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Basins of attraction for several methods to find simple roots of nonlinear equations

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ABSTRACT

There are many methods for solving a nonlinear algebraic equation. The methods are classified by the order, informational efficiency and efficiency index. Here we consider other criteria, namely the basin of attraction of the method and its dependence on the order. We discuss several third and fourth order methods to find simple zeros. The relationship between the basins of attraction and the corresponding conjugacy maps will be discussed in numerical experiments. The effect of the extraneous roots on the basins is also discussed.

1. Introduction

There is a vast literature for the numerical solution of nonlinear equations. The methods are classified by their order of convergence, p, and the number, d, of function (and derivative) evaluations per step. There are two efficiency measures (see Traub [1]) defined as $I = p/d$ (informational efficiency) and $E = p^{1/d}$ (efficiency index). Another measure, introduced recently, is the basin of attraction. See Stewart [2], Scott et al. [3], Amat et al. [4,5], Chun et al. [6], and for methods to find multiple roots, see Neta et al. [7].

Chun et al. [6] have developed a new family of methods for simple roots free from second derivative. The family is of order four and includes Jarratt's method (see [8]) as a special case. They have discussed the dynamics of the family and compared its basin of attraction to three other fourth order methods. Amat et al. [9] discuss the dynamics of a family of third-order methods that do not require second derivatives. In another paper [10] they discuss the dynamics of King and Jarratt's schemes. They do not discuss the best choice of the parameter in King's method as we will do here.

In recent years there has been considerable interest in developing new algorithms with high order convergence. Normally, these high order convergence algorithms contain higher derivatives of the function or multi-step. In the former case, various techniques can be used to eliminate the derivatives. However, the resulting iteration function may be more complex than the original, for example, it can introduce extraneous zeroes. Our study considers several methods of various orders. We include Halley's method, super Halley, modified super Halley, Jarratt's method, and King's family of methods. Newton's method of order $p = 2$ was discussed by Stewart [2] and Scott et al. [3] and thus will not be given here. Halley's method of order three was discussed by Stewart [2] and we included it for comparison with super Halley and modified super Halley (both of order four). Neta et al. [11] have shown that the modified super Halley method is a rediscovered Jarratt's scheme. We also include two other fourth order methods, namely Jarratt's method and King's family. In this study, we will find the

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extraneous fixed points, if any. We will also show how to choose a parameter (in the case of King's family of methods) to get best results.

In the next section we describe the methods to be considered in this comparative study. Section 3 will give the conjugacy maps for each method and discuss the possibility of extraneous fixed points [12]. We will show the relationship between these maps, extraneous fixed points, and the basins of attraction in our numerical experiments detailed in Section 4.

2. Methods for the comparative study

First we list the methods we consider here with their order of convergence.

(1) Halley's method ($p = 3$).
(2) Super Halley optimal method ($p = 4$).
(3) Modified super Halley optimal method ($p = 4$).
(4) King's family of methods ($p = 4$).
(5) Jarratt's method ($p = 4$).

King's family of fourth order methods did not perform well in our previous study [3]. We will show how to choose the family member to get best results based on the location of the extraneous fixed points. We now detail the methods studied here.

- Halley's third order (H3) method [13] is given by

$$x_{n+1} = x_n - u_n \frac{1}{1 - \frac{1}{2} u_n \frac{f''_n}{f'_n}}, \tag{1}$$

where

$$u_n = \frac{f_n}{f'_n}, \tag{2}$$

and $f_n = f(x_n)$ and similarly for the derivatives.

- Super Halley fourth order (SH4) method [14] is given by

$$\begin{aligned} y_n &= x_n - \frac{2}{3} u_n, \\ x_{n+1} &= x_n - \left(1 + \frac{1}{2} \frac{L_f}{1 - L_f}\right) u_n, \end{aligned} \tag{3}$$

where

$$L_f = \frac{f_n f''_n}{(f'_n)^2}. \tag{4}$$

- A modified super Halley fourth order (MSH4) optimal method [15] is given by

$$\begin{aligned} y_n &= x_n - \frac{2}{3} u_n, \\ x_{n+1} &= x_n - \left(1 + \frac{1}{2} \frac{\hat{L}_f}{1 - \hat{L}_f}\right) u_n, \end{aligned} \tag{5}$$

where

$$\hat{L}_f = \frac{f_n}{(f'_n)^2} \frac{f'(y_n) - f'_n}{y_n - x_n}. \tag{6}$$

- King's family of fourth order methods (K4) [16] is given by

$$\begin{aligned} y_n &= x_n - u_n, \\ x_{n+1} &= y_n - \frac{f(y_n)}{f'_n} \frac{f_n + \beta f(y_n)}{f_n + (\beta - 2) f(y_n)}. \end{aligned} \tag{7}$$

