2000

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Abstract

Problems of linear time-dependent dispersive waves in an unbounded domain are considered. The infinite domain is truncated via an artificial boundary $\mathcal{B}$. A high-order Non-Reflecting Boundary Condition (NRBC) is imposed on $\mathcal{B}$, and the problem is solved by a Finite Difference (FD) scheme in the finite domain. The sequence of NRBCs proposed by Higdon is used. However, in contrast to the original low-order implementation, a new scheme is devised which allows the easy use of a Higdon-type NRBC of *any* desired order. In addition, the problem is considered for a *stratified* media. The performance of the scheme is demonstrated via numerical example.

1 Introduction

In various applications one is often interested in solving a dispersive wave problem computationally in a domain which is much smaller than the actual domain where the governing equations hold. One of the several methods that exist for solving a wave problem in a limited computational domain is that of using NRBCs. In this method, the original domain is first truncated by introducing an artificial boundary $\mathcal{B}$, which encloses the computational domain Ω. Then a special boundary condition is applied on $\mathcal{B}$. This boundary condition should not give rise to reflections when waves that propagate from within Ω impinge on it. Boundary conditions that generate no spurious reflection are called "perfectly non-reflecting," "perfectly absorbing," or simply "exact" and are reveiwed in [1]. Most NRBCs are approximate and generate some amount of reflection. However, as long as the reflection is small (e.g. the order of magnitude of the discretization error) the NRBC is considered adequate. The simplest NRBC is the Sommerfeld-like boundary condition, which has the same form as the Sommerfeld radiation condition that holds at infinity. In the last three decades several improved NRBCs that reduce the spurious reflections have been proposed [2].

To design a NRBC, one usually assumes that the governing equations in the exterior are linear. This does not prevent the NRBC from being used with nonlinear equations inside Ω. In terms of the complexity of designing accurate NRBCs, one can distinguish between three types of linear wave problems: time-harmonic wave problems, non-dispersive time-dependent wave problems, and dispersive wave problems. The prototype governing equations for these problems are, respectively, the Helmholtz equation, the scalar wave equation, and the Klein-Gordon equation. Technically more involved equations, but with similar properties, are of interest in each of the three categories.

The case of time-harmonic waves is, to a large extent, solved as far as NRBCs are concerned. Effective, exact, and high-order NRBCs are available; see [3]–[5]. The case of time-dependent waves is much more involved. For three-dimensional waves where $\mathcal{B}$ is a sphere, Grote and Keller [6] and Hagstrom and Hariharan [7] constructed exact NRBCs. In two dimensions, Hagstrom and Hariharan [7] proposed a high-order asymptotic NRBC. Dispersive wave problems, in which waves of different frequencies propagate with different speeds, are the most difficult. High-order NRBCs have been constructed by the authors [8]-[10]. We propose a high-order NRBCs scheme, in the context of the two-dimensional Klein-Gordon equations in stratified media. It is associated with a sequence of NRBCs of increasing order and the Jth-order NRBC is *exact* for any combination of waves that have specified wave number components $(k_x)_j$ and $(k_y)_j$ for $j = 1, \ldots, J$. This methodology originates from the observation that the solution of a dispersive wave problem is an infinite superposition of single waves, each characterized by its wave number components (or, equivalently, by its phase speed component).

We use on the artificial boundary $\mathcal{B}$ one of the *Higdon NRBCs* [11]. For a straight boundary $\mathcal{B}$ normal to the x direction, the Higdon NRBC of order J is

$$H_J : \qquad \left[\prod_{j=1}^{J} \left(\frac{\partial}{\partial t} + C_j \frac{\partial}{\partial x} \right) \right] \eta(x, y, t) = 0 \quad \text{on} \quad \mathcal{B} \, . \tag{1}$$

Here, t is time, and the C_j are parameters which have to be chosen and which signify phase speeds in the x direction. The boundary condition (1) is exact for all combinations of waves that propagate with x-direction phase speeds $C_1, \ldots, C_J$.

2 Statement of the Problem

Consider the shallow water equations (SWEs) in a semi-infinite channel (Figure 1). For simplicity, assume that the channel has a flat bottom and that there is no advection. Rotational (Coriolis) effects are taken into account. A Cartesian coordinate system (x, y) is introduced such that the channel is parallel to the x direction, as shown in Figure 1. The width of the channel is b.

