2001-07

Explicit Analytical Expression for a Lanchester Attrition-Rate Coefficient for þÿ B o n d e r   a n d   F a r r e l l   s   m - P e r i o d Target-Engagement Policy

Explicit Analytical Expression for a Lanchester Attrition-Rate Coefficient for Bonder and Farrell's m-Period Target-Engagement Policy

by

James G. Taylor
MOVES Academic Group
Naval Postgraduate School

and

Beny Neta
Mathematics Department
Naval Postgraduate School

Working Paper #5
DTRA Project
July 9, 2001

1. Introduction.

The purpose of this working paper is to give an explicit analytical expression for a Lanchester-type attrition-rate coefficient for direct-fire combat in a heterogeneous-target environment with serial acquisition of targets for Bonder and Farrell's m-period target-acquisition policy[1]. It develops this result (its main result) from Taylor's [2001d] new important general result (that does not depend on the target-engagement policy of a firer type or even on the particulars of the target-acquisition process) for a Lanchester attrition-rate coefficient for serial acquisition by developing explicit analytical expressions for the two key intermediate quantities on which the coefficient depends: namely,

(1) expected time to acquire a target that will be engaged,

(2) next-target-type-to-be-engaged probability.

An analytical expression for the former quantity (the expect value) was recently developed by one of the authors (Taylor [2001e]), while the paper at hand develops such an expression for the latter probability. These two new important intermediate results have allowed us to develop the explicit analytical expression for a Lanchester attrition-rate coefficient for Bonder and Farrell's target-acquisition policy via Taylor's general expression for direct-fire combat in a heterogeneous-target environment with serial acquisition of targets. These analytical results are then verified against simulation results.

Building on earlier work by Bonder [1967], [1970][2], Bonder and Farrell (see Miller et al. [1978]; also Bonder and Farrell [1970]) did the pioneering work on Lanchester attrition-rate coefficients for direct-fire combat in a heterogeneous-target environment. However, more recent work by Taylor [2001d] has completely revised their earlier results[3]. Bonder and Farrell also introduced a fairly complex target-engagement policy, although they did not use such terminology (nor point the dependence of attrition-rate coefficients on such target-engagement policy[4]). Significantly, Bonder and Farrell did not give any explicit analytical expression for the two key intermediate quantities noted above (namely, the expected time to acquire a target that will be engaged and the next-target-type-to-be-engaged probability) for their target-engagement policy.

Furthermore, this Bonder-Farrell methodology (irrespective of any shortcomings[5]) has been the theoretical basis for direct-fire ground combat in some computer-based combat models widely used by DoD and the U.S. Army (e.g. see CCTC [1979], TRAC-FLVN [1992]). Therefore, it is of considerable theoretical interest to have explicit analytical results available for these two key intermediate quantities. In particular, the theoretical correctness and practical impact of Bonder and Farrell's general expression for a Lanchester attrition-rate coefficient can now be readily investigated (providing that one obtains what expressions are actually used for the two key intermediate quantities noted above). Moreover, having such analytical results readily available allows one to implement simple Lanchester-type models (e.g. for two and three weapon-system types on each side) with such coefficients on a spreadsheet (e.g. Excel spreadsheet). Hence, parametric analyses become

[1] See, for example, Miller et al. [1978] or CCTC [1979].

[2] See also Barfoot [1969].

[3] For example, compare (2) below with the analogous expression given by Bonder and Farrell (e.g. see Miller et al.; also TRAC-FLVN [1992, Section 5.4.2.3]). Moreover, no explicit expressions were given for the two key intermediate quantities noted above.

[4] Therefore, one would not think to investigate the dependence of kill rates in a heterogeneous-target environment on the target engagement policy (or even to try to develop other operationally relevant policies).

[5] Taylor [2000a] has detailed a number of serious flaws in Bonder-Farrell methodology. Among those germane here are the following: (1) incorrect/inconsistent treatment for parallel acquisition of targets, (2) Bonder-Farrell results for parallel acquisition do not apply to cases of preemption by higher-priority target type (see also Taylor [2000c]).

readily accessible via a PC, thereby fostering greater understanding of Lanchester-type models (particularly methodologies for determining numerical values for attrition-rate coefficients).

Thus, this paper takes another step towards establishing theoretically-sound methodology for calculating numerical values for Lanchester attrition-rate coefficients. In particular, it provides the practical means for comparing Taylor's theory and results for Lanchester attrition-rate coefficients with those of Bonder and Farrell. Such numerical values for attrition-rate coefficients are indispensable for representing ground-combat attrition in joint campaign models such as JWARS, ITEM, etc. that are essential for investigating issues concerning issues concerning weapons of mass destruction (WMD). Previously, no theoretically-sound methodology had existed for computing such kill rates in a heterogeneous-target environment, especially for Bonder and Farrell's m-period target-engagement policy.