- Jarratt's fourth order (J4) method [17] is given by

$$\begin{aligned} y_n &= x_n - \frac{2}{3} u_n, \\ x_{n+1} &= x_n - \frac{1}{2} u_n - \frac{1}{2} \frac{u_n}{1 + \frac{3}{2}\left(\frac{f'(y_n)}{f'_n} - 1\right)}. \end{aligned} \tag{8}$$

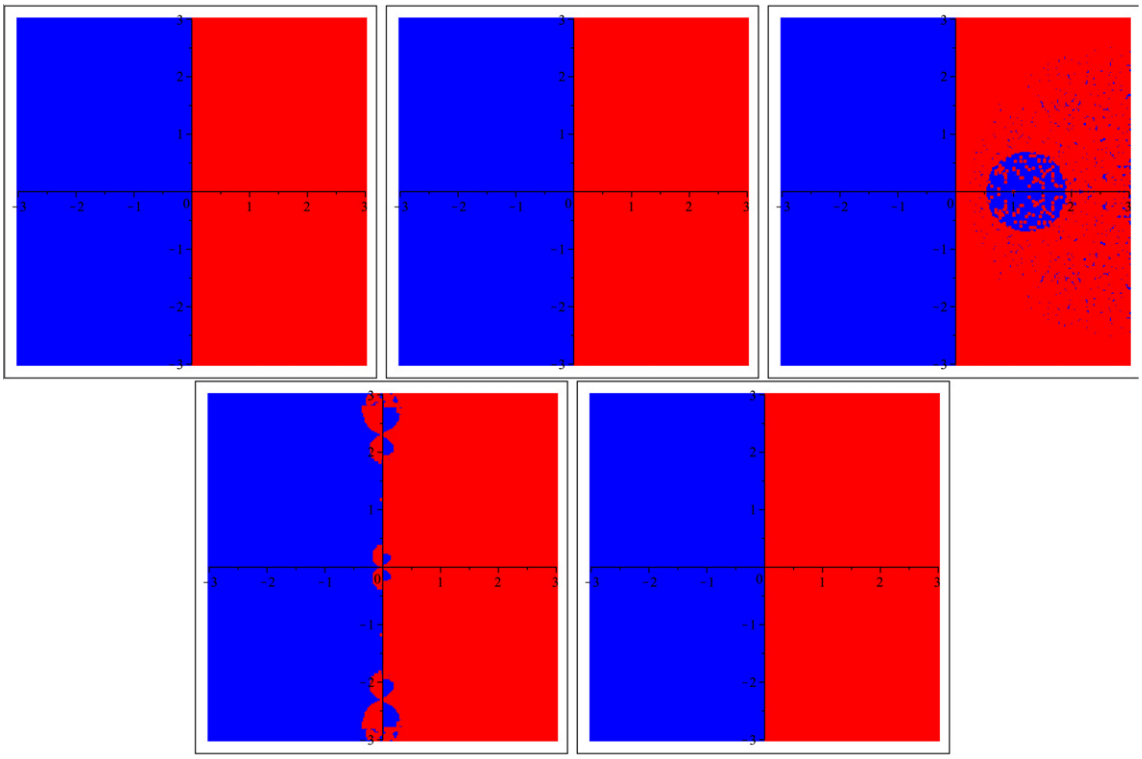

Fig. 1. Top row: Halley's (left), super Halley's method (middle) and modified super Halley (right). Bottom row: King's (left), and Jarratt's (right). The results are for the polynomial $z^2 - 1$.

Note that this is a different method than the one discussed by Amat et al. [10].

3. Corresponding conjugacy maps for quadratic polynomials

For Newton's method the following is well known [4].