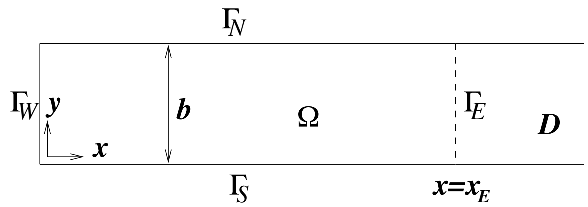

Figure 1: A semi-infinite channel

Stratification with regards to the shallow water model, we are referring to changes in density of the fluid. Van Joolen [12] has shown that for a 2 layer model, the linearized equations about a zero mean flow are given as a system of 2 Klein-Gordon equations

$$\frac{\partial^2 \eta_1}{\partial t^2} - g\theta_1 \nabla^2 (\eta_1 + \eta_2) + f^2 \eta_1 = 0, \tag{2}$$

$$\frac{\partial^2 \eta_2}{\partial t^2} - g\theta_2 \nabla^2 \left(\frac{\rho_1}{\rho_2} \eta_1 + \eta_2 \right) + f^2 \eta_2 = 0 \, . \tag{3}$$

Here t is time, $\eta_i(x, y, t)$ is the unknown water elevation above θ_i, f is the Coriolis parameter, and g is the gravity acceleration. On the north and south boundaries Γ_N and Γ_S we specify the Neumann condition:

$$\frac{\partial \eta_i}{\partial y} = 0 \qquad \text{on} \quad \Gamma_N \quad \& \quad \Gamma_S \, . \tag{4}$$

On the west boundary Γ_W we prescribe η_i using a Dirichlet condition, i.e.,

$$\eta_i(0, y, t) = \eta_{W_i}(y, t) \qquad \text{on} \quad \Gamma_W \, , \tag{5}$$

where $\eta_{W_i}(y, t)$ is a given function (incoming wave). At $x \to \infty$ the solution is known to be bounded and not to include any incoming waves. To complete the statement of the problem, the initial conditions are given:

$$\eta_i(x, y, 0) = \eta_{0_i}, \qquad \frac{\partial \eta_i(x, y, 0)}{\partial t} = v_{0_i}. \tag{6}$$

We now truncate the semi-infinite domain by introducing an artificial east boundary Γ_E, located at $x = x_E$ (see Figure 1). To obtain a well-posed problem in the finite domain Ω we need a single boundary condition on Γ_E. We shall apply a high-order NRBC for the variables η_i. A discussion on this NRBC follows.

3 Higdon's NRBCs

On the artificial boundary Γ_E we use one of the *Higdon NRBCs*. The Higdon NRBC of order J is given by (1) and involves up to Jth-order normal and temporal derivatives. These NRBCs were presented and analyzed in a sequence of papers [13]–[16] for non-dispersive acoustic and elastic waves, and were extended in [11] for dispersive waves.

The first-order condition H_1 is a Sommerfeld-like boundary condition. If we set $C_1 = C_0$ we get the classical Sommerfeld-like NRBC. A lot of work in meteorological literature is based on H_1 with a specially chosen C_1. Pearson [17] used a special but constant value of C_1, while in the scheme devised by Orlanski [18] and in later improved schemes [19] [21] the C_1 changes dynamically and locally in each time-step based on the solution from the previous time-step. See also [22]–[24]. For other parameter choices, the Higdon NRBCs are equivalent to NRBCs derived from rational approximation of the dispersion relation (the Engquist-Majda conditions being the most well-known example). This was proved by Higdon in [11] and in earlier papers.

The Higdon NRBC has many advantages including:

- They are robust. Higdon showed that the *reflection coefficient* is a product of J factors, *each of which is smaller than 1* [11]. This implies that the reflection coefficient becomes smaller as the order J increases. A good choice for the C_j would lead to better accuracy with a lower order J, but even if we miss the correct C_j's considerably, we will still reduce the spurious reflection as we increase the order J.

- They are very *general* and apply to a variety of wave problems, in one, two and three dimensions and in various configurations. They can be used, without any difficulty, for *dispersive* wave problems and for problems with layers. Most other available NRBCs are either designed for non-dispersive media (as in acoustics and electromagnetics) or are of low order (as in meteorology and oceanography).

The scheme used here was developed in [8] and is different than the original Higdon scheme [11]. Discrete Higdon conditions were developed in the literature up to third order only, because of their algebraic complexity. Here we use the implementation to an *arbitrarily high order*.

