x}'|/c\right)}{4\pi|\boldsymbol{x}-\boldsymbol{x}'|} = \int \frac{K_\alpha(\xi'',t')K_{-\alpha}\left(\xi'',t-|\boldsymbol{x}-\boldsymbol{x}'|/c\right)}{4\pi|\boldsymbol{x}-\boldsymbol{x}'|}\mathrm{d}\xi''$$

and the incident field can be written in terms of K_α as

$$u_{\rm inc}(t'',\boldsymbol{y}) = \int S_\alpha^{\rm inc}(\xi'')\frac{K_{-\alpha}\left(\xi'',t''-|\boldsymbol{x}-\boldsymbol{y}|/c\right)}{4\pi|\boldsymbol{x}-\boldsymbol{y}|}\mathrm{d}\xi'' \quad (3)$$

where $S_\alpha^{\rm inc}(\xi'') = \int K_\alpha(\xi'',t)s_{\rm inc}(t)\mathrm{d}t$ is the FrFT of the signal used to establish the field transmitted to the target.

We assume that the target is in uniform motion: specifically, the target's motion (over the duration of the transmitted pulse) obeys $\boldsymbol{y}(t'') = \boldsymbol{y}_0 + t''\boldsymbol{U}(\boldsymbol{y}_0)$ for some function $\boldsymbol{U}(\boldsymbol{y}_0)$ (this will be exact, for example, when a point target is translating at a fixed velocity). Then we can write $V(t'',\boldsymbol{y}) = \tilde{Q}(\boldsymbol{y}(t'') - t''\boldsymbol{U}(\boldsymbol{y}_0))$, where $\tilde{Q}$ is the scattering density represented in a coordinate system fixed to the target. We also assume that the target is situated in the far-field of the radar/sonar transmitter: denote $\boldsymbol{x}-\boldsymbol{y}_0=\boldsymbol{R}$, $R=|\boldsymbol{R}|$, and $\hat{\boldsymbol{R}}=\boldsymbol{R}/R$, so that $|\boldsymbol{x}-\boldsymbol{y}| = |\boldsymbol{R}-t''\boldsymbol{U}(\boldsymbol{y}_0)| = |\boldsymbol{R}|-t''\hat{\boldsymbol{R}}\cdot\boldsymbol{U}(\boldsymbol{y}_0)+O(R^{-1})$. If we let $\tilde{\sigma}(\boldsymbol{y}_0) = 1+\hat{\boldsymbol{R}}\cdot\boldsymbol{U}(\boldsymbol{y}_0)/c$, then $t''-|\boldsymbol{x}-\boldsymbol{y}|/c = \tilde{\sigma}t''-R/c+O(R^{-1})$, and we have

$$u_{\rm inc}(t'',\boldsymbol{y}) = \frac{1}{4\pi R}\int S_\alpha^{\rm inc}(\xi'')K_{-\alpha}(\xi'',\tilde{\sigma}t''-\tau)\mathrm{d}\xi''$$

where $\tau\equiv R/c$. (This approximation retains lowest order terms in the amplitude and first order terms in the phase and is appropriate for small targets located at great distances from the transmitter/receiver; see [3].)

A number of other important properties of the FrFT are derived in [4]. Of particular interest to us is the transform of the derivative: if $F_\alpha(\xi)$ is a member of the space $\mathcal{S}$ of functions of rapid descent (i.e., functions that decay faster than polynomials), then

$$\left\{\mathcal{F}^\alpha\frac{\mathrm{d}^n}{\mathrm{d}t^n}f\right\}(\xi) = \left(-\mathrm{i}\xi\sin\alpha+\cos\alpha\frac{\mathrm{d}}{\mathrm{d}\xi}\right)^n F_\alpha(\xi).$$