Theorem 3.1 (Newton's method). *For a rational map $R_p(z)$ arising from Newton's method applied to $p(z) = (z-a)(z-b)$, $a \neq b$, $R_p(z)$ is conjugate via the Möbius transformation given by $M(z) = \frac{z-a}{z-b}$ to*

$$S(z) = z^2.$$

Theorem 3.2 (Halley's method). *For a rational map $R_p(z)$ arising from Halley's method applied to $p(z) = (z-a)(z-b)$, $a \neq b$, $R_p(z)$ is conjugate via the Möbius transformation given by $M(z) = \frac{z-a}{z-b}$ to*

$$S(z) = z^3.$$

Proof. Let $p(z) = (z-a)(z-b)$, $a \neq b$ and let M be the Möbius transformation given by $M(z) = \frac{z-a}{z-b}$ with its inverse $M^{-1}(u) = \frac{ub-a}{u-1}$, which may be considered as a map from $\mathbb{C} \cup \{\infty\}$. We then have

$$M \circ R_p \circ M^{-1}(u) = M \circ R_p\left(\frac{ub-a}{u-1}\right) = u^3. \qquad \square$$

The proof for the other methods is similar. In the sequel we present only the result.

Theorem 3.3 (*Super Halley's method*). *For a rational map $R_p(z)$ arising from super Halley's method applied to $p(z) = (z-a)(z-b)$, $a \neq b$, $R_p(z)$ is conjugate via the Möbius transformation given by $M(z) = \frac{z-a}{z-b}$ to*

$$S(z) = z^4.$$

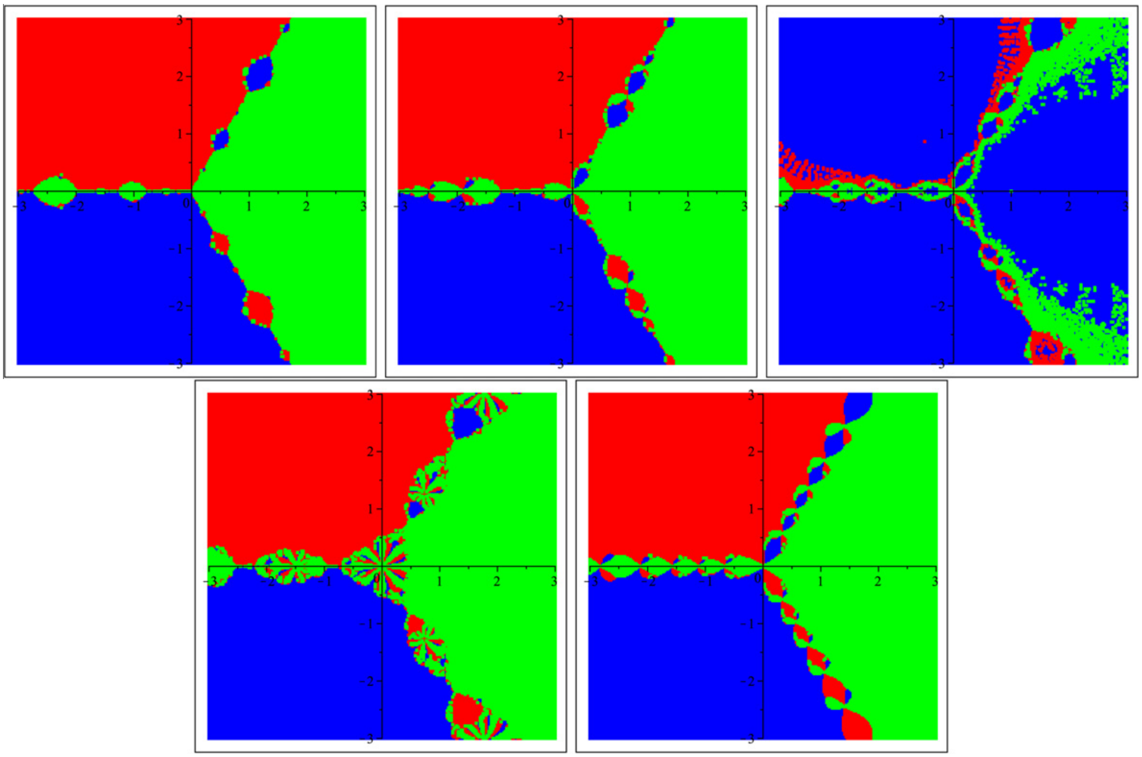

Fig. 2. Top row: Halley's (left), super Halley's method (middle) and modified super Halley (right). Bottom row: King's (left), and Jarratt's (right). The results are for the polynomial $z^3 - 1$.

