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Vertex and edge critical total restrained domination in graphs

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Abstract

A graph G with no isolated vertices is vertex critical with respect to total restrained domination if, for any vertex v of G that is not adjacent to a vertex of degree one, the total restrained domination number of $G - v$ is less than the total restrained domination number

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of G. We call these graphs γ_{tr}-vertex critical. Similarly, a graph with no isolated vertices is edge critical with respect to total restrained domination if for any non-edge e of G, the total restrained domination number of $G + e$ is less than the total restrained domination number of G. We call these graphs γ_{tr}-edge critical. In this paper, we characterize the γ_{tr}-vertex critical trees, as well as those $\gamma_{tr}(G)$-vertex critical graphs G for which $\gamma_{tr}(G) - \gamma_{tr}(G - v) = n - 2$ for some $v \in V(G)$. Moreover, we also characterize the γ_{tr}-edge critical trees, as well as those $\gamma_{tr}(G)$-edge critical graphs G for which $\gamma_{tr}(G) - \gamma_{tr}(G + e) = n - 2$ for some $e \notin E(G)$.

Keywords: Total restrained domination, vertex critical, edge critical.

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1 Introduction

A vertex in a graph G *dominates* itself and its neighbors. A set of vertices S in a graph G is a *dominating set* if each vertex not in S is dominated by some vertex of S. The *domination number* of G, denoted $\gamma(G)$, is the minimum cardinality of a dominating set of G. A dominating set S is called a *total dominating set* if each vertex is dominated by some vertex of S, and the *total domination number* of G, denoted $\gamma_t(G)$, is the minimum cardinality of a total dominating set of G. A *leaf* in a graph G is a vertex of degree one, and a *remote vertex* is a vertex which is adjacent to a leaf. Let $S(G)$ denote the set of remote vertices of G.

Note that the removal of a vertex in a graph may decrease the domination number. A graph G is called *domination vertex critical* if $\gamma(G - v) < \gamma(G)$ for every vertex v in G. For references on domination vertex critical graphs see [1, 4, 8].

Goddard et al. [5] studied the concept of vertex criticality for total domination. They defined a connected graph G of order at least two to be *total domination vertex critical* or just γ_t-vertex critical if, for every vertex $v \in V(G) - S(G)$, we have $\gamma_t(G - v) < \gamma_t(G)$. Note that if G is γ_t-vertex critical and $v \in V(G) - S(G)$, then $\gamma_t(G - v) = \gamma_t(G) - 1$.

Chen et al. [2] and Zelinka [10] introduced the study of *total restrained domination*, which was further studied by Hattingh et al. [6] and Cyman et al. [3]. A set $S \subseteq V(G)$ is a *total restrained dominating set*, denoted TRDS, if every vertex is adjacent to a vertex in S and every vertex in $V(G) - S$ is also adjacent to a vertex in $V(G) - S$. The *total restrained* *domination number* of G, denoted $\gamma_{tr}(G)$, is the minimum cardinality of a total restrained dominating set of G. A total TRDS of cardinality $\gamma_{tr}(G)$ is called a $\gamma_{tr}(G)$-set.

Let G be a connected graph of order at least three. We say that G is *total restrained domination vertex critical* or just γ_{tr}-vertex critical if, for any vertex v of $V(G) - S(G)$, we have $\gamma_{tr}(G - v) < \gamma_{tr}(G)$. Similarly, we say G is *total restrained domination edge critical* or just γ_{tr}-edge critical if for any $e \notin E(G)$, we have $\gamma_{tr}(G + e) < \gamma_{tr}(G)$.

In Section 2, we characterize the γ_{tr}-vertex critical trees, as well as those $\gamma_{tr}(G)$-vertex critical graphs G for which $\gamma_{tr}(G) - \gamma_{tr}(G-v) = n-2$ for some $v \in V(G)$. In Section 3, we characterize the γ_{tr}-edge critical trees, as well as those $\gamma_{tr}(G)$-edge critical graphs G for which $\gamma_{tr}(G) - \gamma_{tr}(G+e) = n-2$ for some $e \notin E(G)$.

