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Closed 3-stop Center and Periphery in Graphs

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Abstract A delivery person must leave the central location of the business, deliver packages at a number of addresses, and then return. Naturally, he/she wishes to reduce costs by finding the most efficient route. This motivates the following:

Given a set of k distinct vertices $\mathcal{S} = \{x_1, x_2, \ldots, x_k\}$ in a simple graph G, the closed k-stop-distance of set $\mathcal{S}$ is defined to be

$$d_k(\mathcal{S}) = \min_{\theta \in \mathcal{P}(\mathcal{S})} (d(\theta(x_1), \theta(x_2)) + d(\theta(x_2), \theta(x_3)) + \cdots + d(\theta(x_k), \theta(x_1))),$$

where $\mathcal{P}(\mathcal{S})$ is the set of all permutations of $\mathcal{S}$. That is the same as saying that $d_k(\mathcal{S})$ is the length of a shortest closed walk through the vertices $\{x_1, \ldots, x_k\}$.

The closed 2-stop distance is twice the standard distance between two vertices. We study the closed k-stop center and closed k-stop periphery of a graph, for $k = 3$.

Keywords Central appendage number, peripheral appendage number, Steiner distance

MR(2000) Subject Classification 05C12, 05C40, 05C07

1 Definitions and Introduction

In this paper, all graphs are simple (i.e., no loops or multiple edges). For vertices u and v of a graph G, let $d(u, v)$ denote the standard distance from u to v (i.e., the length of a shortest path from u to v). Let $G = (V(G), E(G))$ be a graph of order n ($|V(G)| = n$) and size m ($|E(G)| = m$). Let $x \in V(G)$. Recall that the eccentricity $e(x)$ of a vertex x is $\max_{v \in V(G), v \neq x} d(x, v)$.

Let G and H be two graphs. The join of G and H, namely $G + H$, is the graph with $V(G + H) = V(G) \cup V(H)$ and $E(G + H) = E(G) \cup E(H) \cup \{uv : \forall\, u \in V(G), \forall\, v \in V(H)\}$.

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The disjoint union of G and H, namely $G \cup H$, is the graph whose $V(G \cup H) = V(G) \cup V(H)$ and $E(G \cup H) = E(G) \cup E(H)$.

The *closed k-stop distance* of a set with k vertices $\mathcal{S} = \{x_1, x_2, x_3, \ldots, x_k\}$ $(k \geq 2)$ in a connected graph G is defined to be

$$d_k(\mathcal{S}) = \min_{\theta \in \mathcal{P}(\mathcal{S})} (d(\theta(x_1), \theta(x_2)) + d(\theta(x_2), \theta(x_3)) + \cdots + d(\theta(x_k), \theta(x_1))),$$

where $\mathcal{P}(\mathcal{S})$ is the set of all permutations of $\mathcal{S}$ and $x_i \neq x_j, 1 \leq i, j \leq k$. This concept was introduced in [1] and expanded in [2]. That is the same as saying that $d_k(x_1, x_2, \ldots, x_k)$ is the length of a shortest closed walk through the vertices $x_1, x_2, \ldots, x_k$. The *closed k-stop eccentricity* $e_k(x)$ of a vertex x in a connected graph G is $\max\{d_k(\mathcal{S}) | x \in \mathcal{S}, \mathcal{S} \subseteq V(G), |\mathcal{S}| = k\}$. For a connected graph G, the minimum closed k-stop eccentricity among the vertices of G is the *closed k-stop radius*, that is, $rad_k(G) = \min_{x \in V(G)} e_k(x)$. The maximum closed k-stop eccentricity among the vertices of G is the *closed k-stop diameter*, that is, $\text{diam}_k(G) = \max_{x \in V(G)} e_k(x)$. Equivalently, $\text{diam}_k(G) = \max\{d_k(\mathcal{S}) \,|\, \mathcal{S} \subseteq \mathcal{V}(\mathcal{G}), |\mathcal{S}| = k\}$. For our purposes, the definition based on the k-stop eccentricities is more useful.

Note that if $k = 2$, then $d_2(\{x_1, x_2\}) = 2d(x_1, x_2)$. We thus consider $k \geq 3$. In particular, the closed 3-stop distance of x, y and z $(x \neq y,\ x \neq z,\ y \neq z)$ is

$$d_3(\{x, y, z\}) = d(x, y) + d(y, z) + d(z, x).$$

For simplicity, we will write $d_3(x, y, z)$ instead of $d_3(\{x, y, z\})$.

The *closed* 3-*stop eccentricity* $e_3(x)$ of a vertex x in a connected graph G is the maximum closed 3-stop distance of a set of three vertices containing x, that is,

$$e_3(x) = \max_{y, z \in V(G)} (d(x, y) + d(y, z) + d(z, x)).$$

The minimum closed 3-stop eccentricity among the vertices of G is the *closed* 3-*stop radius*, that is, $\text{rad}_3(G) = \min_{x \in V(G)} e_3(x)$. The maximum closed 3-stop eccentricity among the vertices of G is the *closed* 3-*stop diameter*, that is, $\text{diam}_3(G) = \max_{x \in V(G)} e_3(x)$.

The center $C(G)$ of G is the subgraph induced by those vertices of G having minimum eccentricity. For more on standard center of a graph we refer the reader to [3] and [4]. The *closed* 3-*stop center* $C_3(G)$ of G is the subgraph induced by those vertices of G having minimum closed 3-stop eccentricity [2]. For a given graph G, if there exists a graph H such that $C_3(H) \cong G$, we define the *closed* 3-*stop central appendage number of a graph G*, $AC_3(G)$, to be the minimum difference $|V(H)| - |V(G)|$ over all graphs H such that $C_3(H) \cong G$. For more on standard central appendage number of a graph we refer the reader to [5].

The periphery $P(G)$ of a graph G is the subgraph induced by the vertices having maximum eccentricity. For more on standard periphery of a graph we refer the reader to [3, 6] and [7]. The *closed* 3-*stop periphery* $P_3(G)$ of G is the subgraph induced by the vertices having maximum closed 3-stop eccentricities [2]. For a given graph G, if there exists a graph H such that $P_3(H) \cong G$, we define the *closed* 3-*stop peripheral appendage number of G*, $AP_3(G)$, to be the minimum difference $|V(H)| - |V(G)|$ over all graphs H such that $P_3(H) \cong G$. For more on standard peripheral appendage number of a graph we refer the reader to [8] and [9].