Under the change of variables $\xi''\mapsto\xi''-\tau\cos\alpha$, we obtain from (3)

$$\begin{aligned}\partial_{t''}^2 u_{\rm inc}(t'',\boldsymbol{y}) &= \frac{1}{4\pi R}\int K_{-\alpha}(\xi'',\tilde{\sigma}t'')\\ &\times\exp\left(-\mathrm{i}\frac{\tau^2}{2}\sin\alpha\cos\alpha+\mathrm{i}\xi''\tau\sin\alpha\right)\\ &\times\left[-\mathrm{i}(\xi''-\tau\cos\alpha)\sin\alpha+\cos\alpha\mathcal{D}\right]^2 S_\alpha^{\rm inc}(\xi''-\tau\cos\alpha)\mathrm{d}\xi''\end{aligned}$$

where $\mathcal{D}f\equiv f'$. Then the weak scatterer, monostatic far-field scattered from a target undergoing uniform motion induces a receiver signal of the form

$$\begin{aligned}s_{\rm sc}(\boldsymbol{x},t) \approx \frac{1}{(4\pi R)^2}&\int Q(\boldsymbol{y}_0)K_{-\alpha}(\xi'',\tilde{\sigma}t'')\\ &\times\exp\left(-\mathrm{i}\frac{\tau^2}{2}\sin\alpha\cos\alpha+\mathrm{i}\xi''\tau\sin\alpha\right)K_{-\alpha}(\xi',\tau)\\ &\times\left[-\mathrm{i}\left(\xi''-\tau\cos\alpha\right)\sin\alpha+\cos\alpha\mathcal{D}\right]^2\\ &\times S_\alpha^{\rm inc}(\xi''-\tau\cos\alpha)K_\alpha(\xi',t-\tilde{\sigma}t'')\mathrm{d}\xi'\mathrm{d}\xi''\mathrm{d}t''\mathrm{d}\boldsymbol{y}_0\end{aligned}$$

where $Q(\boldsymbol{y}_0) = \tilde{Q}(\boldsymbol{y}_0)/\tilde{\sigma}(\boldsymbol{y}_0)$ ($\tilde{Q}$ and $\tilde{\sigma}$ are defined above).

B. Correlation Receiver

In active radar and sonar systems, the fraction of the transmitted energy scattered by the target and measured by the receiver falls off by a factor of R^{-4} and is often dominated by thermal noise in the instrument. For this reason, radar/sonar systems typically rely on correlation receptions methods: Correlate the scattered signal with one of the form [3] and [7]

$$\begin{aligned}s_{\rm inc}\left(\sigma(t'-t)\right) &= \int S_\alpha^{\rm inc}(\xi)K_{-\alpha}\left(\xi,\sigma(t'-t)\right)\mathrm{d}\xi\\ &= \int S_\alpha^{\rm inc}(\xi)K_{\psi-\alpha}\left(\xi,\zeta\right)K_{-\psi}(\zeta,\sigma(t'-t))\mathrm{d}\xi\mathrm{d}\zeta\\ &= \frac{\sigma\cos\psi}{\cos\gamma}\sqrt{\frac{1-\mathrm{i}\cot\psi}{1-\mathrm{i}\cot\gamma}}\\ &\times\int\exp\left[\mathrm{i}\frac{\zeta^2}{2}\tan\gamma-\frac{\mathrm{i}\sin^2\psi}{2\cos\gamma\sin\gamma}(\zeta-t\cos\gamma)^2\right]S_\alpha^{\rm inc}(\xi)\\ &\times K_{\psi-\alpha}\left(\xi,\sigma\left(\zeta\frac{\cos\psi}{\cos\gamma}-t\cos\psi\right)\right)K_{-\gamma}(\zeta,t')\mathrm{d}\xi\mathrm{d}\zeta\end{aligned}$$

where σ is the Doppler scale factor [3], and this result follows from the variable change $\zeta\mapsto\sigma(\zeta(\cos\psi/\cos\gamma)-t\cos\psi)$ with $\gamma=\arctan(\tan\psi/\sigma^2)$.