2 γ_{tr}-vertex critical graphs

In contrast to total domination, the removal of a vertex may decrease the total restrained domination number by more than one. In fact, if G is a γ_{tr}-vertex critical graph, then $\gamma_{tr}(G) - \gamma_{tr}(G - v) \leq n - 2$ for all $v \in V(G)$. In this section, we characterize $\gamma_{tr}(G)$-vertex critical graphs G for which $\gamma_{tr}(G) - \gamma_{tr}(G - v) = n - 2$ for some $v \in V(G)$. Goddard et. al. [5] have shown that there are no γ_t-vertex critical trees. We will also determine which trees are γ_{tr}-vertex critical.

Let $\mathcal{A}$ be the family of connected graphs G such that G belongs to $\mathcal{A}$ if and only if every edge is incident with a remote vertex or a leaf or G is a cycle on three vertices.

The following result is due to Cyman and Raczek, [3].

Theorem 1 *Let G be a connected graph of order $n \geq 2$. Then $\gamma_{tr}(G) = n$ if and only if G belongs to $\mathcal{A}$.*

Let P_4 be a path with consecutive vertices v_1, v_2, v_3, v_4. Let $m \geq 0$ be an integer and let $G(m)$ be the graph obtained from P_4 by adding m new vertices $u_1, \dots, u_m$ and joining u_i, $i = 1, \dots, m$, to each of the vertices v_2 and v_3.

Proposition 1 *Suppose G is a connected graph of order $n \geq 3$. Then G is* *a γ_{tr}-vertex critical graph for which $\gamma_{tr}(G) - \gamma_{tr}(G - v) = n - 2$ for some $v \in V(G)$ if and only if $G \in \{C_3, K_{1,2}, G(n-4)\}$.*

Proof. Let $G = G(n-4)$. By Theorem 1, $\gamma_{tr}(G) = n$, and so G is a γ_{tr}-vertex critical graph. Let $v = v_1$. Then $\{v_3, v_4\}$ is a TRDS of $G - v_1$, and so $\gamma_{tr}(G - v_1) = 2$. It follows similarly for $G = C_3$ or $G = K_{1,2}$ that G is a γ_{tr}-vertex critical graph such that $\gamma_{tr}(G) = 3$. Moreover, for C_3, any vertex may be chosen for v, while for $K_{1,3}$ a leaf may be chosen for v.

For the converse, suppose G is a γ_{tr}-vertex critical graph for which $\gamma_{tr}(G) - \gamma_{tr}(G-v) = n-2$ for some $v \in V(G)$. Then $\gamma_{tr}(G) = n$, while $\gamma_{tr}(G-v) = 2$.

Suppose $n = 3$. By Theorem 1, either G is C_3 or each vertex is incident with a remote vertex. In the latter case, $G = K_{1,2}$. We henceforth assume $n \geq 4$. By Theorem 1, we may assume that each edge of G is incident with a remote vertex of G.

If each remote vertex u of G is adjacent to at least two leaves or $\deg(u) = 2$, then, for every $v \in V(G)$, each edge of $G - v$ is still incident with a remote vertex, and so, by Theorem 1, $\gamma_{tr}(G - v) = n - 1 \geq 3$, which is a contradiction.

Thus, there exists a remote vertex u of G such that $\deg(u) \geq 3$ and u is adjacent to exactly one leaf ℓ of G. Let S_v be a γ_{tr}-set of $G - v$.

Case 1. $v \neq \ell$.

As ℓ is also a leaf of $G - v$, we have $S_v = \{u, \ell\}$, and so each vertex of $R = V(G) - \{u, \ell, v\}$ is adjacent to u. Moreover, each vertex of R is adjacent to another vertex of R. Thus, no vertex in R is a remote vertex of $G - v$. However, in G, each edge in $< R >$ must be incident with a remote vertex of G. Thus, some vertex w in R is remote, which implies that v is the leaf adjacent to w in G. Note that v is not adjacent to any of the vertices of $R - \{w\}$, and so each vertex of $R - \{w\}$ is adjacent to only w in $< R >$. Thus, $G = G(n-4)$.