When more than one graph is discussed, such as G and H, we use the notation $d_3^G(x, y, z)$ and $e_3^G(x)$ to represent the closed 3-stop distance of x, y and z and the closed 3-stop eccentricity of x, respectively, in the graph G, and $d_3^H(x, y, z)$ and $e_3^H(x)$ for the corresponding distance and eccentricity in H.

Recall that the Steiner distance of a set $\mathcal{S}$ of vertices is the number of edges in a minimum connected subgraph containing all of the vertices in $\mathcal{S}$. The closed k-stop distance can be viewed as an alternative method of defining distance for a set of vertices. For references on Steiner distance, see [10–14]. The relationship between Steiner distance and closed k-stop distance was explored in [2].

For other graph theory terminology we refer the reader to [15]. In this paper we study the closed 3-stop central appendage number and the closed 3-stop peripheral appendage number.

We end this section with the following propositions that have appeared in [2] and which will be used in this paper.

Proposition A *Let G be a connected graph of order at least 3. Then*

$$|V(P_3(G))| \geq 3.$$

Proposition B *For any connected graph G, we have*

$$\text{rad}_3(G) \leq \text{diam}_3(G) \leq \frac{3}{2}\text{rad}_3(G).$$

Observation C *If u and v are adjacent vertices in a connected graph, then*

$$|e_3(u) - e_3(v)| \leq 2.$$

For the rest of the paper we consider graphs with at least 3 vertices.

2 The Closed 3-stop Peripheral Appendage Number

We start with the closed 3-stop peripheral appendage number. This number is zero if and only if G is its own closed 3-stop periphery, i.e., G is closed 3-stop self-peripheral. This occurs exactly when every vertex of G has the same closed 3-stop eccentricity, so a closed 3-stop self-peripheral graph is also closed 3-stop self-centered, that is, the graph is its own closed 3-stop center.

Recall that a graph G is *vertex-transitive* if for every pair of vertices $u, v \in V(G)$, there is an automorphism of $V(G)$ which maps u to v.

Observation 2.1 *If G is connected and vertex-transitive, then G is closed 3-stop self-centered and closed 3-stop self-peripheral.*

However, if some vertex v in G has $e_3(v) \leq 4$ and G is not a closed 3-stop self-peripheral graph, then G is not the closed 3-stop periphery of any supergraph H. To see this, first notice that G cannot be a complete graph. Also since G is not complete, for all $x \in V(H) - V(G)$, we will have $e_3^H(x) \geq 4$, while there exists $v \in G$ such that $e_3^H(v) \leq 4$ in the supergraph H. Thus, we may assume that $\text{rad}_3(G) \geq 5$.

The next three propositions characterize the graphs G with $\text{rad}_3(G) \geq 5$ for which the closed 3-stop peripheral appendage number is defined. Recall that for a vertex $u \in V(G)$, the open neighborhood of u is $N(u) = \{v : uv \in E(G)\}$ and the closed neighborhood of u is $N[u] = N(u) \cap \{u\}$.

Proposition 2.2 *Assume that* $\mathrm{rad}_3(G) \geq 5$. *Suppose that either* (1) *for every vertex* v *in* G, *the vertices of* $V(G) - N[v]$ *induce a complete graph, or* (2) *for every vertex* v *in* G, $V(G) - N[v]$ *contains two nonadjacent vertices. Then* G *is the closed* 3*-stop periphery of some graph* H, *with* $AP_3(G) \in \{0, 1\}$, *both being realizable.*

Proof First, suppose that for every $v \in V(G)$, the subgraph induced by $V(G) - N[v]$ contains two non-adjacent vertices. Let H be the graph obtained from G, $H = K_1 + G$, by adding one extra vertex x that forms the K_1. Then $e_3^H(x) = 4$, and $e_3^H(y) = 6$, for all $y \in V(G)$.

Suppose now that for every vertex $u \in V(G)$, the vertices in $V(G) - N[u]$ induce a complete graph. Since this is true for every vertex in G, the standard $\mathrm{diam}(G) \leq 3$. Since $\mathrm{rad}_3(G) \geq 5$, it follows that $|V(G) - N[u]| \neq \emptyset$ for all $u \in V(G)$. We can conclude that $\mathrm{diam}(G)$ is either 2 or 3. First, suppose $\mathrm{diam}(G) = 2$. Let u be an arbitrary vertex in $V(G)$. If there is exactly one vertex in $V(G) - N[u]$, it is not adjacent to every vertex of $N(u)$; otherwise, $e_3(u) = 4$. Consider $H = G + K_1$, with a new vertex x corresponding to K_1. Then $e_3^H(x) = 4$. We now show that $e_3^H(u) = 5$ for every $u \in V(G)$. If $v, w \in N(u)$, then $d_3^H(u, v, w) \leq 4$. If $v, w \in V(G) - N[u]$, then $d_3^H(u, v, w) = 5$. If $v \in N(u)$ and $w \in V(G) - N[u]$, then $d_3^H(u, v, w)$ is either 4 or 5, depending on whether v and w are adjacent. (Note that $d(v, w) \leq \mathrm{diam}(G) = 2$.) If either v or w equals x, then $d_3^H(u, v, w)$ is 3 or 4. Thus, $e_3^H(u) = 5$ for every $u \in V(G)$.

Now, consider the case that for every vertex $u \in V(G)$, the vertices in $V(G) - N[u]$ induce a complete graph, and $\mathrm{diam}(G) = 3$. Let u and v be two vertices of G such that $d_G(u, v) = 3$. Since $N[u] \subseteq V(G) - N[v]$, it follows that $N[u]$ must induce a complete graph. Similarly, $N[v]$ must induce a complete graph, $N[u] \cup N[v] = V(G)$, and $N[u] \cap N[v] = \emptyset$. Let $H = G + K_1$, and label the new vertex x. Then $e_3^H(x) = 4$. If w is any vertex other than u, v and x, then w is adjacent to x and exactly one of u and v, so $d_3^H(u, v, w) = 5$. It follows that $d_3^H(u) = d_3^H(v) = d_3^H(w) = 5$.