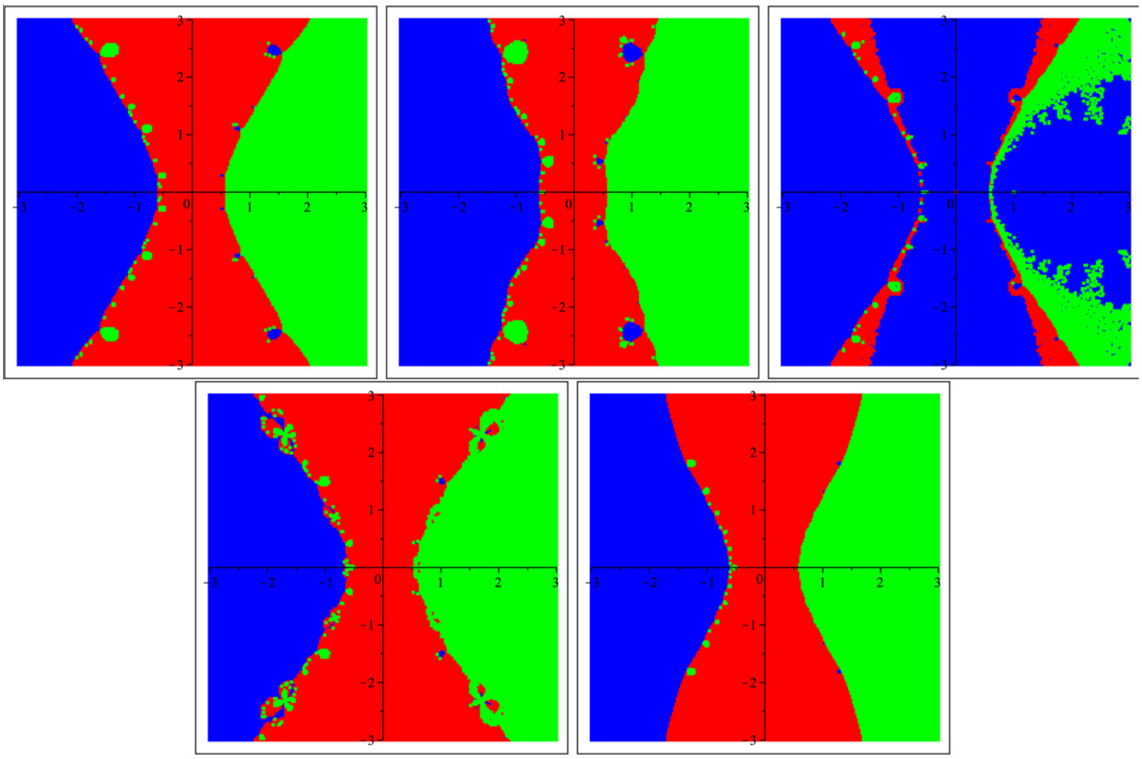

Fig. 3. Top row: Halley's (left), super Halley's method (middle) and modified super Halley (right). Bottom row: King's (left), and Jarratt's (right). The results are for the polynomial $z^3 - z$.