Case 2. $v = \ell$.

If $u \in S_v$, then $S_v \cup \{\ell\}$ is a TRDS of G, and so $\gamma_{tr}(G) \leq 3$, which is a contradiction. We assume $u \notin S_v$. Let $S_v = \{x, y\}$ and suppose, without loss of generality, that u is adjacent to x. Note that each vertex of $R = V(G) - \{x, y, \ell\}$ is adjacent to another vertex of R, and so R does not contain any leaves. Since the edge xy is incident with a remote vertex of G, either x or y is a remote vertex. But y cannot be a remote vertex, and so x is remote, while y is a leaf of G. Since y is also a leaf of $G - v$, each vertex of R is adjacent to x. However, in G, each edge in $< R >$ must be incident with a remote vertex of G. The only remote vertex in R is the vertex u, and so each vertex of $R - \{u\}$ is adjacent to only u in $< R >$. Thus, $G = G(n-4)$, as required. □

We next characterize γ_{tr}-vertex critical trees, and then determine which paths are γ_{tr}-vertex critical.

Let P be a diametrical path of T, and suppose r and r' are the leaves of T which form the two endpoints of P. Root T at r', and consider a nonleaf vertex u on a path from r' to a leaf of T. A path $u = u_0, u_1, \ldots, u_t$ from u to a leaf u_t is called a *maximal reference* path if every path $u = u_0, u_1, u'_2, \ldots, u'_s$ has the property $s \leq t$. Let $\mathcal{R}_{t,u}$ be the set of all maximal reference paths of length t originating from u which do not contain the parent of u. An element of $\mathcal{R}_{t,u}$ will be called a u-**Rt**-*path* (or just an **Rt**-path if the context is clear), and denoted by $u = u_0^i, \ldots, u_t^i$ for some $i \in \{1, \ldots, |\mathcal{R}_{t,u}|\}$.

The set S will denote a $\gamma_{tr}-$set of T, while S' will denote a $\gamma_{tr}-$set of T', where T' will be defined later.

Theorem 2 *Let T be a tree of order $n \geq 2$. T is $\gamma_{tr}-$vertex critical if and only if $\gamma_{tr}(T) = n$.*

Proof. Suppose first that $\gamma_{tr}(T) = n$. Then $\gamma_{tr}(T - v) \leq n - 1$ for every $v \notin S(T)$, and so T is $\gamma_{tr}-$vertex critical. Suppose now that T is $\gamma_{tr}-$vertex critical. We will employ induction on the $n(T)$, the order of T, to show that $\gamma_{tr}(T) = n$. If $1 \leq \text{diam}(T) \leq 3$, then $\gamma_{tr}(T) = n$. Thus, the result is true for all trees of order $n \in \{2, 3, 4\}$. Suppose T is a tree of order $n \geq 5$, and suppose that for any γ_{tr}-vertex -critical tree T' of order $2 \leq n(T') = n' < n$ we have that $\gamma_{tr}(T') = n'$. By the above, we may assume that $\text{diam}(T) \geq 4$.

Claim 1. Let $t \in \{2, 3\}$, and consider the **Rt**-path $u = u_0, u_1, \ldots, u_t$. If $u \in S(T)$, then $\gamma_{tr}(T) = n$.

Proof. Suppose $u \in S(T)$, and let $T' = T - u_t$. Since u_{t-1} is either a leaf or a support vertex of T', we have that $u_{t-1} \in S'$. Thus, $S' \cup \{u_t\}$ is a TRDS of T, and so $\gamma_{tr}(T) \leq \gamma_{tr}(T') + 1$.

We first show that $\gamma_{tr}(T') = \gamma_{tr}(T) - 1$: (*)

Since u is a remote vertex of T, we have that $u \in S$. Also, $\{u_{t-1}, u_t\} \subseteq S$. Moreover, if $t = 3$, every vertex in $N(u_1) - \{u\}$ is either a leaf or a remote vertex, and so $N(u_1) \subseteq S$, which implies that $u_1 \in S$. Thus, $S - \{u_t\}$ is a

TRDS of T', and so $\gamma_{tr}(T') \leq |S| - 1 = \gamma_{tr}(T) - 1$.