A class with $AP_3(G) = 0$ is P_n, with $n \geq 3$, and a class $AP_3(G) = 1$ is an extended star, formed by subdividing each edge of $K_{1,n}$, with $n \geq 3$. Notice that P_n satisfies hypothesis (2) for $n \geq 6$ and the extended star satisfies hypothesis (2) for $n \geq 3$. A class which satisfies hypothesis (1) and has $AP_3(G) = 0$ is formed by starting with a complete graph on at least 5 vertices and removing the edges of a hamiltonian cycle. A class which satisfies (1) and has $AP_3(G) = 1$ is formed by starting with two complete graphs K_r and K_s with $r \geq 2$ and $s \geq 3$ and joining a vertex of K_r with at least one and at most $s - 2$ vertices of K_s. □

Notice that in a connected graph G if v is a vertex such that $V(G) - N[v]$ induces a complete graph, then the standard eccentricity of v is at most 3.

Proposition 2.3 *Suppose that* G *is a graph with* $\mathrm{diam}_3(G) > \mathrm{rad}_3(G) \geq 5$. *Furthermore, suppose that* G *contains at least one vertex* v *such that* $V(G) - N[v]$ *induces a complete graph and* $e(v) \leq 2$, *and at least one vertex* u *such that* $V(G) - N[u]$ *contains a pair of nonadjacent vertices. Then* G *is not the closed* 3*-stop periphery of any supergraph* H.

Proof Suppose, to the contrary, that G is the closed 3-stop periphery of some supergraph H.

Claim 1 $e_3^H(v) \leq 5$.

In the graph G, $e_3^G(v) \leq 6$. For $e_3^H(v)$ to be larger than $e_3^G(v)$, there would have to be vertices in $V(H) - V(G)$ with the same closed 3-stop eccentricity as v, which contradicts G being the closed 3-stop periphery of H.

Furthermore, if $e_3^G(v) = 6$, then there must be vertices t and s in $V(G)$ such that $d(v,t) = 2$, $d(v,s) = 1$, and $d(t,s) = 3$. If $e_3^H(v) = 6$, then there must exist vertices t' and s' in $V(G)$ such that $d(v,t') = 2$, $d(v,s') = 1$, and $d(t',s') = 3$ in H as well as in G. Let $x \in V(H) - V(G)$. If x is adjacent to both s' and t', then $d(s',t')$ is reduced to 2. Otherwise, $d_3(x,s',t') = d(x,s') + d(s',t') + d(t',x) \geq 2d(s',t') \geq 6$, which implies that $e_3(x) \geq 6 = e_3(v)$ and contradicts the fact that $v \in P_3(H)$ and $x \notin P_3(H)$. Thus, we may assume that $e_3(v) \leq 5$ in H, and hence, $e_3^H(x) \leq 4$ for all $x \in V(H) - V(G)$.

Claim 2 Every $x \in V(H) - V(G)$ is adjacent to every $y \in V(G)$.

Suppose there is a vertex $x \in V(H) - V(G)$ and a vertex $y \in V(G)$ such that x and y are not adjacent in H. Since $\text{rad}_3(G) \geq 5$, there must be some vertex z in $V(G)$ which is not adjacent to y. Thus, $d_3^H(x,y,z) = d^H(x,y) + d^H(y,z) + d^H(z,x) \geq 5$. This contradicts $e_3^H(x) \leq 4$. Therefore, we may assume that every vertex $x \in V(H) - V(G)$ is adjacent to every vertex $y \in V(G)$.

Finally, consider the vertex u in $V(G)$ such that $V(G) - N[u]$ contains two vertices q and r which are not adjacent in G. In H, $d_3^H(u,q,r) = 6$. Thus, $e_3^H(u) > e_3^H(v)$, so v cannot be in the closed 3-stop periphery of H. This is a contradiction. □

We have the following partial result for the remaining cases.

Proposition 2.4 *Suppose that G is a graph with $\text{diam}_3(G) > \text{rad}_3(G) \geq 5$. Furthermore, suppose that G contains at least one vertex u such that $V(G) - N[u]$ contains a pair of nonadjacent vertices and at least one vertex v such that $V(G) - N[v]$ induces a complete graph, but $e(v) = 3$. Let $A = \{w \in V(G) - N[v] \mid d(v,w) = 2\}$, $B = \{w \in V(G) - N[v] \mid d(v,w) = 3\}$, $C = \{c \in N(v) \mid \text{There is some } w \notin N[v] \text{ such that } wc \in E(G)\}$, and $D = N(v) - C$. If there exists $a \in A$ such that $d(a,d) = 2$ for all $d \in D$, then G is not the closed 3-stop periphery of any supergraph H. Otherwise, $AP_3(G) \in \{0,1,2\}$.*

Proof Suppose G has vertices u and v as described above and suppose G is $P_3(H)$ for some H, with $G \subseteq H$. Each $y \in V(G)$ must have 3-stop eccentricity $\text{diam}_3(H)$ in H.

Claim 1 First we will show that $\text{diam}_3(H) = 7$ and $\text{rad}_3(H) = 6$ and that G contains a subgraph isomorphic to G_0, with $V(G_0) = \{a,b,c,d,e\}$ and $E(G_0) = \{ab, bc, ac, cd, de\}$.

Let q and r be non-adjacent vertices in $V(G) - N[u]$. Then, since none of u, q, or r is adjacent in G or in H, $e_3^H(u) \geq 6$. Now, consider v. Define sets A and B as above. Given any two vertices s and t in $N[v] \cup A$, $d_3(v,s,t) \leq 6$. If $s \in A$ and $t \in B$, then $d_3(v,s,t) = 6$. If $s,t \in B$, then $d_3(v,s,t) = 7$. If $s \in N[v]$ and $t \in B$, then $d_3(v,s,t) \leq 8$. Thus, $\text{diam}_3(H) \in \{6,7,8\}$.

Let s and t be vertices such that $e_3^H(v) = \text{diam}_3(H) = d_3(v,s,t) \geq 6$. Thus, $d^H(v,s) +$

$d^H(s,t)+d^H(t,v) \geq 6$. These distances cannot all be 2, since if s and t are both in $V(G)-N[v]$, then they would be adjacent to each other. At least one of these three distances must be 3. It follows that, given $x \in V(H)-V(G)$, x cannot be adjacent to every vertex in $V(G)$. If, for example, $d^H(s,t) \geq 3$, then x could be adjacent to at most one of s and t, say s, and $d_3(x,s,t) = d(x,s)+d(s,t)+d(t,x) \geq 1+3+2 = 6$. The other cases are similar. We have $e_3(x) \geq 6$. Since x is not in $P_3(H)$, $\mathrm{diam}_3(H) \geq 7$.