In this representation for $s_{\rm inc}$, we set $\tan\psi=\sigma^2\tan\alpha$ and obtain

$$\begin{aligned}\eta(t,\sigma) &= \int s_{\rm sc}(\boldsymbol{x},t')\overline{s}_{\rm inc}\left(\sigma(t'-t)\right)\mathrm{d}t'\\ &= \frac{1}{\sigma}\frac{C_\alpha}{(4\pi R)^2}\int Q(\boldsymbol{y}_0)\overline{S}_\alpha^{\rm inc}(\xi)\\ &\times\Big\{\left[-\mathrm{i}(\xi''-\tau\cos\alpha)\sin\alpha+\cos\alpha\mathcal{D}\right]^2\\ &\quad\times S_\alpha^{\rm inc}(\xi''-\tau\cos\alpha)\Big\}K_{-\alpha}(\xi'',t'')K_\alpha(\xi',t'-t'')\\ &\times K_{-\alpha}(\xi',\tau)K_\alpha(\zeta,t')K_{\alpha-\psi}\left(\xi,\sigma\left(\zeta\frac{\cos\psi}{\cos\gamma}-t\cos\psi\right)\right)\\ &\times\exp\Big\{-\frac{\mathrm{i}\tan\alpha}{2}\Big[\zeta^2-2\xi''\tau\cos\alpha+\tau^2\cos^2\alpha\\ &\qquad-\left(\frac{\sin\psi}{\sin\alpha}\right)^2(\zeta-t\cos\alpha)^2\Big]\Big\}\\ &\times\mathrm{d}\xi\mathrm{d}\zeta\mathrm{d}\xi'\mathrm{d}\xi''\mathrm{d}t'\mathrm{d}t''\mathrm{d}\boldsymbol{y}_0\end{aligned} \quad (4)$$

where $C_\alpha = (\sin\psi/\sin\alpha)\sqrt{(1+\mathrm{i}\cot\psi/1+\mathrm{i}\cot\alpha)}$.

Integrating over ξ' and t' and making the variable change $\zeta \mapsto \zeta + \tau\cos\alpha$ yields

$$\begin{aligned}\eta(t,\sigma) = &\frac{1}{\sigma}\frac{C_\alpha}{(4\pi R)^2}\int Q(\boldsymbol{y}_0)\overline{S}_\alpha^{\rm inc}(\xi)\\ &\times\Big\{\big[-{\rm i}(\xi''-\tau\cos\alpha)\sin\alpha+\cos\alpha\mathcal{D}\big]^2\\ &\quad\times S_\alpha^{\rm inc}(\xi''-\tau\cos\alpha)\Big\}K_{-\alpha}(\xi'',t'')K_\alpha(\zeta,t'')\\ &\times K_{\alpha-\psi}\left(\xi,\frac{1}{\sigma}\frac{\sin\psi}{\sin\alpha}(\zeta-(t-\tau)\cos\alpha)\right)\\ &\times\exp\Bigg\{-\frac{{\rm i}\tan\alpha}{2}\Big[\zeta^2-2\xi''\tau\cos\alpha+\tau^2\cos^2\alpha\\ &-\left(\frac{\sin\psi}{\sin\alpha}\right)^2(\zeta-(t-\tau)\cos\alpha)^2\Big]\Bigg\}\times{\rm d}\xi{\rm d}\zeta{\rm d}\xi''{\rm d}t''{\rm d}\boldsymbol{y}_0.\end{aligned}$$

From the requirement $\sigma^2 = \tan\psi/\tan\alpha$, we obtain

$$\begin{aligned}\cot(\alpha-\psi) &= \frac{1+\cot\alpha\cot\psi}{\cot\psi-\cot\alpha} = \frac{1+\sigma^2\tan^2\alpha}{1-\sigma^2}\cot\alpha\\ \frac{\sin\psi}{\sin\alpha} &= \sigma^2\sqrt{\frac{1+\tan^2\alpha}{1+\sigma^4\tan^2\alpha}}\\ \frac{\sin\psi}{\sin\alpha}\csc(\alpha-\psi) &= \frac{\sigma^2}{1-\sigma^2}\sec\alpha\csc\alpha\\ C_\alpha\sqrt{\frac{1+{\rm i}\cot(\alpha-\psi)}{2\pi}} &= \sqrt{\frac{\sigma^2\sec\alpha\csc\alpha}{2\pi(1-\sigma^2)}}{\rm e}^{{\rm i}\pi/4}.\end{aligned}$$

