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On the parameter B = Re/KC = D²/vT

Sarpkaya, T.

Short communication

On the parameter $\beta = \mathrm{Re}/\mathrm{KC} = D^2/\nu \mathrm{T}$

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Abstract

This brief communication is a summary of the facts regarding the universalization in 1976 of a parameter β to fully nonlinear unsteady separated flows about bluff bodies nearly 125 years after its first appearance in a *linearized* analysis of *unseparated* viscous flow with very slow oscillations about a cylindrical rod and sphere by Sir George Gabriel Stokes [1851. On the effect of the internal friction of fluids on the motion of pendulums. Transactions of the Cambridge Philosophical Society 9(II), 8–106]. The primary purpose of Stokes was to show that "*the index of friction*" (the kinematic viscosity), in the equations of motion may be deduced from experiments for the vindication of the heuristic reasoning that went into the derivation by Navier [1827. Mémoire sur les lois du movement des fluides. Mémoires de l'Académie de Sciences 6, 389–416.], Poisson [1831. Nouvelle théorie de l'action capillaire Bachelier, Paris.], de Saint-Venant [1843. Note á joindre un memoire sur la dynamique des fluides. Comptes Rendus 17, 1240–1244.] and Stokes [1845. On the theories of internal friction of fluids in motion. Transactins of the Cambridge Philosophical Society 8, 287–305.] of what are now called the Navier–Stokes equations.
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1. On the contributions of Stokes and the background of *m* and *β*

As noted by Stokes (1851) in a truly remarkable contribution (at the age of 32), "The great importance of the results obtained by means of pendulum has induced philosophers to devote so much attention to the subject, and to perform the experiments with such a scrupulous regard to accuracy in every particular, that pendulum observations may justly be ranked among those distinguished by modern exactness." Large numbers of pendulum experiments have been performed with disks, wires, and spheres beginning with du Buat (1786) and continuing through 1840s in air, hydrogen, water and vacuum. In fact, according to Stokes (1851), "The attention of the scientific world having been called to the subject by the publication of Bessel's memoir (1826), fresh researches both theoretical and experimental soon appeared." Poisson (1831) calculated the exact inviscid-flow value of the added mass coefficient for a sphere and Green (1833) for an ellipsoid. Stokes (1851) recognized that the fundamental shortcoming of all previous analyses was that "they have failed to make a distinction between the action of different fluids, except what arises from their difference of density." In other words, the effect of viscosity (he called it *index of friction*) or "the existence of a tangential action between the pendulum and the air, and between one layer of air and another" has been ignored.

Stokes (1851) first attempted to solve the problem of a long cylindrical rod, thinking "the labour would not be ill-bestowed on the subject." But, he was "stopped by a difficulty relating to the determination of the arbitrary constants,

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which appeared as the coefficients of certain infinite series by which the integral of a certain differential equation was expressed." "Having failed in the case of a cylinder", he tried a sphere and found that "the solution of the problem could be completely affected."

Subsequently, Stokes (1851) was able to solve the cylinder problem also, but without using the then well-known asymptotic expansion of Bessel functions. It is important to note here that, in both cases, he solved the *linearized* version of the Navier–Stokes equations (negligible convective acceleration, *no flow separation*, no vortex shedding, no transition, and no turbulence). The word 'separation' occurs only once in his long paper in connection with the 'separation of symbols.' It must be noted in all fairness that the fluid mechanics of 1851 had not yet realized separation, stability, transition, boundary layer, turbulence, vortex shedding, Reynolds number (Re $= U_m D/v$), Strouhal number, von Karman's vortex street, fluid–structure interaction, Keulegan–Carpenter (1956) number (hereafter referred to as KC($= U_m T/D$) with the velocity $U = U_m e^{i\omega t}$, period $T = 2\pi/\omega$, and D, the diameter of a circular cylinder). When Stokes passed away in 1903, the boundary layer concept was in the horizon of the coming year.