We next establish the following fact.

Fact 1. T' is γ_{tr}-vertex critical.

Proof. Suppose, to the contrary, that there exists $v \notin S(T')$ such that $\gamma_{tr}(T') \leq \gamma_{tr}(T' - v)$. Let w be the leaf adjacent to u. We first show that $v \neq w$. For suppose, to the contrary, that $v = w$. Note that $N_{T'}[u_{t-1}] - \{u_{t-2}\} \subseteq S'$, while $\{w, u\} \subseteq S'$. Moreover, if $t = 3$, every vertex in $N(u_1) - \{u\}$ is either a leaf or a remote vertex, and so $N(u_1) \subseteq S'$, which implies that $u_1 \in S$. Thus, $S' - \{w\}$ is a TRDS of $T' - v$, and so $\gamma_{tr}(T' - v) \leq |S'| - 1 = \gamma_{tr}(T') - 1 \leq \gamma_{tr}(T' - v) - 1$, which is a contradiction.

Thus, $v \neq w$ and either $v \notin S(T)$ or the only leaf adjacent to v is u_t.

We eliminate the possibility that the only leaf adjacent to v is u_t. For suppose, to the contrary, that the only leaf adjacent to v is u_t. Note that $N_{T'}[u_{t-1}] \subseteq S'$, while $\{w, u\} \subseteq S'$. Thus, $S' - \{v\}$ is a TRDS of $T' - v$, and so $\gamma_{tr}(T' - v) \leq |S'| - 1 = \gamma_{tr}(T') - 1 \leq \gamma_{tr}(T' - v) - 1$, which is a contradiction.

Thus, $v \neq w$ and $v \notin S(T)$. As $v \notin S(T)$, $\gamma_{tr}(T - v) \leq \gamma_{tr}(T) - 1$. If we can show that $\gamma_{tr}(T' - v) \leq \gamma_{tr}(T - v) - 1$, then, referring to (*), we have $\gamma_{tr}(T) - 1 = \gamma_{tr}(T') \leq \gamma_{tr}(T' - v) \leq \gamma_{tr}(T - v) - 1 \leq \gamma_{tr}(T) - 2$, which will produce a contradiction, and establish our fact.

Let U be a $\gamma_{tr}(T - v)$-set. Note that $v \notin \{u_{t-1}, u_t\}$. Also, $\{u_{t-1}, u_t\} \subseteq U$.

Suppose $\deg(u_{t-1}) \geq 3$. Suppose $v \in N_T(u_{t-1}) - \{u_{t-2}, u_t\}$. Since u is a remote vertex of $T - v$, we have that $u \in U$. Moreover, if $t = 3$, every vertex in $N_{T-v}(u_1) - \{u\}$ is either a leaf or a remote vertex, and so $N_{T-v}(u_1) \subseteq U$, which implies that $u_1 \in U$. Thus, $U - \{u_t\}$ is a TRDS of $T - v - u_t$, and so $\gamma_{tr}(T' - v) \leq |U| - 1 = \gamma_{tr}(T - v) - 1$.

If $t = 3$ and $v = u_1$, then, since $v \notin S(T')$, every vertex in $N_{T-v}(v) - \{u\}$ is a remote vertex, but not a leaf, in T, and so $N_{T-v}(v) \subseteq U$, which implies that $U - \{u_t\}$ is a TRDS of $T - v - u_t$, and so $\gamma_{tr}(T' - v) \leq |U| - 1 = \gamma_{tr}(T - v) - 1$.

Thus, $v \notin N_T(u_{t-1}) \cup \{u_{t-2}\}$, and so $N_{T-v}[u_{t-1}] \subseteq U$. Thus, $U - \{u_t\}$ is a TRDS of $T - v - u_t$, and so $\gamma_{tr}(T' - v) \leq |U| - 1 = \gamma_{tr}(T - v) - 1$.