Then, integrating over t'' and ξ'' and substituting these last results, we obtain the data model as

$$\begin{aligned}\eta(t,\sigma) = &\sqrt{\frac{\sigma^2\sec\alpha\csc\alpha}{2\pi(1-\sigma^2)}}\frac{{\rm e}^{{\rm i}\pi/4}}{(4\pi R)^2}\int Q(\boldsymbol{y}_0)\\ &\times\overline{S}_\alpha^{\rm inc}(\sigma\xi)\left\{[-{\rm i}\zeta\sin\alpha+\cos\alpha\mathcal{D}]^2S_\alpha^{\rm inc}(\zeta)\right\}\\ &\times\exp\left[-\frac{{\rm i}\sigma^2\sec\alpha\csc\alpha}{2(1-\sigma^2)}(\zeta-\xi-(t-2\tau)\cos\alpha)^2\right]\\ &\times\exp\left[-\frac{{\rm i}\tan\alpha}{2}(\zeta^2-\sigma^2\xi^2)\right]{\rm d}\xi{\rm d}\zeta{\rm d}\boldsymbol{y}_0\end{aligned}\tag{5}$$

where we have made the dummy variable substitutions $\zeta - \tau\cos\alpha \mapsto \zeta$ and $\xi \mapsto \sigma\xi$.

III. Ambiguity Function

Expanding the derivative factor in (5) yields $[-{\rm i}\zeta\sin\alpha + \cos\alpha\mathcal{D}]^2S_\alpha^{\rm inc}(\zeta) = -(\zeta^2\sin^2\alpha + {\rm i}\sin\alpha\cos\alpha)S_\alpha^{\rm inc}(\zeta) - 2{\rm i}\zeta\sin\alpha\cos\alpha\,\mathcal{D}S_\alpha^{\rm inc}(\zeta) + \cos^2\alpha\,\mathcal{D}^2S_\alpha^{\rm inc}(\zeta)$. Integration by parts yields

$$\int {\rm e}^{{\rm i}\Phi_\alpha}\zeta\mathcal{D}S_\alpha^{\rm inc}(\zeta){\rm d}\zeta = {\rm e}^{{\rm i}\Phi_\alpha}\zeta S_\alpha^{\rm inc}(\zeta)|_{-\infty}^{\infty} - \int(1+{\rm i}\zeta\mathcal{D}\Phi_\alpha){\rm e}^{{\rm i}\Phi_\alpha}S_\alpha^{\rm inc}(\zeta){\rm d}\zeta \tag{6}$$

and, similarly

$$\int {\rm e}^{{\rm i}\Phi_\alpha}\mathcal{D}^2S_\alpha^{\rm inc}(\zeta){\rm d}\zeta = {\rm e}^{{\rm i}\Phi_\alpha}\left[\mathcal{D}S_\alpha^{\rm inc}(\zeta) - S_\alpha^{\rm inc}(\zeta)\mathcal{D}\Phi_\alpha\right]\Big|_{-\infty}^{\infty} + \int\left[-(\mathcal{D}\Phi_\alpha)^2+{\rm i}\mathcal{D}^2\Phi_\alpha\right]{\rm e}^{{\rm i}\Phi_\alpha}S_\alpha^{\rm inc}(\zeta){\rm d}\zeta. \tag{7}$$

Since for any function $S_\alpha^{\rm inc}(\zeta)\in\mathcal{S}$, we have $\zeta^mS_\alpha^{\rm inc}(\zeta)\in\mathcal{S}$, then, using (6) and (7) with

$$\Phi_\alpha(\zeta,\xi,t-2\tau) \equiv -\frac{\tan\alpha}{2}(\zeta^2-\sigma^2\xi^2) - \frac{\sigma^2\sec\alpha\csc\alpha}{2(1-\sigma^2)}(\zeta-\xi-(t-2\tau)\cos\alpha)^2 \tag{8}$$