2. Background.

Both DoD and DTRA extensively use combat models for analysis of significant issues and also development of policy. Essentially all aggregated-force models (both ITEM [a joint campaign model] and others such as CEM and VIC) currently in use for analysis, or planned for the future (e.g. JWARS, AWARS), base their attrition calculations on some type of underlying Lanchester-type force-on-force model. The basic Lanchester-type attrition paradigm (see Taylor [1983]) (out of which such computer-based complex operational models have been developed by the process of model enrichment) is given by (see Fig. 1)

$$\begin{cases} \frac{dx}{dt} = -a\,y & \text{with } x(0) = x_0, \\ \frac{dy}{dt} = -b\,x & \text{with } y(0) = y_0, \end{cases} \tag{1}$$

where t = 0 denotes the time at which the battle begins and x(t) and y(t) denote the numbers of X and Y at time t. Here, for example, ***a*** denotes the rate at which a single typical Y firer kills X targets and is called a Lanchester attrition-rate coefficient (single-weapon-system-type kill rate (see Taylor [1982]).

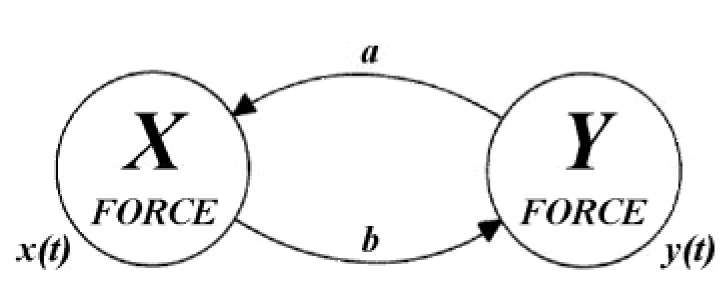

Fig. 1. Combat between two homogeneous forces.

The practical use of such differential-equation models in defense analysis depends (in essence) on one's ability to obtain realistic values for the Lanchester attrition-rate coefficients. Two general approaches that have been used to develop numerical values for Lanchester attrition-rate coefficients (i.e. single-weapon-system-type kill rates) are

(1) the freestanding-analytical-model approach (which generates these values from an analytical model, independent of any high-resolution model),
(2) the hierarchy-of-models approach (which estimates parameter values for such an attrition-rate coefficient from the output of a high-resolution Monte-Carlo combat simulation).

The first approach was pioneered by Bonder and Farrell [1970] (see also Bonder [1967], [1970] and Barfoot [1969]) and for this reason is frequently called the Bonder-Farrell approach. It will be the approach used in the paper at hand. It conceptually consists in considering (for the case of homogeneous forces depicted in Fig. 1) a single typical firer on a particular side and then computing the rate at which this firer type kills enemy targets according to a micro-combat model[6]. A mathematical formula for such a rate is developed from this micro-combat model.

Recently, one of the authors (Taylor [2001b]) developed a new principle for computing a numerical value for a Lanchester attrition-rate coefficient: namely, computing it as the ratio of the expected number of kills for a given firer type against a specific target type per target-engagement cycle divided by the expected length of this target-engagement cycle. Taylor [2001d] then used this principle to develop a general expression (not depending on the firer type's target-engagement policy) for a Lanchester attrition-rate coefficient for direct-fire combat in a heterogeneous-target environment with serial acquisition of targets. Concrete results for this attrition-rate coefficient were then obtained for Taylor's constant-probability-of-engagement-for-a-given-target-type target-engagement policy and verified by Monte-Carlo simulation of the target-engagement cycle.

Taylor's [2001d] general expression for a Lanchester attrition-rate coefficient for direct-fire combat in a heterogeneous-target environment for serial acquisition of targets depends of the two key intermediate quantities
(1) next-target-type-to-be-engaged probability,
(2) expected time to acquire a target that will be engaged.

For Taylor's constant-probability-of-engagement-for-a-given-target-type target-engagement policy, these two quantities are easily computed[7]. However, this state of affairs is not true for Bonder and Farrell's m-period target-engagement policy. In fact, the authors know of no other such explicit analytical results for the Bonder-Farrell policy. A previous working paper (Taylor [2001e]), however, has developed an explicit analytical expression for the expected time to acquire a target that will be engaged for this policy. It remains to develop an explicit analytical result for the next-target-type-to-be-engaged probability. Such an explicit analytical result would allow one to use the author's general expression for a Lanchester attrition-rate coefficient to develop explicit analytical attrition-rate-coefficient results for Bonder and Farrell's target-engagement policy. Thus, the paper at hand will develop an explicit analytical expression for the next-target-type-to-be-engaged probability for Bonder and Farrell's m-period target-engagement policy.