Furthermore, since $d_3^H(v,s,t) = \mathrm{diam}_3(v,s,t) \geq 7$, either s and t are both in B or without loss of generality $s \in N[v]$ and $t \in B$. Suppose that s and t are both in B. There is a v-s path of length 3 in G, say v,q,r,s. Since r and $t \in V(G)-N[v]$, r and s are both adjacent to t, and $\{v,q,r,s,t\}$ induces a subgraph isomorphic to G_0. Notice that in this case, $e_3^H(v) = 7 = \mathrm{diam}_3(H)$, so $e_3^H(x) = \mathrm{rad}_3(H)$ must be 6.

Suppose that $s \in N[v]$ and $t \in B$. Then $d^H(v,s) = 1$ and $d^H(v,t) = 3$. We must have $d^H(s,t) = 3$ or 4. If $d^H(s,t) = 4$, then $d_3(x,s,t) = d^H(x,s)+d^H(s,t)+d^H(t,x) \geq 2d^H(s,t) = 8$, which is not possible. Thus, $d^H(s,t) = 3$ and $d^H(v,t) = 3$. Again in this case, $\mathrm{diam}_3(H) = 7$ and $\mathrm{rad}_3(H)$ must be 6. Consider a shortest s-t path, s,q,r,t. Notice that q must be in $N[v]$, since otherwise, q would be adjacent to t, and q cannot be v, since $d^H(v,t) = 3$. Thus, both q and s are adjacent to v, and $\{v,s,q,r,t\}$ induces a subgraph isomorphic to G_0.

Claim 2 Next we show that, for every vertex $x \in V(H)-V(G)$, there is at least one vertex in $V(G)$ not adjacent to x in H. If $V(H)-V(G) = \{x\}$, then there are at least two vertices in $V(G)$ not adjacent to x in H, but any two vertices in $V(G)$ not adjacent to x must be within distance 2 of each other.

Since there must be vertices $s,t \in V(G)$ such that $d^H(s,t) = 3$, it follows that for any vertex $x \in V(H)-V(G)$, there is at least one vertex, say $w \in V(G)$, not adjacent to x. Suppose $V(H)-V(G) = \{x\}$. If x is not adjacent to any vertex in $N(w) \cap V(G)$, then since $e_3^H(x) = 6$, every vertex of H must lie on some x-w geodesic. But then $e_3^H(w) = 6$, which is a contradiction. Thus, x is adjacent to some $z \in V(G)$ such that z is adjacent to w. If x is adjacent to every vertex of $V(G)$ except for w, we would have $e_3^H(z) \leq 6$, which is a contradiction. There must be another vertex $y \in V(G)$ which is not adjacent to x in H. Now, $6 = e_3^H(x) \geq d_3^H(x,y,w) = 2+d(y,w)+2$, so $d(y,w) \leq 2$. Thus, y and w are either adjacent or share a common neighbor in G. If they share a common neighbor, such as z, then there must be a third vertex not adjacent to x which is not adjacent to z.

Define sets A, B, C, and D as in the statement of the proposition. Suppose, for all $a \in A$, there exists $d \in D$ such that $d(a,d) = 3$. Notice that $D \neq \emptyset$ and that for every $c \in C$, there must exist $d \in D$ such that $cd \notin E(G)$. Define H by $V(H) = V(G) \cup \{x,x'\}$ and $E(H) = E(G) \cup \{xy|y \in N[v]\} \cup \{x'y|y \in C \cup A \cup B\} \cup \{xx'\}$. Then $e_3^H(x) = e_3^H(x') = 6$. Notice that $d_3^H(b,c,d) = 7$ for every $b \in B$, $c \in C$ and $d \in D$ with $cd \notin E(G)$, $d_3^H(v,b,d) = 7$ for every $b \in B$ and $d \in D$, and $d_3^H(a,b,d) = 7$ for every $a \in A$, $b \in B$ and $d \in D$ with $d(a,d) > 2$. We have $e_3^H(y) = 7$ for every $y \in V(G)$.

However, if there exists $a \in A$ with $d(a,d) = 2$ for all $d \in D$, then $e_3^H(a) \leq 6$. This contradicts the fact that $e_3^H(x) = \mathrm{diam}_3(H) = 7$ for all $x \in V(G)$ (see Claim 1). □

We have seen examples of classes with $AP_3(G)$ equal to 0 or 1. We now show that $AP_3(G) = 2$ from Proposition 2.4 is realizable. Let G be the graph with $V(G) = \{v, v', w, w', y, y', u\}$ and $E(G) = \{vw', vy', w'y', vu, v'u, v'w, v'y, wy\}$ (see Figure 1). We claim that $AP_3(G) = 2$. First, we calculate $e_3^G(v) = e_3^G(u) = e_3^G(v') = 8$ and $e_3^G(w) = e_3^G(y) = e_3^G(w') = e_3^G(y') = 9$, so $AP_3(G) \geq 1$. Suppose $AP_3(G) = 1$, with $V(H) - V(G) = \{x\}$. From the proof of Proposition 2.4, there must be at least two vertices not adjacent to x, and any two vertices not adjacent to x must be at distance at most two in G. Since by the proof of Proposition 2.4, $e_3^H(v) = 7$, at least one of v, w, and y is not adjacent to x. Similarly, at least one of v', w', and y' is not adjacent to x. If v and v' are both not adjacent to x, then w, y, w' and y' must all be adjacent to x, since each one is distance 3 from one of v and v'. But then $e_3^H(u) \leq 6$, which is not possible. The only other possibility, without loss of generality, is that v', w, and y are not adjacent to x, while v, w' and y' are all adjacent to x. But then $e_3^H(x) \geq d_3^H(x, y, y') \geq 8$, which is also not possible.

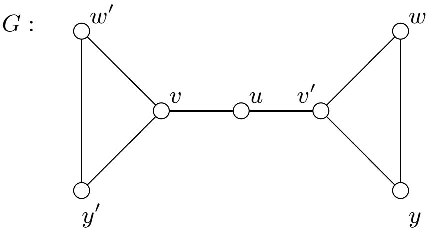

Figure 1 A graph G with $AP_3(G) = 2$

Now, consider the graph H formed by adding two vertices x and x' to G. Add edges $\{xx', xv, xw', xy', xu, x'v', x'w, x'y, x'u\}$. Notice that in H, $e_3(x) = e_3(x') = 6$, while every other vertex has 3-stop eccentricity 7.