We henceforth assume that $\deg(u_{t-1}) = 2$. Note that if $t = 3$, then, since $v \notin S(T')$, $v \neq u_1$. Moreover, every vertex in $N_{T-v}(u_1) - \{u\}$ is either a leaf or a remote vertex, and so $N(u_1) \subseteq U$, which implies that $u_1 \in U$. Thus, $U - \{u_t\}$ is a TRDS of $T - v - u_t$, and so $\gamma_{tr}(T' - v) \leq |U| - 1 = \gamma_{tr}(T - v) - 1$.

◇

By the induction assumption and Fact 1, $\gamma_{tr}(T') = n - 1$, and, since $\gamma_{tr}(T') = \gamma_{tr}(T) - 1$, we have $\gamma_{tr}(T) = n$. ◇

Since $\text{diam}(T) \geq 4$, let $r' = v_k, \ldots, v_1, u = u_0, u_1, u_2, u_3 = r$ be a diametrical path. By our Claim, u is not a remote vertex. Consider the tree $T' = T - u$. Hence, by the criticality of T, it follows that $|S'| \leq |S| - 1$. For $i = 1, \ldots, m$, let u, u_1^i, u_2^i, u_3^i be the **R3**-paths originating from u. By our Claim, u_1^i is not a remote vertex for $i = 1, \ldots, m$. Thus, all the vertices of the subtree of T' induced by u_1^i and its descendants must be contained in S'. Hence, $N(u) - \{v_1\} \subseteq S'$. If $v_1 \in S'$, then $S'' = S' - \cup_{i=1}^{m}\{u_1^i\}$ is a TRDS of T, and so $\gamma_{tr}(T) \leq |S''| = |S'| - m \leq |S| - m - 1 \leq \gamma_{tr}(T) - 2$, which is a contradiction. Thus, $v_1 \notin S'$, and S' is a TRDS of T of size at most $\gamma_{tr}(T) - 1$, which is a contradiction. □

As an immediate consequence (cf. Theorem 1), we obtain:

Corollary 1 *Let T be a tree of order $n \geq 2$. Then T is γ_{tr}−vertex critical if and only if T belongs to $\mathcal{A} - \{C_3\}$.*

Corollary 2 *The path P_n of order $n \geq 3$ is γ_{tr}-vertex critical if and only if $n \in \{3, 4, 5\}$.*

Proof. The only paths in which every edge is incident with a remote vertex or a leaf, are P_3, P_4 and P_5. Thus, $(\mathcal{A} - \{C_3\}) \cap \{P_n | n \geq 1\} = \{P_3, P_4, P_5\}$, and so P_3, P_4 and P_5 are the only γ_{tr}-vertex critical paths. □

A *caterpillar* is a tree with the property that the removal of its leaves results in a path $v_1, \ldots, v_s$ as the *spine* of the caterpillar. A caterpillar T is uniquely determined by the sequence of nonnegative integers $(t_1, \ldots, t_s)$, where t_i is the number of leaves adjacent to v_i, for $s \geq 2$, and $t_1 \geq 1$ and $t_s \geq 1$. For example, the sequence $(1, 0, 0, 1)$ determines the caterpillar path P_6.

Let W be a caterpillar with sequence $(a_1, a_2, \ldots, a_n)$ such that whenever $a_i = 0$ for some $2 \leq i \leq n - 1$, then $a_{i-1} \geq 1$ and $a_{i+1} \geq 1$. Then $\text{diam}(W) = n + 1$, and, by Corollary 1, W is a γ_{tr}-vertex critical tree. Hence, $\gamma_{tr}(W) - \text{diam}(W) = (\sum_{i=1}^{n} a_i) + n - (n + 1) = (\sum_{i=1}^{n} a_i) - 1$, and so there exists a γ_{tr}-vertex critical tree W such that the difference $\gamma_{tr}(W) - \text{diam}(W)$ can be made arbitrarily large.