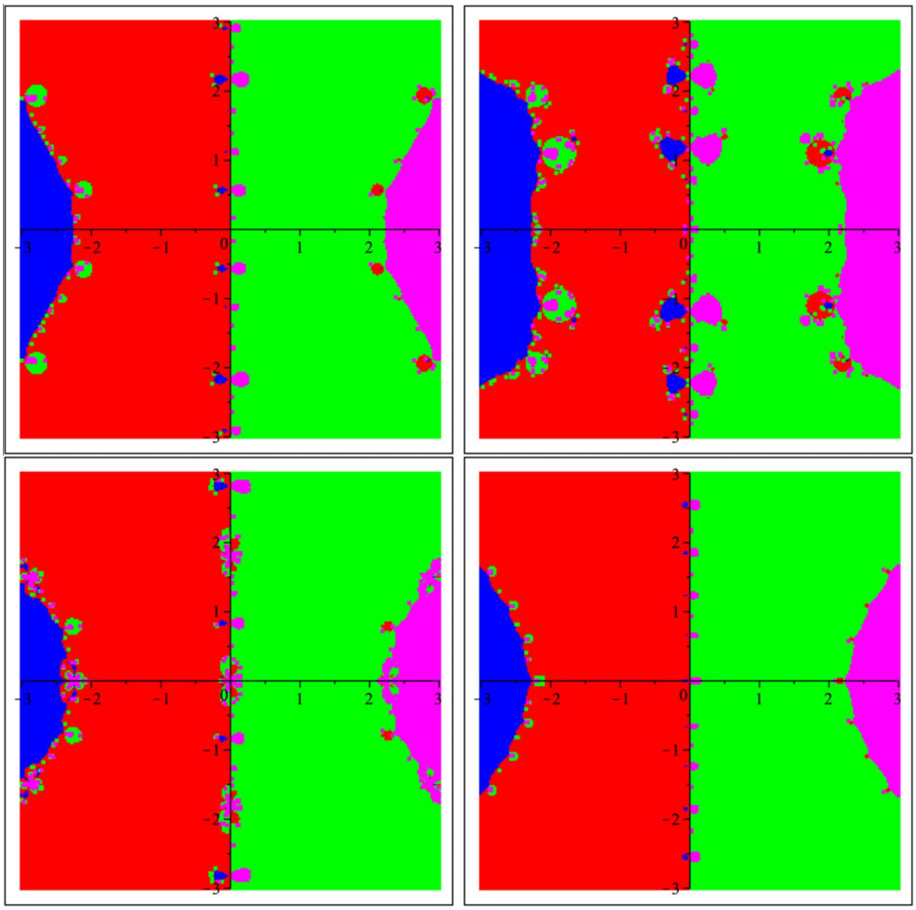

Fig. 4. Top row: Halley's (left), and super Halley's method (right). Bottom row: King's (left), and Jarratt's (right). The results are for the polynomial $z^4 - 10z^2 + 9$.

Theorem 3.4 (*Modified super Halley method*). *For a rational map $R_p(z)$ arising from the method* (5) *applied to $p(z) = (z-a)(z-b),\ a \neq b,\ R_p(z)$ is conjugate via the Möbius transformation given by $M(z) = \frac{z-a}{z-b}$ to*

$$S(z) = z^4.$$

Theorem 3.5 (*King's fourth-order family of methods*). *For a rational map $R_p(z)$ arising from the method* (7) *applied to $p(z) = (z-a)(z-b),\ a \neq b,\ R_p(z)$ is conjugate via the Möbius transformation given by $M(z) = \frac{z-a}{z-b}$ to*

$$S(z) = \frac{1 + 2\beta + (2+\beta)z + z^2}{1 + (2+\beta)z + (1+2\beta)z^2} z^4.$$

For Jarratt's method we have the following result.

Theorem 3.6 (*Jarratt's fourth order optimal method*). *For a rational map $R_p(z)$ arising from the method* (8) *applied to $p(z) = (z-a)(z-b),\ a \neq b,\ R_p(z)$ is conjugate via the Möbius transformation given by $M(z) = \frac{z-a}{z-b}$ to*

$$S(z) = z^4.$$

Note that the maps are of the form $S(z) = z^p R(z)$ where $R(z)$ is either unity or a rational function and p is the order of the method.