We next show that it is possible to have the closed 3-stop periphery and the standard periphery as $P_3(G) \subseteq P(G)$, or $P(G) \subseteq P_3(G)$, or even $P(G) \cap P_3(G) = \emptyset$. For instance, for a path, $P(G) \subseteq P_3(G)$. A C_6 with a pendant edge and vertex added to each of three nonadjacent vertices has $P_3(G) \subseteq P(G)$.

Proposition 2.5 *Let F be a graph with at least two components and let G be a graph with at least three components. Then for every integer $k \geq 3$, there exists a connected graph H such that $P(H) \cong F$, $P_3(H) \cong G$, and $d(P(H), P_3(H)) = k + 3$.*

Proof Let $V(G) = \{x_0, x_1, x_2, x_0', x_1', x_2', u, v, w\} \cup \{u_i, u_i', v_i, v_i', w_i, w_i' \mid 1 \leq i \leq k-1\} \cup \{u_k, v_k, w_k\}$ and let $E(G) = \{x_0x_1,\ x_1x_2,\ x_0'x_1',\ x_1'x_2',\ x_2u_1,\ x_2v_1,\ x_2w_1,\ x_2'u_1',\ x_2'v_1',\ x_2'w_1',\ u_ku,\ v_kv,\ w_kw\} \cup \{u_iu_{i+1},\ v_iv_{i+1},\ w_iw_{i+1},\ u_i'u_{i+1}',\ v_i'v_{i+1}',\ w_i'w_{i+1}' \mid 1 \leq i \leq k-1\}$, where $u_k' = u_k$, $v_k' = v_k$ and $w_k' = w_k$. Then $P(G) = \{x_0, x_0'\}$, with $e(x_0) = \mathrm{diam}(G) = 2k + 4$, while $P_3(G) = \{u, v, w\}$ with $e_3(v) = 6k + 6$. Each of these vertices could be replaced with one or more components of the appropriate graph. Figure 2 shows an example with $k = 3$. In the example, $e(x_0) = e(x_0') = 10$, $e(x_1) = e(x_1') = 9$, $e(x_2) = e(x_2') = e(u) = e(v) = e(w) = 8$,

$e(u_1) = e(v_1) = e(w_1) = e(u_1') = e(v_1') = e(w_1') = e(u_3) = e(v_3) = e(w_3) = 7$, and $e(u_2) = e(v_2) = e(w_2) = e(u_2') = e(v_2') = e(w_2') = 6$. The closed 3-stop eccentricities are $e_3(x_0) = e_3(x_0') = e_3(u_3) = e_3(v_3) = e_3(w_3) = 22$, $e_3(u) = e_3(v) = e_3(w) = 24$, and the closed 3-stop eccentricity of each of the remaining vertices is 20. □

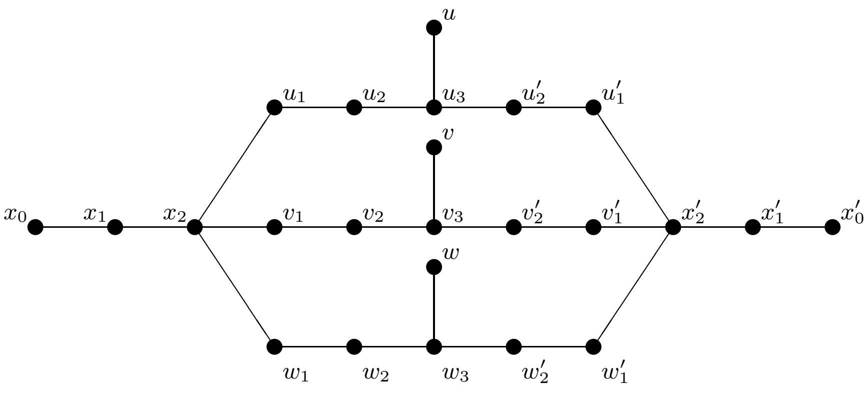

Figure 2 A graph with $P(G) = \{x_0, x_0'\}$ and $P_3(G) = \{u, v, w\}$ at distance 6

3 The Closed 3-stop Central Appendage Number

We now turn our attention to the center of a graph. We first show that every graph can be the closed 3-stop center of some graph, and the closed 3-stop central appendage number is at most 5.

Proposition 3.1 *Let G be any graph. Then there is a supergraph H of G such that $C_3(H) = G$. In general, $|V(H)| - |V(G)| \leq 5$.*

Proof Let G be a graph. We obtain H by adding three new vertices x, y, and z and joining each of them to every vertex in G. Then add vertices u and v and edges uv, ux, and vy. It is straightforward to check that for every $w \in V(G)$, $e_3^H(w) = 6$ in the new graph, while $e_3^H(x) = e_3^H(y) = e_3^H(z) = 7$ and $e_3^H(u) = e_3^H(v) = 7$. □

As a quick corollary of Proposition 3.1 and Proposition A, we have the following:

Corollary 3.2 *The closed 3-stop central appendage number of a graph G is $AC_3(G) \in \{0, 3, 4, 5\}$.*

A class of graphs with $AC_3(G) = 0$ is the class of paths of order at least 3. We say that a graph is *closed 3-stop self-centered* if every vertex has the same closed 3-stop eccentricity. For any closed 3-stop self-centered graph G, we have $AC_3(G) = 0$. We study the closed 3-stop self-centered graphs, and first we make a few observations.

Observation 3.3 *If G has the property that for every vertex $v \in V(G)$, the vertices in $V(G) - N[v]$ induce a graph with at least two non-adjacent vertices, then $G + \overline{K_n}$ is closed 3-stop self-centered for every integer $n \geq 3$.*

It is straightforward to check that for every vertex $v \in V(G)$, $e_3(v) = 6$. Our next observation illustrates that not every graph is closed 3-stop self-centered (as already observed in

Section 2).

Observation 3.4 *If G has a cut-vertex v such that $G-v$ has at least three components, then G is not closed 3-stop self-centered.*

To see this, suppose $e_3(v) = d(v, y, z)$ and let x be a vertex that is not in the same component of $G - v$ as either y or z. Notice that $e_3(x) \geq d(x, y, z) > d(v, y, z) = e_3(v)$.

Recall that an x-y geodesic is a shortest path between vertex x and vertex y, and the interval $I[x, y]$ is the set of all vertices which lie on some x-y geodesic. That is, $I[x, y] = \{v : v$ belongs to some x-y geodesic$\}$.