Stokes' work has been most aptly summarized by Stuart [in Rosenhead (1963, pp. 349–408)]. Here, we will refer to Stuart's assessment rather than to the original work of Stokes, primarily because of the need to use additional concepts brought into existence by the brilliant discoveries of Prandtl in 1904. As noted by Stokes as well as by Stuart, the linearization of the N–S equations (here 2-D) is permissible *if the spatial amplitude of oscillations is small compared with the diameter D*. No boundary-layer assumption is made partly because the concept did not exist in 1851 and partly because, as noted by Stuart, the accurate determination of pressure is not possible if both the linearization and the boundary-layer simplification are applied simultaneously. In short, Stokes formulated the total force using the linearized forms of the equations in two dimensions with the usual boundary conditions (no slip, no penetration on the wall) for a sinusoidally oscillating cylinder ($U = U_0 e^{i\omega t}$) and expressed the real part of the complex force $[-M' U_o i\omega(k-ik')e^{i\omega t}]$, as

$$F = M' U_o \omega(k \sin \omega t - k' \cos \omega t) \tag{1}$$

in which M' is the mass of fluid displaced by a length L of the cylinder of radius a and k and k', in Stokes' notation, are given by

$$k = 1 + \sqrt{2}m^{-1} \text{ and } k' = \sqrt{2}m^{-1} + \frac{1}{2}m^{-2} \text{ with } m = \frac{a}{2}\sqrt{n/\mu'}, \tag{2}$$

where $n = \omega$ and μ' (Stokes' kinematic viscosity) $= v$. It should be noted in passing that k and k' may be derived simply by using the Bessel function K_1 [see, e.g., Stuart (1963)]. However, Stokes derived the properties of K_1, and hence k and k', directly from first principles. This relatively lengthy derivation was partly responsible for the manner the parameter **m** appeared in his two-term series. Had he used the Bessel functions (well known in 1850s), he would have directly obtained the complex force as (Stuart, 1963)

$$F = -M' U_o i\omega \left\{ 1 - \frac{4K_1(\sqrt{i\beta_s})}{K_1(\sqrt{i\beta_s}) + \sqrt{i\beta_s}K_1'(\sqrt{i\beta_s})} \right\}, \tag{3}$$

in which $\beta_s = \omega R^2/v$, after replacing a by R. This is a mathematical consequence of the fact that the velocity on the cylinder, parallel to the direction of oscillation, must be $U_o e^{i\omega t}$ to satisfy the boundary conditions. In other words, β_s cannot be anything but $\omega R^2/v$ if the boundary conditions and the linearized equations of motion are to be satisfied for an imposed cylindrical motion in quiescent water of $U_o e^{i\omega t}$. Obviously, β_s is not a consequence of a dimensional analysis or a phenomenological reasoning based on experimental data and, it does not reveal anything about its limitations.

In all subsequent studies of oscillatory flow in tubes [see, e.g., Grace (1928), Sexl (1930), Uchida (1956), Sarpkaya (1966), and the references cited therein], the equation of motion was first written in cylindrical coordinates and then the oscillatory motion parallel to the axis of the tube was designated by $w = f(x, y)e^{i\omega t}$. The solution was then written in terms of the Bessel function J_o which, when expanded in terms of the ber and bei functions, leads to ber $R(\omega/v)^{1/2}$ and bei $R(\omega/v)^{1/2}$ or, once again, to $\beta_s = \omega R^2/v$, as in other works noted above. Stuart (1963) called β_s the "*frequency parameter.*"

However, to the best of our knowledge, $A/D = F(fA^2/v)$, or (fD^2/v), after multiplying with $(A/D)^2$, was first introduced by Andrade (1931) in connection with his work on acoustic streaming. Stuart (1963, p. 386 & 629) referred to Andrade (1931) but made no mention of Andrade's derivation of the dimensionless term consisting of a frequency, a length-squared, and a kinematic viscosity. It is very interesting to record here the conclusions of Andrade (1931, p. 461): "The conditions for initiation of vortex motion are therefore represented by finding experimentally A_c/D as a function of $f_c A^2/v$ where A_c and f_c are the critical amplitude and frequency for which vortex motion just begins with a given diameter D of the obstacle." Apparently, the frequency parameter appears in diverse phenomena such as the frequency

of vortex shedding in laminar flows, wave frequency in oscillatory flows, and streaming motion about vibrating cylinders in compressible as well as incompressible fluids.