3.1. Extraneous fixed points

Note that all these methods can be written as

$$x_{n+1} = x_n - u_n H_f(x_n, y_n).$$

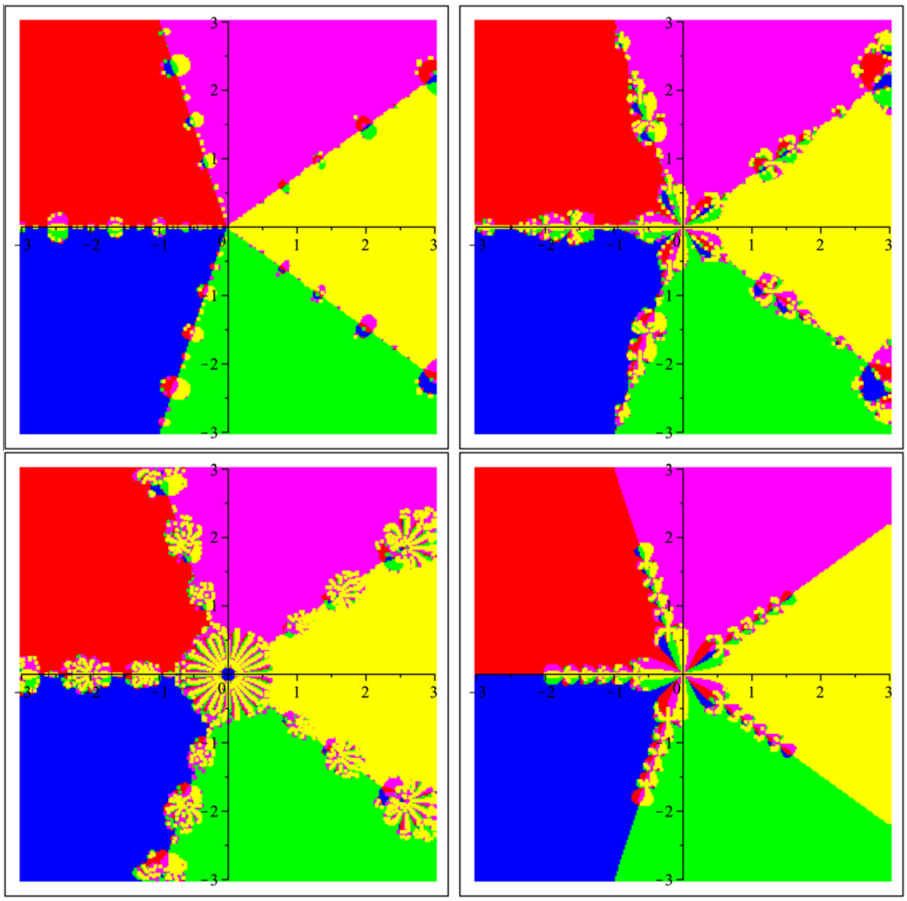

Fig. 5. Top row: Halley's (left), and super Halley's method (right). Bottom row: King's (left), and Jarratt's (right). The results are for the polynomial $z^5 - 1$.

Clearly the root α is a fixed point of the method. The points $\xi \neq \alpha$ at which $H_f(\xi) = 0$ are also fixed points of the method, since the second term on the right vanishes. These points are called extraneous fixed points (see [12]).

Theorem 3.7. *There are no extraneous fixed points for Halley's method.*

Proof. For Halley's method (1) we have

$$H_f = \frac{1}{1 - \frac{1}{2} u_n \frac{f''_n}{f'_n}}.$$

This function does not vanish and therefore there are no extraneous fixed points. □

Theorem 3.8. *The extraneous fixed points of super Halley's, modified super Halley's and Jarratt's methods are* $\pm i\frac{\sqrt{3}}{3}$.

Proof. For super Halley's method, the extraneous fixed point can be found by solving $L_f = 2$. This leads to the equation

$$\frac{1}{2}\frac{3z^2 + 1}{z^2 + 1} = 0$$

for which the roots are $\pm i\frac{\sqrt{3}}{3}$. These fixed points are repulsive.

The poles are at $z = \pm\frac{\sqrt{2}}{2} i$.

For modified super Halley's method, the extraneous fixed points are the solution of $\hat{L}_f = 2$. This leads to the same equation as above and therefore the same extraneous fixed points. These fixed points are repulsive. For Jarratt's method the extraneous fixed points are those for which