3 γ_{tr}-edge critical graphs

Note if G is a γ_{tr}-edge critical graph, then $\gamma_{tr}(G) - \gamma_{tr}(G+e) \leq n-2$ for all $e \notin E(G)$. In this section, we characterize those $\gamma_{tr}(G)$-edge critical graphs G for which $\gamma_{tr}(G) - \gamma_{tr}(G+e) = n-2$ for some $e \notin E(G)$. We also determine which trees are γ_{tr}-edge critical.

Let the graph $G(m)$ be defined as before.

Proposition 2 *Suppose G is a connected graph of order $n \geq 3$. Then G is a γ_{tr}-edge critical graph for which $\gamma_{tr}(G) = n$ and $\gamma_{tr}(G+e) = 2$ for some $e \in E(\bar{G})$ if and only if $G \in \{K_{1,3}, G(n-4)\}$.*

Proof. Let $G = G(n-4)$. By Theorem 1, $\gamma_{tr}(G) = n$, while $2 \leq \gamma_{tr}(G+e) \leq n-2$ for every $e \notin E(G)$. Thus, G is a γ_{tr}-edge critical. Moreover, $\gamma_{tr}(G + v_1v_4) = 2$. If $G = K_{1,3}$, then $\gamma_{tr}(G) = n$, while $\gamma_{tr}(G+e) = 2$ for every $e \notin E(G)$, as required.

For the converse, suppose G is a γ_{tr}-edge critical graph for which $\gamma_{tr}(G) = n$ and $\gamma_{tr}(G+e) = 2$ for some $e = xy \in E(\bar{G})$. Let $H = G + xy$, and let $\{u, v\}$ be a $\gamma_{tr}(G+e)$-set. Then every vertex of H is adjacent to either u or v, while every vertex of $R = V(G) - \{u, v\}$ is adjacent to another vertex of R. We have the following fact that will be used repeatedly in the proof.

Fact 2. $\deg_G(a) \geq 2$, $\forall a \in R$

We proceed with the following cases.

Case 1. $\{u, v\} = \{x, y\}$.

Without loss of generality, assume $u = x$ and $v = y$. It follows from Fact 2 and Theorem 1, that every vertex in R is either a remote vertex or adjacent to a remote vertex. Moreover, also by Fact 2 no vertex of R is a leaf of G. Let $w \in R$ be a remote vertex of G. Then w is adjacent to a leaf, which must be either x or y. Without loss of generality assume it is x. Let $w' \in R$ be a vertex which is adjacent to w. Then w' must be adjacent to y, as x is a leaf. Since at least one of the endpoints of yw' is a remote vertex of G, and since $\deg(r) \geq 2$ for every $r \in R$, vertex y is not a remote vertex, whence w' must be remote. But then y is also a leaf of G. Hence, $G = P_4 = G(0) = G(n-4)$.

Case 2. $x = u$ and $y \in R$.

Again, every vertex in R is adjacent to a vertex of R, whence $\deg(z) \geq 2$

for every $z \in R - \{y\}$. By Theorem 1, at least one of u or v is a remote vertex.

Suppose u is a remote vertex. Then v is a leaf, and every vertex of $R - \{y\}$ is adjacent to u. If y is adjacent to at least two vertices of $R - \{y\}$, then no vertex in R can be remote. Thus, y is a leaf of G. Let $w \in R - \{y\}$ be the vertex adjacent to y. No vertex in $R - \{y, w\}$ is a remote vertex G, and so, by Theorem 1, $R - \{y, w\}$ is an independent set of G. Thus, every vertex in $R - \{y, w\}$ is adjacent to w. Since $\{u, v\}$ is a minimum TRDS, it follows that $uw \in E(G)$, and so $G = G(n-4)$.