3.1 Discretization of Higdon's NRBCs

The Higdon condition H_J is a product of J operators of the form $\frac{\partial}{\partial t} + C_j \frac{\partial}{\partial x}$. Consider the following Finite Difference (FD) approximations:

$$\frac{\partial}{\partial t} \simeq \frac{3I - 4S_t^- + (S_t^-)^2}{2\Delta t} \quad , \quad \frac{\partial}{\partial x} \simeq \frac{3I - 4S_x^- + (S_x^-)^2}{2\Delta x} \ . \tag{7}$$

In (7), Δt and Δx are, respectively, the time-step size and grid spacing in the x direction, I is the identity operator, and S_t^- and S_x^- are shift operators defined by

$$S_t^- \eta_{i,pq}^n = \eta_{i,pq}^{n-1} \quad , \quad S_x^- \eta_{i,pq}^n = \eta_{i,p-1,q}^n \ . \tag{8}$$

Here and elsewhere, $\eta_{i,pq}^n$ is the FD approximation of $\eta_i(x, y, t)$ at grid point (x_p, y_q) and at time t_n. We use (7) in (1) to obtain:

$$\left[\prod_{j=1}^{J} \left(\frac{3I - 4S_t^- + (S_t^-)^2}{2\Delta t} + C_j \frac{3I - 4S_x^- + (S_x^-)^2}{2\Delta x} \right) \right] \eta_{i,Eq}^n = 0 \ . \tag{9}$$

Here, the index E correspond to a grid point on the boundary Γ_E. Higdon has solved this difference equation (and also a slightly more involved equation that is based on time- and space-averaging approximations for $\frac{\partial}{\partial t}$ and $\frac{\partial}{\partial x}$) for $J \leq 3$ to obtain an explicit formula for $\eta_{i,Eq}^n$. This formula is used to find the current values on the boundary Γ_E *after* the solution in the interior points and on the other boundaries has been updated. The algebraic complexity of these formulas increases rapidly with the order J. We have implemented the Higdon NRBCs *to any order* using a simple algorithm [8].

4 The Interior Scheme

Higdon [11] has proved, in the context of the scalar Klein-Gordon equation (2), that the discrete NRBCs (9) are stable if the interior scheme is the standard *second-order centered* difference scheme

$$\begin{aligned}
\eta_{1,pq}^{n+1} &= 2\eta_{1,pq}^n - \eta_{1,pq}^{n-1} + \left(\tfrac{C_{01}\Delta t}{\Delta x}\right)^2 \left(\eta_{1,p+1,q}^n - 2\eta_{1,pq}^n + \eta_{1,p-1,q}^n\right) \\
&+ \left(\tfrac{C_{01}\Delta t}{\Delta x}\right)^2 \left(\eta_{2,p+1,q}^n - 2\eta_{2,pq}^n + \eta_{2,p-1,q}^n\right) \\
&+ \left(\tfrac{C_{01}\Delta t}{\Delta y}\right)^2 \left(\eta_{1,p,q+1}^n - 2\eta_{1,pq}^n + \eta_{1,p,q-1}^n\right) \\
&+ \left(\tfrac{C_{01}\Delta t}{\Delta y}\right)^2 \left(\eta_{2,p,q+1}^n - 2\eta_{2,pq}^n + \eta_{2,p,q-1}^n\right) - (f\Delta t)^2 \eta_{1,pq}^n
\end{aligned} \tag{10}$$

$$
\begin{aligned}
\eta_{2,pq}^{n+1} &= 2\eta_{2,pq}^{n} - \eta_{2,pq}^{n-1} + \left(\tfrac{C_{02}\Delta t}{\Delta x}\right)^2 \left(\eta_{2,p+1,q}^{n} - 2\eta_{2,pq}^{n} + \eta_{2,p-1,q}^{n}\right) \\
&+ \left(\tfrac{C_{02}\Delta t}{\Delta x}\right)^2 \tfrac{\rho_1}{\rho_2} \left(\eta_{1,p+1,q}^{n} - 2\eta_{1,pq}^{n} + \eta_{1,p-1,q}^{n}\right) \\
&+ \left(\frac{C_{02}\Delta t}{\Delta y}\right)^2 \left(\eta_{2,p,q+1}^{n} - 2\eta_{2,pq}^{n} + \eta_{2,p,q-1}^{n}\right) \\
&+ \left(\frac{C_{02}\Delta t}{\Delta y}\right)^2 \tfrac{\rho_1}{\rho_2} \left(\eta_{1,p,q+1}^{n} - 2\eta_{1,pq}^{n} + \eta_{1,p,q-1}^{n}\right) - (f\Delta t)^2 \eta_{2,pq}^{n}
\end{aligned} \tag{11}
$$

where $C_{0i} = \sqrt{gH_i}$. We use this interior scheme in the numerical experiments presented in the next section. Since (9)-(11) are explicit, the whole scheme is explicit.