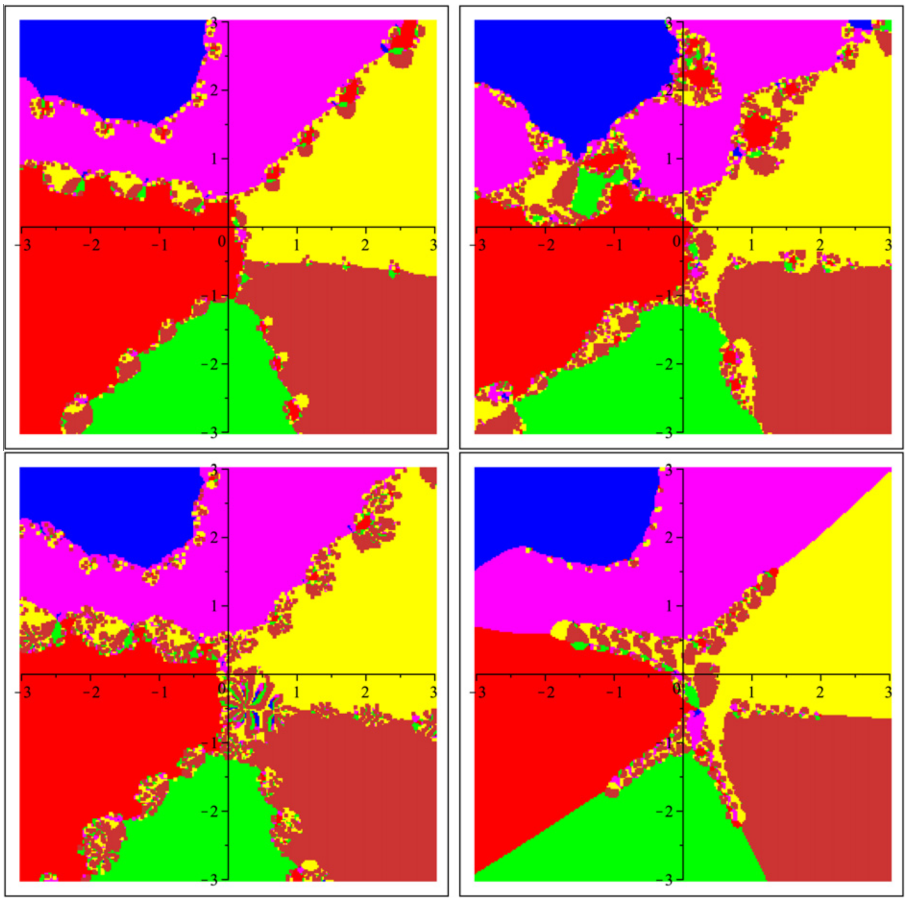

Fig. 6. Top row: Halley's (left), and super Halley's method (right). Bottom row: King's (left), and Jarratt's (right). The results are for the polynomial $z^6 - \frac{1}{2}z^5 + \frac{11(i+1)}{4}z^4 - \frac{3i+19}{4}z^3 + \frac{5i+11}{4}z^2 + \frac{i-11}{4}z + \frac{3}{2} - 3i$.

$$1 + \frac{1}{1 + \frac{3}{2}\left(\frac{f'(y(z))}{f'(z)} - 1\right)} = 0$$

where $y(z) = z - \frac{2}{3}u(z)$. Upon substituting $f(z) = z^2 - 1$, we get the equation

$$1 + \frac{1}{-\frac{1}{2} + \frac{3}{2}(1 - \frac{z^2-1}{3z^2})} = 0.$$

The solution is again $z = \pm i\frac{\sqrt{3}}{3}$. These fixed points are repulsive.

The poles are at $z = \pm\frac{\sqrt{2}}{2}i$. □

Theorem 3.9. *There are four extraneous fixed points of King's family of methods. For $\beta = 3 - 2\sqrt{2}$ we get the roots very close to the imaginary axis.*

Proof. The extraneous fixed point of King's family of methods are those for which $1 + \frac{y(z)^2-1}{z^2-1}v(z) = 0$ where $v(z) = \frac{z^2-1+\beta(y(z)^2-1)}{z^2-1+(\beta-2)(y(z)^2-1)}$. Upon substituting $y(z) = z - \frac{f(z)}{f'(z)}$ we get the equation

$$\frac{(5\beta+12)z^4 + (4-6\beta)z^2 + \beta}{4z^2((\beta+2)z^2 - \beta + 2)} = 0.$$

In order to get roots on the imaginary axis, we choose $\beta = 3 - 2\sqrt{2}$ and then the four roots are $\pm.2074660892e-4 \pm .3398755690i$. Note that the real parts are negligible, so the roots are not on the imaginary axis but very close. These four fixed points are slightly repulsive (the derivative at those point is $1.000000172 \pm .0005549846764i$ and its magnitude is 1.000000326.