it is straightforward to show that (5) can be written as

$$\eta(t,\sigma) = \frac{1}{(4\pi R)^2}\int Q(\boldsymbol{y}_0)\mathcal{A}_\alpha(\xi,t-2\tau(\boldsymbol{y}_0),\zeta;\sigma)\,{\rm d}\xi{\rm d}\zeta{\rm d}\boldsymbol{y}_0 \tag{9}$$

where

$$\begin{aligned}\mathcal{A}_\alpha(\xi,t,\zeta;\sigma) \equiv &\sqrt{\frac{\sigma^2\sec\alpha\csc\alpha}{2\pi(1-\sigma^2)}}{\rm e}^{{\rm i}\pi/4}D_\alpha(\zeta)\\ &\times\overline{S}_\alpha^{\rm inc}(\sigma\xi)S_\alpha^{\rm inc}(\zeta)\exp\left\{-\frac{{\rm i}\tan\alpha}{2}(\zeta^2-\sigma^2\xi^2)\right\}\\ &\times\exp\left\{-{\rm i}\frac{\sigma^2\sec\alpha\csc\alpha}{2(1-\sigma^2)}(\zeta-\xi-t\cos\alpha)^2\right\}\end{aligned}$$

and $D_\alpha(\zeta)\equiv-(\sigma^2\csc\alpha/(1-\sigma^2))^2(\zeta-\xi-(t-2\tau)\cos\alpha)^2$.

The quantity

$$A_\alpha^{\rm wb}(t;\sigma) \equiv \int\frac{\mathcal{A}_\alpha(\xi,t,\zeta;\sigma)}{D_\alpha(\zeta)}{\rm d}\zeta{\rm d}\xi$$

is the "fractional form" of the (wideband) ambiguity function, which can be seen to be formed from the product of $\overline{S}_\alpha^{\rm inc}(\sigma\xi){\rm e}^{{\rm i}\tan\alpha\sigma^2\xi^2/2}$ and a Gauss–Weierstrass transform[1] (or "chirp convolution") $\mathcal{G}^b$ of the function $\{\mathcal{C}^{\tan\alpha}S_\alpha^{\rm inc}\}(\zeta) = S_\alpha^{\rm inc}(\zeta)\exp\{-({\rm i}/2)\zeta^2\tan\alpha\}$ with $b = ((1-\sigma^2)/\sigma^2)\sin\alpha\cos\alpha$ (where $\mathcal{C}^d$ denotes "chirp multiplication"). We have

$$A_\alpha^{\rm wb}(t;\sigma) = \int\overline{\{\mathcal{C}^{\tan\alpha}S_\alpha^{\rm inc}\}}(\sigma\xi)\times\left\{\mathcal{G}^b\left\{\mathcal{C}^{\tan\alpha}S_\alpha^{\rm inc}\right\}\right\}(\xi+t\cos\alpha){\rm d}\xi \tag{10}$$

(this definition of wideband ambiguity $A_\alpha^{\rm wb}$ should be compared with that in [7]).

A. Narrowband Ambiguity Function

In radar systems, the "narrowband ambiguity function" is much more common. This limiting form occurs when we can write

$$s_{\rm inc}(t) = a(t){\rm e}^{{\rm i}\omega_0t} \tag{11}$$

where $a(t)$ is the "signal envelope" that is slowly varying in time, and ω_0 is the "signal carrier frequency." The narrowband approximation is appropriate when the bandwidth of $s_{\rm inc}(t)$ [i.e., the support of $S_{\pi/2}^{\rm inc}(\xi)$] is small in comparison with ω_0.