Stokes concluded his analysis by noting "It may be remarked that these approximate expressions, regarded as functions of the radius a, have precisely the same form as the exact expressions obtained for a sphere, the coefficients only being different." Reverting to present-day notation, Stuart (1963) reduced Eqs. (2) to

$$k = 1 + 2\sqrt{2/\beta_s} \text{ and } k' = 2\sqrt{2/\beta_s} + 2/\beta_s. \tag{4}$$

Stokes did not reduce them to the expressions shown directly above and made no discussion of m or β_s ($ = 4m^2$) anywhere in his long paper. He was content with the realization of his and his contemporaries' (Bessel, 1826; Poisson, 1831; Baily, 1832; de Saint Venant, 1843) expectations that "*the index of friction* determined by one observation, giving the effect of the fluid either on the time or on the arc of observation of any one pendulum of one of the above forms, and then the theory ought to predict the effect both on the time and on the arc of vibration of all such pendulums." Stokes noted that "The agreement of theory with the experiments of Baily (1832) on the time of vibration is remarkably close," and added "Even the rate of decrease of the arc of vibration, which it formed no part of Baily's object to observe, except so far as was necessary for making the small correction for reduction to indefinitely small vibrations, agrees with the result calculated from theory as nearly as could reasonably be expected under the circumstances."

Over a century later, Wang (1968) extended Stokes' solution to the order of $[(\pi\beta)^{-3/2}]$ using the method of inner and outer expansions. The force or the drag and inertia coefficients, C_d and C_m, are given in Wang (1968) and Sarpkaya (1986a). Wang's expanded solution and Stokes' two-term series for the drag and inertia coefficients yield virtually identical results in the region of their validity ($KC \ll 1$, $\mathrm{Re}\,KC \ll 1$ and $\beta \gg 1$) provided that the nonlinear terms are negligible and *there is no separation*, e.g., for KC values smaller than a number between 1.5 and 2 for β less than about 1000 (Sarpkaya, 1986b, 2006). In concluding this section, it must be noted that Stokes repeatedly drew attention to the linearized nature of the equations leading to his solution and the limitations imposed on the size, frequency and amplitude of oscillations, and the "*index of friction*."

2. On the contributions of Keulegan and Carpenter and of Sarpkaya

It is clear from the preceding section that the history of a cylinder oscillating sinusoidally in a fluid otherwise at rest with small amplitude to diameter ratios or high frequencies is extensive and rich. However, recent studies of this problem, or its kinematical equivalent (flow oscillating sinusoidally about a cylinder at rest), have focused on relatively large amplitudes of oscillation or small frequencies, or both, leading to an entirely different set of flow phenomena, e.g., separation and synchronization found in problems of wavy motion past fixed or flexible cylinders found in the offshore industry. The Stokes linearized solution for very small values of KC and large values of β is restricted to *unseparated* flows, unlike those encountered in the offshore industry. Consequently, Stokes' m remained irrelevant (at least until 1976) to all nonlinear flow phenomena and had no obvious connection to Reynolds number (known since 1888) and to KC number (known since 1956). Thus, it is appropriate to review historically some studies by KC and by Sarpkaya to show how the connection between Re, KC, β, and m was established through the use of extensive experiments and a new definition of β as Re/KC.

While at the University of Iowa, Sarpkaya was given the preliminary data of KC and asked to try to delineate the effect of the Reynolds number ($U_m D/\nu$) on C_d and C_m (Morison et al., 1950). This effort led to the plots shown in Figs. 1(a) and 1(b) where the drag coefficient is plotted as a function of KC, first without and then with the identification of the corresponding m values. Similar plots were made for the inertia coefficient (not shown here for sake of brevity).