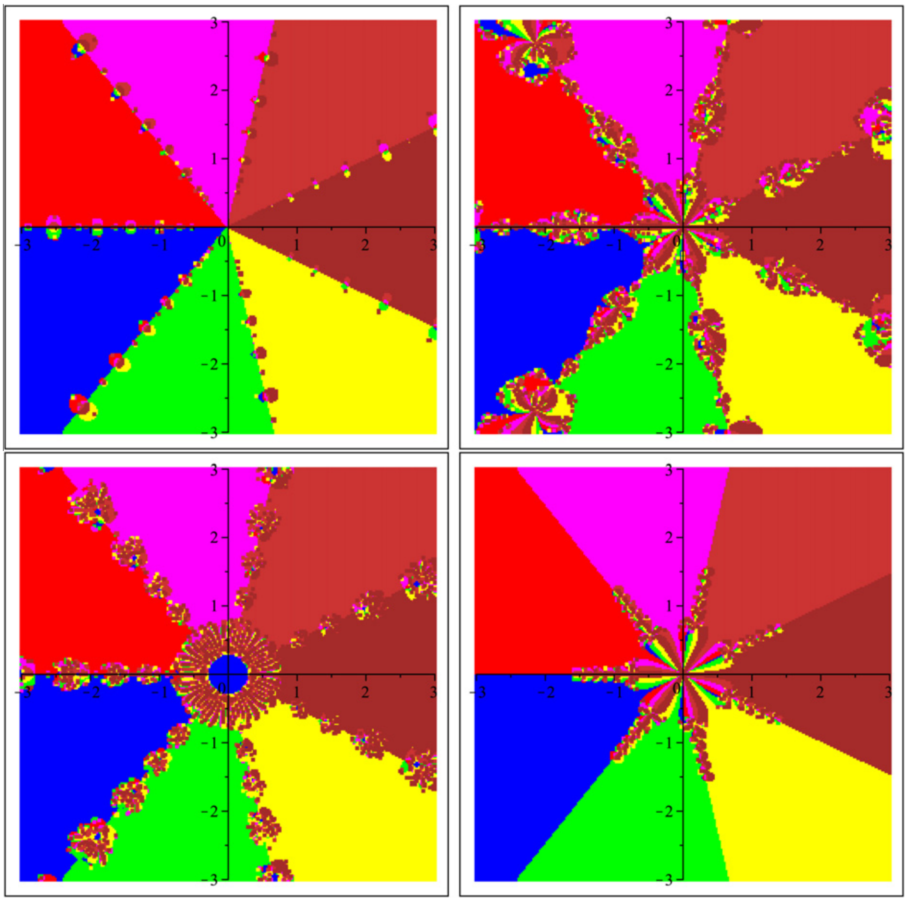

Fig. 7. Top row: Halley's (left), and super Halley's method (right). Bottom row: King's (left), and Jarratt's (right). The results are for the polynomial $z^7 - 1$.

The poles are at $z = 0$ (double) and $z = \pm\sqrt{\frac{\beta-2}{\beta+2}}$. For $\beta = 3 - 2\sqrt{2}$ the poles are at $z = \pm i\sqrt{\frac{2\sqrt{2}-1}{5-2\sqrt{2}}}$ all on the imaginary axis. □

4. Numerical experiments

In our first experiment, we have run all the methods to obtain the real simple zeros of the quadratic polynomial $z^2 - 1$. The results of the basin of attractions are given in Fig. 1. It is clear that Halley, super Halley and Jarratt are the best followed by King's method. The member of King's family of methods we have used is the one with $\beta = 3 - 2\sqrt{2}$ so that the extraneous fixed points are very close to the imaginary axis. In our previous study [3] we have used other values and the results were not as good as these. The modified super Halley does not give good results even though we proved that it is a rediscovered Jarratt's scheme. The maps are identical and the extraneous roots are identical, of course. This means that it does matter how one organizes the calculation.

In our next experiment we have taken the cubic polynomial $z^3 - 1$. The results are given in Fig. 2. Again the modified super Halley's method did not give the same results as Jarratt's method. Halley's method is best followed by super Halley's and Jarratt's methods.

In the next examples we have taken polynomials of increasing degree. The results for the cubic polynomial $z^3 - z$ are given in Fig. 3. Here Jarratt's scheme is best followed by Halley's and King's methods.

We have decided not to show the results for the modified super Halley method for the rest of the experiments. Fig. 4 shows the results for the polynomial $z^4 - 10z^2 + 9$. In this case Jarratt's method is best followed by Halley's and King's methods.

The fifth order polynomial, $z^5 - 1$, results are shown in Fig. 5. Halley's and Jarratt's methods are best. The next example is for a polynomial of degree 6 with complex coefficients, $z^6 - \frac{1}{2}z^5 + \frac{11(i+1)}{4}z^4 - \frac{3i+19}{4}z^3 + \frac{5i+11}{4}z^2 + \frac{i-11}{4}z + \frac{3}{2} - 3i$. The results are presented in Fig. 6. Again Halley's and Jarratt's methods performed better than the other two. The last example for a polynomial of degree 7, $z^7 - 1$. In all these Figs. 2–7 we find that Halley's and Jarratt's methods are better than the other schemes.

5. Conclusions

In this paper we have considered several third and fourth order methods for finding simple zeros of a nonlinear equation. Note that the conjugacy maps do not tell the whole story as one can see from comparing Jarrat's method to the super Halley method. We have studied all of the extraneous fixed points and they are repulsive. We have shown how to find the best parameter for the King's family of methods so that its performance is improved. Unfortunately, Halley's third order method and Jarratt's fourth order methods performed even better. We should also mention that since Ostrowski's method [18] is a special case of King's family with $\beta = 0$, then we cannot expect it to perform better than King's method with the choice of $\beta = 3 - 2\sqrt{2}$.

Acknowledgements

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