The Doppler scale factor is related to target range rate $v=\dot{R}$ by $\sigma=(c-v)/(c+v)=1-2v/(c+v)$, where c denotes the speed of signal propagation. In the radar case, a target's radial speed will typically obey $v\ll c$, and so $\sigma\approx1-2v/c$. Then, since $a(t)$ is slowly varying in t, we can write $s_{\rm inc}(\sigma t) = a(\sigma t){\rm e}^{{\rm i}\omega_0\sigma t} = a((1-2\beta)t){\rm e}^{{\rm i}(\omega_0+\omega_{\rm D})t}\approx s_{\rm inc}(t){\rm e}^{{\rm i}\omega_{\rm D}t}$, where $\beta\equiv v/c$ and $\omega_{\rm D}\equiv-2\beta\omega_0$. The FrFT of $s_{\rm inc}(\sigma t)$ can therefore be approximated as [1] $\{\mathcal{F}^\alpha s_{\rm inc}(\sigma t)\}(\xi)\approx$

[1]The Gauss–Weierstrass transform $\mathcal{G}^b$ of a function f is defined by

$$\{\mathcal{G}^bf\}(\xi) = \frac{1}{\sqrt{-2\pi{\rm i}b}}\int\exp\left\{-{\rm i}(\zeta-\xi)^2/2b\right\}f(\zeta){\rm d}\zeta$$

$S_\alpha^{\rm inc}(\xi+\omega_{\rm D}\sin\alpha)\exp({\rm i}(\omega_{\rm D}^2/2)\sin\alpha\cos\alpha+{\rm i}\xi\omega_{\rm D}\cos\alpha)$ so that

$$s_{\rm inc}(\sigma(t'-t))\approx\int K_{-\alpha}(\xi,t'-t)S_\alpha^{\rm inc}(\xi+\omega_{\rm D}\sin\alpha)\times\exp\left({\rm i}\frac{\omega_{\rm D}^2}{2}\sin\alpha\cos\alpha+{\rm i}\xi\omega_{\rm D}\cos\alpha\right){\rm d}\xi.$$

Inserting this result into (4) and performing the (now greatly simplified) subsequent integrations yields the usual (narrowband) form $A_\alpha^{\rm nb}$ for the ambiguity function as [1] and [5]

$$A_\alpha^{\rm nb}(t,\omega_{\rm D})={\rm e}^{{\rm i}\omega_{\rm D}t/2}\int\overline{S}_\alpha^{\rm inc}\left(\xi-\frac{1}{2}t\cos\alpha+\frac{1}{2}\omega_{\rm D}\sin\alpha\right)\times S_\alpha^{\rm inc}\left(\xi+\frac{1}{2}t\cos\alpha-\frac{1}{2}\omega_{\rm D}\sin\alpha\right)\times\exp\left\{-{\rm i}\xi(t\sin\alpha+\omega_{\rm D}\cos\alpha)\right\}{\rm d}\xi$$

where we have made the variable change $\xi\mapsto\xi-(1/2)(t\cos\alpha+\omega_{\rm D}\sin\alpha)$. From this result, it is easy to see that $A_0^{\rm nb}(t,\omega_{\rm D})$ and $A_{\pi/2}^{\rm nb}(t,\omega_{\rm D})$, respectively, satisfy the usual time and frequency domain definitions of narrowband ambiguity [3].

If we set $x=t\cos\alpha-\omega_{\rm D}\sin\alpha$ and $y=t\sin\alpha+\omega_{\rm D}\cos\alpha$, then

$$\tilde{A}_\alpha^{\rm nb}(x,y)=\int\overline{S}_\alpha^{\rm inc}\left(\xi-\frac{1}{2}x\right)S_\alpha^{\rm inc}\left(\xi+\frac{1}{2}x\right){\rm e}^{-{\rm i}\xi y}{\rm d}\xi. \tag{12}$$

Let $\mathcal{R}_\theta$ denote the rotation operator acting on functions of two variables by $\mathcal{R}_\theta\{f(x,y)\}=f(x\cos\theta+y\sin\theta,-x\sin\theta+y\cos\theta)$. We can conclude

$$\tilde{A}_\alpha^{\rm nb}(x,y)=\mathcal{R}_{-\alpha}\left\{A_0^{\rm nb}(t,\omega_{\rm D})\right\} \tag{13}$$

which is a well-known result relating the narrowband ambiguity function formed from $S_\alpha(\xi)$ to the rotation of the narrowband ambiguity function formed from $s(t)$ (c.f., [1], [2], [5], and [6]).