Proposition 3.5 *If a graph G has an end-vertex x' and G is closed 3-stop self-centered, then there must exist a vertex $y \in V(G)$ such that $d(x', y) = \mathrm{diam}(G)$ and the interval $I[x', y] = V(G)$.*

Proof Suppose that G has an end-vertex x' adjacent to a vertex x, and suppose that G is closed 3-stop self-centered. Let w and z be vertices such that $e_3(x) = d_3(x, w, z)$. If neither w nor z is equal to x', then $d_3(x', w, z) = d_3(x, w, z) + 2$, which is a contradiction. Thus, $e_3(x) = d_3(x, w, x') = 2d(x, w) + 2$ for some vertex w.

Let y be a vertex furthest from x', so necessarily $e(x') = d(x', y)$ and $e(x) = d(x, y)$. If there is a vertex $z \notin I[x', y]$, then $d_3(x, y, z) \geq 2d(x, y)+1 \geq 2d(x, w)+1 = e_3(x)-1$. However, then $d_3(x', y, z) = d_3(x, y, z) + 2 \geq e_3(x) + 1$, which is a contradiction. Thus, there is no vertex $z \notin I[x', y]$. □

By Proposition 3.5, a graph with at least 3 pendant edges cannot be closed 3-stop self-centered. We concentrate next on graphs with one or two pendant edges.

The converse of Proposition 3.5 is not true. The graph in Figure 3 has an end-vertex x' and a vertex y such that $d(x', y) = \mathrm{diam}(G)$ and $I[x', y] = V(G)$, yet $e_3(z) \geq d(z, w, v) = 12$, while $e_3(x') = e_3(x) = 10$.

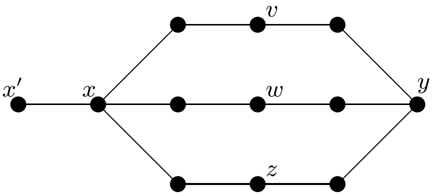

Figure 3 Counterexample to the converse of Proposition 3.5

Corollary 3.6 *If G has two end-vertices x and y and G is closed 3-stop self-centered, then $I[x, y] = V(G)$.*

Corollary 3.7 *If G has two end-vertices x and y and $\mathrm{diam}(G) > d(x, y)$, then G is not closed 3-stop self-centered.*

And so, we now consider graphs that are not closed 3-stop self-centered.

Remark 3.8 For every positive integer $n \geq 3$, $AC_3(K_{1,n}) = 3$.

To see this, we obtain a connected graph H from G by adding 3 vertices x, y, z, so that every pendant of G is either adjacent to x and y, or to z, such that H is a connected graph. And so the degree of each pendant of the star becomes 2 or 3 in H. The closed 3-stop eccentricities in H are 10 for the vertices x, y, and z, and 8 for the vertices in $V(G)$.

Proposition 3.9 *If G is a graph with no isolated vertices, then $AC_3(G) \leq 3$.*

Proof Consider any spanning forest F of G and let A and B be the partite sets of a bipartition of F. Notice that since G (and F) has no isolated vertices, every vertex of A has at least one neighbor in B and every vertex in B has at least one neighbor in A, and both A and B are nonempty.

Now, add three new vertices x, y and z to G. Join x to every vertex in A, join y to every vertex in B, and join z to every vertex in $V(G)$. Notice that $d_3^H(x,y,z) = 7$, $d_3^H(a,x,y) = 6$, and $d_3^H(b,x,y) = 6$ for every $a \in A$ and $b \in B$. We claim that these distances produce the eccentricities. We can check that $d_3^H(x,b,b') \leq 6$, $d_3^H(x,a,a') \leq 4$, $d_3^H(x,a,b) \leq 6$, $d_3^H(x,a,z) \leq 4$, and $d_3^H(x,z,b) \leq 5$ for every $a, a' \in A$ with $a \neq a'$ and every $b, b' \in B$ with $b \neq b'$. Similarly, every closed 3-stop distance involving y is at most 6 except for $d_3^H(x,y,z)$. Every closed 3-stop distance involving z is at most 6 except for $d_3^H(x,y,z)$. Finally, $d_3^H(a,a',b) \leq 6$ and $d_3^H(a,b,b') \leq 6$ for all $a, a' \in A$ and $b, b' \in B$ using vertex z. □

Thus, if G has no isolated vertices, then $AC_3(G)$ is either 0 or 3. The graph $\overline{K}_2$ has closed 3-stop central appendage number 3, since the closed 3-stop center of $K_{2,3}$ consists of the smaller partite set. Similarly, $AC_3(K_1 \cup K_m) = 3$ for any positive integer m, since the graph formed by adding three new vertices x, y and z and joining each of them to every vertex of $K_1 \cup K_m$ has closed 3-stop center $K_1 \cup K_m$. The next result shows that there exist graphs with $AC_3(G) > 3$.

Proposition 3.10 $AC_3(\overline{K}_3) = 5$.

Proof By Proposition 3.1, we have $AC_3(\overline{K}_3) \leq 5$. Since $\overline{K}_3$ is not connected, $C_3(\overline{K}_3)$ is undefined, and $\overline{K}_3$ cannot be closed 3-stop self-centered. Thus, by Corollary 3.2, $AC_3(\overline{K}_3) \geq 3$.

Case I Suppose $AC_3(\overline{K}_3) = 3$.

Let H be a (connected) graph of order 6 with $C_3(H) = \overline{K}_3$. Let u, v and w be the vertices of $\overline{K}_3$ and let x, y, and z be the vertices of $V(H) - V(\overline{K}_3)$. Since u, v, and w form an independent set in H, $d_3^H(u,v,w) \geq 6$. Thus, $d_3^H(x,y,z) = \mathrm{diam}_3(H) \geq 7$. One of $d(x,y)$, $d(y,z)$, and $d(x,z)$ must be at least 3, say without loss of generality $d(y,z) \geq 3$. Furthermore, if any two of x, y, and z are adjacent, say $d(x,y) = 1$, then since H is connected, z must be distance 2 from one of x or y, and $d_3(x,y,z) \leq 6$. Thus, we may assume that no two of x, y, or z are adjacent.

Case IA x is adjacent to u, v, and w.

Notice that y and z cannot be adjacent and cannot have a common neighbor. Since H is a connected graph, each of y and z must be adjacent to at least one of x, u, v, and w, and at most one can be adjacent to each of x, u, v, or w. Thus, we may assume, without loss of generality, that y is adjacent to u and that z is not adjacent to u.