We may therefore assume that u is a leaf, and every vertex of $R - \{y\}$ is adjacent to v. If y is adjacent to v, then $\deg(z) \geq 2$ for every $z \in R$. Since y must be also adjacent to a vertex not in a TRDS, no vertex of R can be remote, which is a contradiction. Thus, y is not adjacent to v. If y is adjacent to at least two vertices of $R - \{y\}$, then no vertex in R can be remote. Thus, y is a leaf of G. Let $w \in R - \{y\}$ be the vertex adjacent to y. No vertex in $R - \{y, w\}$ is a remote vertex G, and so, by Theorem 1, $R - \{y, w\}$ is an independent set of G. Thus, every vertex in $R - \{y, w\}$ is adjacent to w, and so $G = G(n-4)$.

Case 3. $\{x, y\} \subseteq R$.

Suppose x is adjacent to both u and v. Then, by Theorem 1, either x or v is a remote vertex. If x is a remote vertex, then x is adjacent to a leaf in $R - \{y\}$, which is impossible, since $\deg(z) \geq 2$ for every $z \in V(G) - \{y\}$. Thus, x is not a remote vertex, whence v is a remote vertex of G. Since $\deg(z) \geq 2$ for every $z \in V(G) - \{y\}$, it follows that y must be a leaf of G. Now, considering the edge ux, vertex u must be adjacent to a leaf in $R - \{y\}$ since ux must be incident to a remote vertex. This produces a contradiction.

Thus, x (y, respectively) is adjacent to exactly one of the vertices in the set $\{u, v\}$.

Suppose u is adjacent to both x and y.

Suppose v is adjacent to a vertex in $w \in R$. Then, by the above, $w \in R - \{x, y\}$. As before, either v or w is a remote vertex of G. But v cannot be remote, since then a leaf exists in $R - \{x, y\}$, which is a contradiction. Thus, w must be adjacent to a leaf in R, which is a contradiction. Hence, v is a leaf of G. Since no vertex in R is a remote vertex of G, Theorem 1 implies $R = \{x, y\}$. Thus, $G = K_{1,3}$.

We may therefore, without loss of generality, assume that u is adjacent to only x in $\{x, y\}$, while v is adjacent to only y in $\{x, y\}$. Moreover, since no vertex in R is a remote vertex of G, we must have that $R = \{x, y\}$. Thus, $G = P_4 = G(0) = G(n-4)$. □

Proposition 3 *Suppose G is a γ_{tr}-edge critical graph. If R is the set of remote vertices, then $\langle R \rangle$ is complete.*

Proof. Let $\{u, v\} \subseteq R$ such that $uv \in \bar{G}$. Let S be a $\gamma_{tr}(G+uv)$-set. Then $\{u, v\} \subseteq S$, and so S is also a TRDS of G, whence $\gamma_{tr}(G) \leq \gamma_{tr}(G+uv)$, which is a contradiction. □

Proposition 4 *Suppose G is a γ_{tr}-edge critical graph. Let $\{r_1, \ldots, r_\ell\}$ be the remote vertices of G, and let L_i be the leaves adjacent to r_i for $i = 1, \ldots, \ell$. If $\ell \geq 2$, then $|L_i| = 1$ for $i = 1, \ldots, \ell$.*

Proof. Suppose $\ell \geq 2$ and, without loss of generality, that $\{u, v\} \subsetneq L_1$. Moreover, let $w \in L_2$. Let $e = r_2 v$, and let S be a γ_{tr}-set of $G + e$. Then $\{u, r_1, r_2, w\} \subseteq S$, whence $v \in S$, and so S is a TRDS of G, whence $\gamma_{tr}(G) \leq \gamma_{tr}(G+uv) \leq \gamma_{tr}(G) - 1$, which is a contradiction. Thus, $|L_i| = 1$ for $i = 1, \ldots, \ell$, as required. □

Proposition 5 *The only γ_{tr}-edge critical tree T is P_4.*

Proof. Note that $\text{diam}(T) \leq 3$, since otherwise (cf. Proposition 3) the two remote vertices on a diametrical path are adjacent, implying that T has a cycle. If $\text{diam}(T) = 3$, then, by Proposition 4, both support vertices on a diametrical path has degree two, implying that T is isomorphic to P_4. Lastly, P_3 is not γ_{tr}-edge critical. □

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