5 A Numerical example

We apply the new scheme to a simple test problem using the wave-guide depicted in Figure 1. The channel width b is 5 and the channel depth is 1. The medium is stratified with two layers. The upper layer has a thickness of $\theta_1 = .2$ and a uniform density $\rho_1 = 1$. The lower layer has a thickness of $\theta_2 = .8$ and a uniform density of $\rho_2 = 1.25$. A gravitational parameter $g = 9.8$ and a dispersion parameter $f = 1$ are used.

The initial values are zero everywhere, and the boundary function η_W on the west boundary Γ_W is zero everywhere for the first layer. A wave pulse is generated at Γ_W in the second layer and given by:

$$
\eta_W(y,t) = \begin{cases} .15\theta_2 \cos\left[\frac{\pi}{2r}(y - y_0)\right] & \text{if } |y - y_0| \le r \quad \& \quad 0 \le t \le t_0 \ , \\ 0 & \text{otherwise,} \end{cases} \tag{12}
$$

where the pulse center, radius and duration are $y_0 = 2.5$, $r = 1.5$, and $t_0 = 0.75$ respectively.

An artificial boundary $\mathcal{B}$ is introduced at $x = 5$, thus defining as the computational domain Ω a 5×5 square. In Ω a mesh of 20×20 is used, with linear interpolation for all the variables. The extended domain for the reference solution η_{ref} is a 15×5 rectangle, with a mesh of 60×20 elements. No artificial boundary is imposed on the extended domain and therefore η_{ref} is not polluted by spurious reflections.

Two cases with artificial boundaries are investigated and juxtaposed to η_{ref}. In the first case a Higdon NRBC with $J = 5$ is constructed with parameters $C_j = \sqrt{g}$. The respective numerical solution η_5 is compared to η_{ref} to obtain a measurement of error $||e||$ at time t which was calculated by the following formula:

$$
||e|| = \sum_{i=1}^{N_x} \sum_{j=1}^{N_y} \sqrt{\frac{[\eta_{ref}(x_i, y_j, t) - \eta_5(x_i, y_j, t)]^2}{N_x N_y}}, \tag{13}
$$

where N_x and N_y are determined by the grid spacing. In a second example, a Higdon NRBC on $\mathcal{B}$ with $J = 1$ and $C_j = \sqrt{g}$ is constructed and its numerical solution η_1 is compared to η_{ref} to obtain a second error measurement. In both cases, the total error and the error for each layer is reported.