After a careful examination of the data in both plots, the effort to correlate the KC data (1956) using m was abandoned for a number of reasons: (a) the KC data exhibited too much scatter because the rectangular free-surface tank used by KC to generate a standing wave did not yield a purely sinusoidal motion. There was both horizontal and vertical harmonic oscillations in the test-section, the horizontal velocity was not uniform in the *vertical* direction, the distance of the cylinders to the free surface was rather small, and the L/D ratio varied widely; (b) the plotting of the coefficients C_d and C_m (from the KC data) with respect to m (serving as the x-axis) produced definitely *larger* scatter than that with respect to KC. This was disappointing but not surprising *then*, in view of the linearization of the equations of motion and the 'inapplicability' of m to *separated* flows; (c) For KC between 25 and 120, the drag coefficients were nearly confined to a region between 1.2 and 1.5 regardless of the value of m (*each of which had only a few points,* as shown in Fig. 1(b)). For KC between 2 and 25, the data were more compactly situated and appeared to be a function of KC only and provided no discernable variation with m. The inertia coefficient C_m was affected in the opposite direction since the larger values of C_d are associated with the smaller values of C_m.
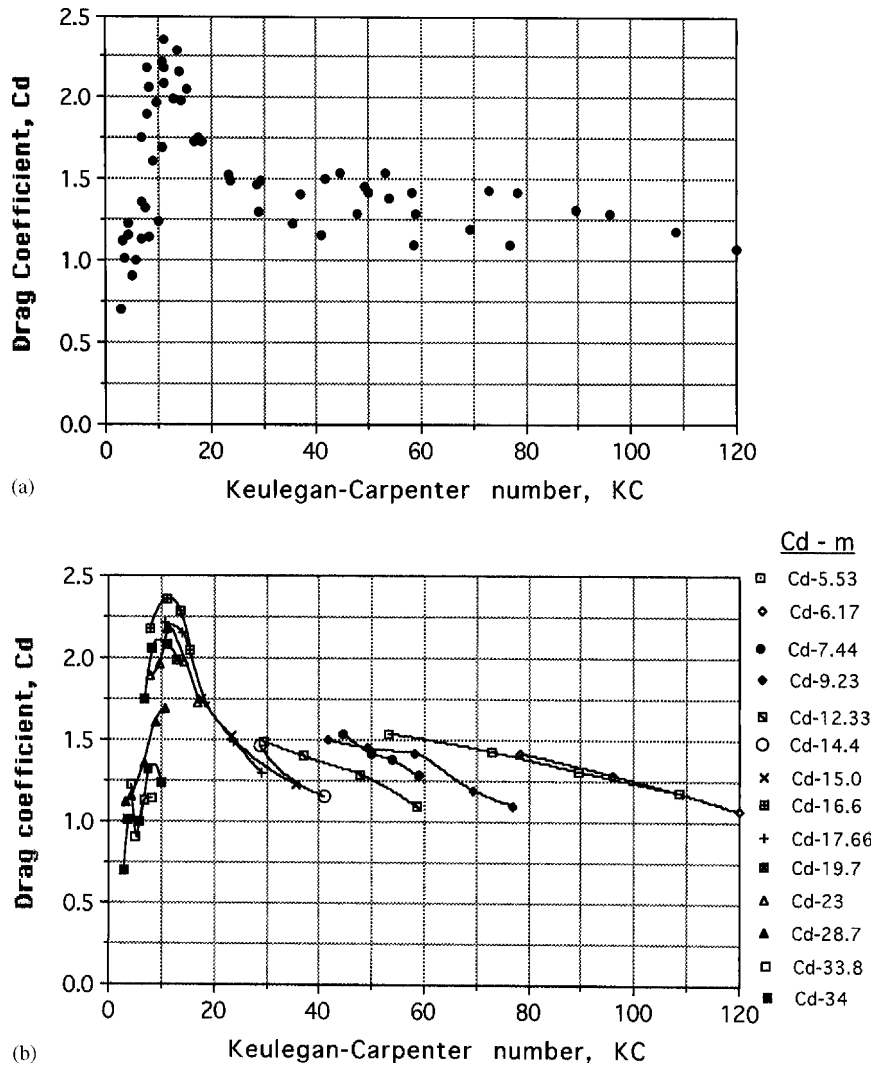
Fig. 1. Drag coefficient C_d versus KC: (a) without the identification of the specific values of *m*; and (b) with specific values of *m*.