B. Wideband Signals on Slow Moving Targets

An increasingly important situation in both radar and sonar is one for which (11)—with $a(t)$ a slowly varying envelope—is *not* a valid approximation. This condition occurs, for example, in so-called ultra-wideband systems, which are of interest because of their increased range resolution in comparison with their narrowband counterparts.

In general, a complete analysis of such systems requires application of (10). When the target is slow moving (in comparison with the free-space propagation speed c), however, the collected data $\eta(t,\sigma)$ will have appreciable values only for $\sigma\approx 1$. With $\sigma=(1-\beta)/(1+\beta)$, we can write $\sigma^2/(1-\sigma^2)=1/4((1/\beta)-2+\beta)$.

Equation (9) is an oscillatory integral, and, when $\beta\ll 1$, this equation can be readily analyzed by the method of stationary phase. Substituting $\sigma\approx 1-2\beta$ into the phase term of (8) and expanding and retaining only those terms of degree 1 or less in β yields $\Phi_\alpha(\zeta,\xi,t)=-((1/\beta)-2+\beta)((\zeta-\xi-t\cos\alpha)^2/(8\cos\alpha\sin\alpha))-(\tan\alpha/2)[\zeta^2-(1-4\beta)\xi^2]$. From this last result, we can perform the standard stationary phase integration with "large parameter" $1/\beta$ and obtain the fractional ambiguity function as

$$A_\alpha^{\rm wb}(t,\beta)\approx\int\overline{S}_\alpha^{\rm inc}((1-2\beta)\xi)\,S_\alpha^{\rm inc}(\xi+t\cos\alpha)\times\exp\left\{-\frac{{\rm i}\tan\alpha}{2}(t^2\cos^2\alpha+2\xi t\cos\alpha+4\beta\xi^2)\right\}{\rm d}\xi. \tag{14}$$

Under the variable change $\xi\mapsto(1+\beta)(\xi-(1/2)t\cos\alpha)$, (14) becomes (to first order in β)

$$A_\alpha^{\rm wb}(t,\beta)\approx\int\overline{S}_\alpha^{\rm inc}\left((1-\beta)\xi-\frac{1}{2}(1-\beta)t\cos\alpha\right)\times S_\alpha^{\rm inc}\left((1+\beta)\xi+\frac{1}{2}(1-\beta)t\cos\alpha\right)\times\exp\{-{\rm i}(1-\beta)\xi t\sin\alpha\}\exp\{-2{\rm i}\beta\xi^2\tan\alpha\}{\rm d}\xi.$$

Now let $\tilde{x}=(1-\beta)t\cos\alpha$ and $\tilde{y}=(1-\beta)t\sin\alpha$. Then

$$A_\alpha^{\rm wb}=\int\overline{S}_\alpha^{\rm inc}\left((1-\beta)\xi-\frac{1}{2}\tilde{x}\right)S_\alpha^{\rm inc}\left((1+\beta)\xi+\frac{1}{2}\tilde{x}\right)\times{\rm e}^{-{\rm i}\xi\tilde{y}}{\rm e}^{-2{\rm i}\xi^2\beta\tan\alpha}{\rm d}\xi \tag{15}$$

which is an extended form of (12) appropriate for wideband signals when the target is slowly moving. Note that (15) includes the chirp factor ${\rm e}^{-2{\rm i}\xi^2\beta\tan\alpha}$ in the integrand.

IV. Discussion

Relation (13) fails in the wideband case. This breakdown occurs because Doppler scaling can no longer be interpreted as Doppler shifting. Instead, we are required to apply the complete representation (10), which, in turn, can be approximated as (15) when the target is moving slowly in comparison with the free-space wave propagation speed c.

Since (9) is a functional relation between an object function $Q(\boldsymbol{y}_0)$ and a set of measurements η, it can be interpreted as an imaging equation with kernel $\mathcal{A}_\alpha$. Recovery of Q from η is an inverse problem and has been extensively addressed in the literature for the narrowband case. For rotating targets, (13) relates the fractional form of the ambiguity function to inverse synthetic aperture imaging. When the interrogating signals are wideband, however, these simple relationships fail, and alternate image recovery algorithms must be developed.

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