Case IAi $e_3^H(u) = \mathrm{rad}_3(H) \leq 6$.

If z is not adjacent to x, then $d_3^H(u,y,z) = d(u,y) + d(y,z) + d(z,u) \geq 1 + 3 + 3 = 7$, which contradicts $e_3^H(u) = 6$. If z is adjacent to x, then x is adjacent to every vertex except y and $e_3^H(x) \leq 6$, which contradicts $e_3^H(x) = \mathrm{diam}_3(H)$.

Case IAii $e_3^H(u) = \mathrm{rad}_3(H) \geq 7$.

Thus, $e_3^H(x) = d_3^H(x, y, z) = \mathrm{diam}_3(H) \geq 8$. We have $d(x, y) \leq 2$, and since z must be adjacent to x, v, or w, $d(x, z) \leq 2$. Thus, we must have $d(y, z) = 4$ and z adjacent to v, w, or both. Thus, $d_3^H(x, y, z) = 8$, and $e_H(u) \geq d_3^H(u, y, z) \geq d(y, u)+d(y, z)+d(u, z) \geq 2d(y, z) = 8$. We have a contradiction.

Case IB x is adjacent to at most two of u, v and w.

In this case, without loss of generality, we may assume that x is not adjacent to w. Since $d(y, z) \geq 3$, at most one of y and z is adjacent to w, say y. Thus, $\deg_H(w) = 1$. It follows that $d_3^H(x, w, z) = d_3^H(x, y, z) + 2 = \mathrm{diam}_3(H) + 2$, which is not possible.

Case II Suppose $AC_3(\overline{K}_3) = 4$.

Let H be a connected graph of order 7 with $C_3(H) = \overline{K}_3$, let u, v, and w be the vertices of $\overline{K}_3$ and let a, b, c, and d be the remaining vertices of H. Without loss of generality, assume that $d^H(a, b)$ is a maximum among $d^H(a, b)$, $d^H(a, c)$, $d^H(a, d)$, $d^H(b, c)$, $d^H(b, d)$, and $d^H(c, d)$. Set $k = d^H(a, b)$. Notice that u, v, and w form an independent set in H, so $\mathrm{rad}_3(H) = d_3^H(u, v, w) \geq 6$ and $\mathrm{diam}_3(H) \geq 7$. It follows that $k \geq 3$. Notice that $\mathrm{rad}_3(H) = e_3(u) \geq d_3^H(u, a, b) \geq 2d(a, b) \geq 2k$.

Since each shortest a-b path has length at least 3 and since u, v, and w are mutually non-adjacent, each shortest a-b path must contain at least one vertex other than a, b, u, v, and w. If both c and d lie on shortest a-b paths, then $\mathrm{diam}_3(H) = 2k$ and $C_3(H) = H$. This is a contradiction. Thus, without loss of generality, c lies on every shortest a-b path, and d does not lie on any shortest a-b path. It also follows that $k \leq 4$.

Case IIA $\mathrm{rad}_3(H) = 2k$.

In this case, each of u, v, and w must lie on some shortest a-b path, since, for example, $e_3^H(u) \geq d_3^H(u, a, b)$.

We now present a proof by contradiction both in the case of $k = 3$ and $k = 4$. If $k = 3$, then without loss of generality, u, v, and w are each adjacent to both a and c, and c is adjacent to b. If $k = 4$, then without loss of generality, u is adjacent to a and c, while v and w are adjacent to b and c. (Any other possibility involves a different partition of $\{u, v, w\}$.) In either case, if d is adjacent to any of c, u, v, or w, then $e_3^H(c) \leq 2k$, which is not possible since $c \notin C_3(H)$. (In the $k = 3$ case, if d is adjacent to c, then $d(a, c) = 2$ and every other vertex is within distance 1 of c, so $e_3^H(c) \leq 6$. In the $k = 3$ case, if d is adjacent to u, v, or w, then only a and d are distance 2 from c, and $d_3^H(c, a, d) = 6$. In the $k = 4$ case, every vertex is within distance 2 of c, so $e_3^H(c) \leq 8$.) Otherwise, d is adjacent only to a or to b, but not to both. But then $d(d, a)$ or $d(d, b)$ is greater than $d(a, b)$, which contradicts our choice of a and b.

Case IIB $\mathrm{rad}_3(H) \geq 2k + 1$.

Thus, $\mathrm{diam}_3(H) \geq 2k + 2$. If $k = 3$, then without loss of generality, a, u, c, b is a shortest

a-b path. Since $e_3^H(c) \geq 8$ and $d^H(c,a) = 2$, $d^H(c,b) = 1$, and each pair of a, b, c, and d is at distance at most 3, we must have $d^H(c,d) = 3$ and $d^H(d,a) = 3$. Now, d cannot be adjacent to a, b, or u, so d is adjacent only to w or v. Say d is adjacent to w; then w cannot be adjacent to a or c, so w must be adjacent to b. Notice that w cannot be adjacent to anything other than b and d, and d cannot be adjacent to any vertex other than w and possibly v. If d is adjacent to v, then v is also adjacent to only b and d. Now, since $d^H(a,d) = 3$, and b is the only vertex distance 2 from d, we must have a adjacent to b. However, this contradicts our choice of a and b so that $d^H(a,b)$ is a maximum among $d^H(a,b)$, $d^H(a,c)$, $d^H(a,d)$, $d^H(b,c)$, $d^H(b,d)$ and $d^H(a,d)$.

If $k = 4$, then without loss of generality, a shortest a-b path is a, u, c, v, b. We need $e_3^H(c) \geq 10$, but $d^H(a,c) = d^H(b,c) = 2$. We must have $d^H(c,d) \geq 3$, so d is not adjacent to c, u, or v. Now, d can be adjacent to at most one of a or b, and if d is not adjacent to any vertex other than a or b, then $d^H(a,d)$ or $d^H(b,d)$ is greater than $d^H(a,b)$, contradicting our choice of a and b. Thus, d must be adjacent to w, and w is not adjacent to c. If w is adjacent to both a and b, then $d^H(a,b) < 4$. Otherwise, w is adjacent to one of a and b, say b. Since $d^H(a,d) \leq 4$, we must have d adjacent to a. There is now a Hamiltonian cycle in the graph, so $e_3^H(u) \leq 7$. This is not possible. □

Regarding the standard center and the closed 3-stop center of a graph G, it is also possible that $C_3(G) \subseteq C(G)$, or $C(G) \subseteq C_3(G)$, or even $C(G) \cap C_3(G) = \emptyset$. For instance, $C(P_n) \subseteq C_3(P_n)$ and $C_3(K_{2,n}) \subseteq C(K_{2,n})$ for $n \geq 3$, while $C(G) \cap C_3(G) = \emptyset$ for the graph in Figure 4.