Numerous plots to uncover additional correlations with respect to *m* were not inspiring, primarily because the correlation which Sarpkaya (1976a) has discovered about 20 years later was too deeply buried in the scatter. One had to have *better and more extensive data*, in addition to further experience, to universalize *m*. It was only during the period of 1974–1976 and only after analyzing his extensive data from a large U-shaped water tunnel *in terms of constant* Re/KC that Sarpkaya (1976a, b, 1977) was able to trace the existence of a weak dependence of the drag and inertia coefficients in KC data on Re/KC.

The data obtained in the ocean environment, using towers and platforms, were not too helpful in view of the variation of the height, shape, and period of waves, orbital motion, and the omni-directionality of the waves and currents [for an engineering survey, see, e.g., Hogben et al. (1977)]. It is important to note that the KC data (1956), Stokes' *m* (then available to all for 107 years), Stuart's review (1963), and some ocean data (if not kept proprietary) were available to all during the period (1956–1976) and yet *nobody* was able to bring some order into the scatter in the plots of the force coefficients with respect to Re or KC. There were two important reasons for this: (i) as noted above, it was presumed that Stokes' linear analysis and the parameter *m* stemming from it were applicable only to unseparated flows and the effort to introduce Re or *m* into the plots was not pursued; (ii) each data point obtained in the ocean environment represented a new wave height and period (in addition to other changes), for a given fully submerged pipe mounted on a platform.

The writing of Re/KC to eliminate the velocity between the two numbers was not inspired by Stokes' m or by the "frequency parameter" β. The use of a tunnel of constant frequency and the difficulty of changing (rapidly enough) the diameter of the test cylinder, dictated that the velocity U_m *must be eliminated* between Re and KC. This led to Re/KC $= D^2/\nu T$. Sarpkaya (1976a) called it the 'frequency parameter' β, after Stuart (1963).

Clearly, $\beta = fD^2/\nu$ does not impel one to think of writing $\beta = $ Re/KC, *but* Re/KC immediately impels one to write $\beta = fD^2/\nu$. Some, but not all, researchers rushed to call it 'Stokes number' as soon as they became aware of Re/KC, *but not before*, not at least during the time period of 125 years from 1851 to 1976, especially since Re did not exist as a parameter until the1900 s. Obviously, Re/KC is more than a change of notation. It removed all the restrictions associated with Stokes' m (linearization, separation, laminar or turbulent flow, etc.) and rendered $\beta = fD^2/\nu$ a *universal parameter* wherever KC and Re are definable.

Knowing m or β alone could *not* have enabled one to obtain our results, as evidenced by the fact that from the time (1956) the KC data were first made generally available, to 1976 (i.e., for 20 years), *no one* had found a method to bring order into the scatter of the KC data or to the chaotic nature of the existing ocean data.

A careful perusal of some of the classical texts on fluid mechanics by Sir Horace Lamb (1945), Batchelor (1967), Rosenhead (1963), and Schlichting (1979) did not ever refer to a "Stokes number". Schlichting (1979, p. 428) referred to it as "the dimensionless group." In Sarpkaya (1966, 1977) and in Sarpkaya and Isaacson (1981), β has been consistently referred to as the *frequency parameter* in deference to Stuart (1963) and others before him.

It follows from the foregoing that regardless of the inconsequential differences in their appearance, if not in their universality, the symbols m, $\omega D^2/\nu$, $(D/2)(\omega/\nu)^{1/2}$ *and* $\beta = $ Re/KC $= D^2/\nu T$ *or* fD^2/ν should simply be called the "*frequency parameter*" in recognition of the theoretical, experimental, and phenomenological insights and contributions of many.

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