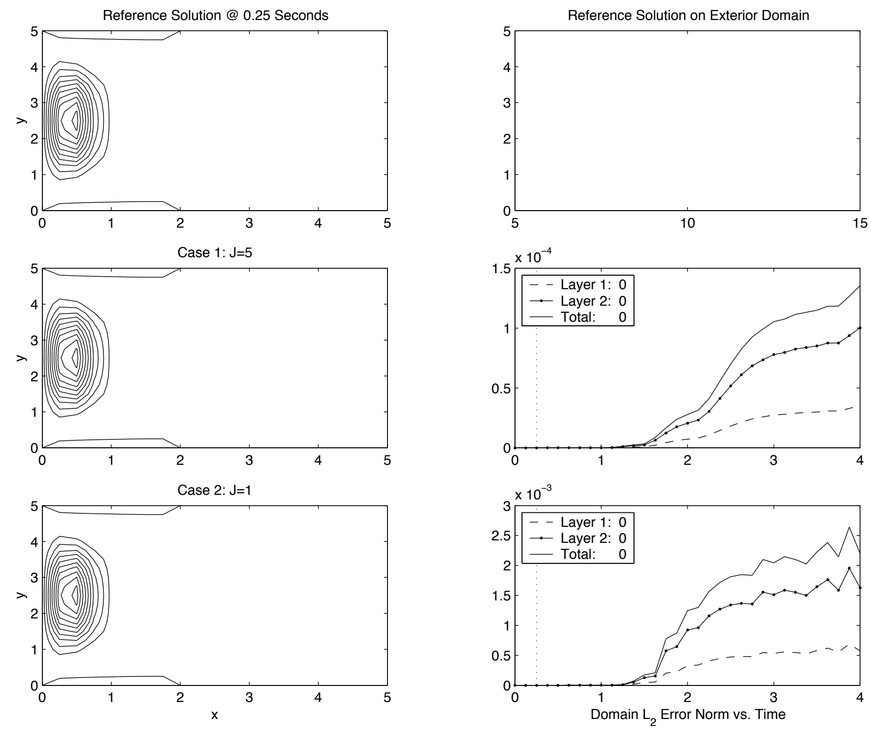

Figure 2: Solution at t=.25

Figures 2 and 3 show the solutions for η_{ref}, η_5 and η_1 at times t=.25 and 3. The top-left and top-right plots depict η_{ref} on the truncated domain Ω and extended domain D respectively (note that the domain for η_{ref} is continuous with no artificial boundary $\mathcal{B}$, but it has been separated in the figure so that η_{ref} in Ω may be better contrasted with η_5 and η_1). The middle-left and bottom-left plots correspond to η_5 and η_1 respectively. Two graphs on center- and bottom-right present the error measures as a function of time that resulted from spurious reflections on $\mathcal{B}$.

At time $t = .25$ (Fig. 2) the wave packet η_W is still close to Γ_W. The solution at and near $\mathcal{B}$ is still zero, hence no spurious reflection has occured. The wave plots are identical and, as expected, the measured error is 0.

At time $t = 3$ (Fig. 3), most of the wave packet has left the truncated domain Ω and is now visible in the extended domain D. The solution for η_5 exhibits wave traces similar to those in η_{ref}. On the other hand, η_1

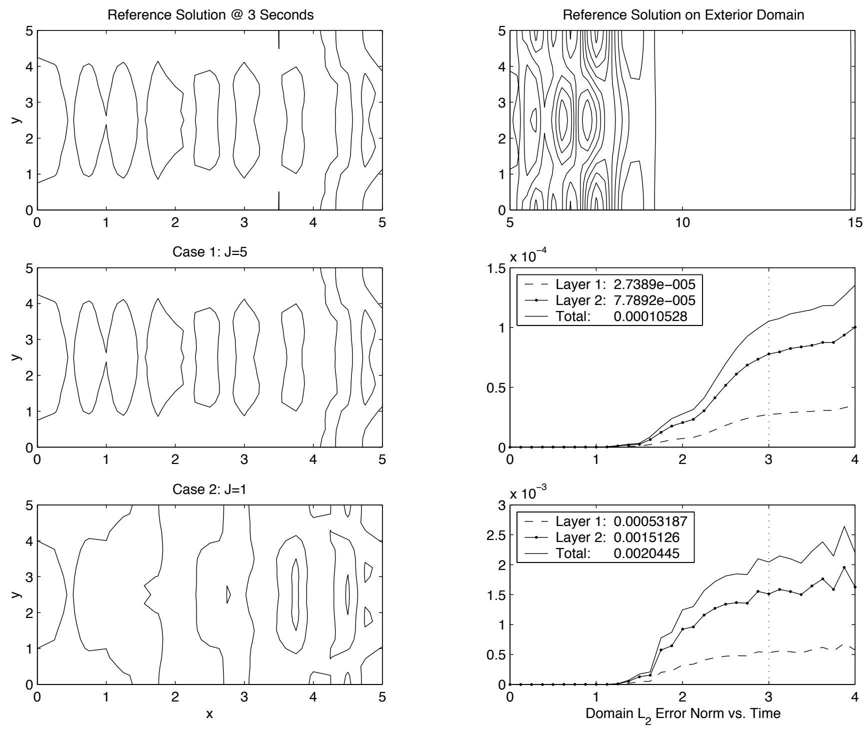

Figure 3: Solution at t=3

reveals a reflected wave that moves backwards in Ω polluting the computational domain. The error norm plots reveal an improvement of one order of magnitude for η_5. This was achieved with minimal computational expense.

Acknowledgments

The authors acknowledge the support from ONR (Marine Meteorology and Atmospheric Effects) grant N0001402WR20211 managed by Dr. Simon Chang and by the Naval Postgraduate School.

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