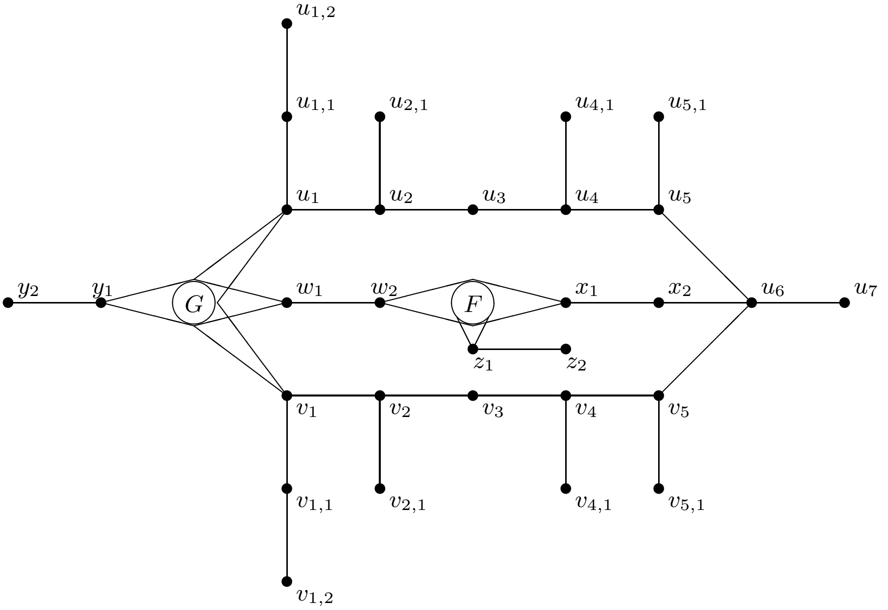

Figure 4 Example of a graph H with $C(H) \cong F$, $C_3(H) \cong G$, and $d(C(H), C_3(H)) = 3$

Proposition 3.11 *For any graphs G and F and any integer $k \geq 3$, there exists a connected graph H such that $C(H) \cong F$, $C_3(H) \cong G$ and $d(C(H), C_3(H)) = k$.*

Proof For $k \geq 3$, we define the graph H as $V(H) = V(G) \cup V(F) \cup \{u_i \,|\, 1 \leq i \leq 2k+1\} \cup \{v_i \,|\, 1 \leq i \leq 2k-1\} \cup \{w_i, x_i \,|\, 1 \leq i \leq k-1\} \cup \{y_i, z_i, u_{1,i}, v_{1,i} \,|\, 1 \leq i \leq 2\} \cup \{u_{i,1}, v_{i,1} \,|\, 2 \leq i \leq 2k-1, i \neq k\}$, and $E(H) = E(F) \cup E(G) \cup \{xx_1, xz_1, xw_{k-1} \,|\, x \in V(F)\} \cup \{xu_1, xv_1, xw_1, xy_1 \,|\, x \in V(G)\} \cup \{u_iu_{i+1} \,|\, 1 \leq i \leq 2k\} \cup \{v_iv_{i+1} \,|\, 1 \leq i \leq 2k-2\} \cup \{w_iw_{i+1}, x_ix_{i+1} \,|\, 1 \leq i \leq k-2\} \cup \{u_iu_{i,1}, v_iv_{i,1} \,|\, 1 \leq i \leq 2k-1, i \neq k\} \cup \{u_{1,1}u_{1,2}, v_{1,1}v_{1,2}, y_1y_2, z_1z_2, v_{2k-1}u_{2k}\}$. From the construction of graph H, we have the following:

$$e(x) = 2k \text{ for } x \in V(F),$$
$$e(x) > 2k \text{ for } x \in V(H) - V(F),$$
$$e_3(x) = 4k+6 \text{ for } x \in V(G),$$
$$e_3(x) > 4k+6 \text{ for } x \in V(H) - V(G).$$

See Figure 4 for an example with $k = 3$. □

4 Open Questions

Propositions 2.2, 2.3, and 2.4 characterize the graphs G for which the closed 3-stop central appendage number exists and show that $AP_3(G)$ is 0, 1, or 2 when it exists. An open question is to characterize which graphs are closed 3-stop self-peripheral graphs, which graphs have $AP_3(G) = 1$, and which graphs have $AP_3(G) = 2$. By Proposition 2.5, we know that the periphery and closed 3-stop periphery of a graph may be arbitrarily far apart and that the periphery and closed 3-stop periphery may be any graphs provided that the periphery has at least two components and the closed 3-stop periphery has at least 3 components. However, we have no general construction showing how the periphery and closed 3-stop periphery can overlap. Specifically, given a graph J with subgraphs F and G such that $V(F) \cup V(G) = V(J)$, F has at least 2 components, G has at least 3 components, and $V(F) \cap V(G) \neq \emptyset$, does there exist a graph H with $P(H) \cong F$ and $P_3(H) \cong G$?

By Proposition 3.1 and Corollary 3.2, the closed 3-stop central appendage number exists for every graph G and $AC_3(G) \in \{0, 3, 4, 5\}$. Furthermore, by Proposition 3.9, if G has no isolated vertices, then $AC_3(G) \in \{0, 3\}$. Observations 3.3 and 3.4, Proposition 3.5 and Corollaries 3.6 and 3.7 identify some particular classes of graphs as being closed 3-stop self-centered or not closed 3-stop self-centered, though more work is needed to fully characterize the closed 3-stop self-centered graphs. We have examples of graphs G with $AC_3(G) = 0$, 3, and 5, but have not found an example with $AC_3(G) = 4$. Is it possible for a graph G to have $AC_3(G) = 4$? Finally, Proposition 3.11 shows that the center and closed 3-stop center of a graph can be any graphs and can be arbitrarily far apart. However, we have no general construction showing how the center and closed 3-stop center might overlap. Given a graph J with subgraphs F and G such that $V(F) \cup V(G) = V(J)$ and $V(F) \cap V(G) \neq \emptyset$, does there exist a graph H with $C(H) \cong F$ and $C_3(H) \cong G$?

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