3. Taylor's New General Expression for a Lanchester Attrition-Rate Coefficient for Serial Acquisition.

Recently, Taylor developed[8] the following general expression for the rate a which a single

[6] Here micro-combat model refers to an entity-level (i.e. single-shooter) model in which all the details of the process by which this individual combatant acquires and engages an enemy target are considered.
[7] That is why such a policy was considered in the first place.
[8] Revising earlier work (Taylor [1999], [2000b], [2000f]) that was flawed for the case of direct-fire combat in a heterogeneous-target environment. Equation (2) is based on the principle that such a rate should be computed as the ratio of the expected number of kills against a particular target type per target-engagement cycle divided by its expected length (see Taylor [2001d] for further details). Other existing work suffers from even more serious flaws.

typical Y_j firer type kills X_i target types with direct fire in a heterogeneous-target environment with serial acquisition of targets[9]

$$\alpha_{ij}^{ser} = \frac{\left\{ \dfrac{P^{eng}_{X_iY_j}}{\alpha_{ij} + \mu} \right\} \alpha_{ij}}{\left\{ E\left[T_{a_{Y_j}} \right] + \sum_{k=1}^{n_X} \left\{ \dfrac{P^{eng}_{X_kY_j}}{\alpha_{kj} + \mu} \right\} \right\}}, \qquad (2)$$

where α_{ij} denotes the rate at which an individual Y_j firer type kills acquired X_i targets, μ denotes the rate at which LOS is lost between a X_i-target/Y_j-observer pair, $E\left[T_{a_{Y_j}}\right]$ denotes the expected time for the Y_j firer type to acquire the next target that will be engaged, and $P^{eng}_{X_iY_j}$ denotes the probability that the next target type to be engaged by the Y_j firer type will be an X_i one. The latter two key intermediate quantities carry all the dependence on the target-engagement policy with them. Assumptions have been made for only the line-of-sight (LOS) and the attacking-of-an-acquired-target processes for the development of (2) above (see Taylor [2001d, Section 7.1] for further details; also Section 7 below).

4. Bonder and Farrell's m-Period Target-Engagement Policy.

For serial acquisition of targets in a heterogeneous-target environment (m different target types) with stochastic LOS, Bonder and Farrell considered the following m-period target-engagement policy[10] (also referred to as rules of engagement). In the first such period of time, for a particular firer type, only its highest-priority target type will be immediately engaged when acquired, although all lower-priority target types are remembered as long as they are visible. This first period lasts from $t=0$ until $t=t_1$. At time t_1, any second-priority target that is still visible will be immediately engaged. In the second such period of time, either of the top-two-priority target types will be immediately engaged when acquired, although all lower-priority target types are remembered as long as they are visible. This second period lasts from $t= t_1$ until $t=t_2$. At time t_2, any third-priority target that is still visible will be immediately engaged, etc. The last such period of time starts at $t=t_{m-1}$ and does not end at any finite time. At time t_{m-1}, any second-lowest-priority target that is still visible will be immediately engaged. In this last period of time, any target type that is acquired will be immediately engaged. The same type of m-period target-engagement policy holds for each firer type. The situation for 3 periods (and, of course, 3 target types) is depicted below in Fig. 2.

Bonder and Farrell's m-period target-engagement policy is an open-loop policy that does not involve any feedback about the battlefield state[11]. In the case of battlefield feedback (e.g. having knowledge about how many enemy targets are available for acquisition) one might want to adopt a policy of always waiting for one's highest-priority target to be acquired (as long as the expected waiting time is not too long). It is well known that such a closed-loop policy is always more efficient than an open-loop one (e.g. see Padulo and Arbib [1974], Luenberger [1979]).

---

[9] Serial acquisition (as opposed to parallel acquisition) of targets means that no new target can be acquired while an acquired target is being attacked by a firer (see Taylor [2001d, Section 5] for further details).
[10] See CCTC [1979, pp. 53 and 55] or Miller et al. [1978, pp. 49 and 51]; see also Taylor [1982, p. 113]).
[11] See Luenberger [1979, Section 8.9] for a lucid discussion of the concepts of open-loop and closed-loop control.

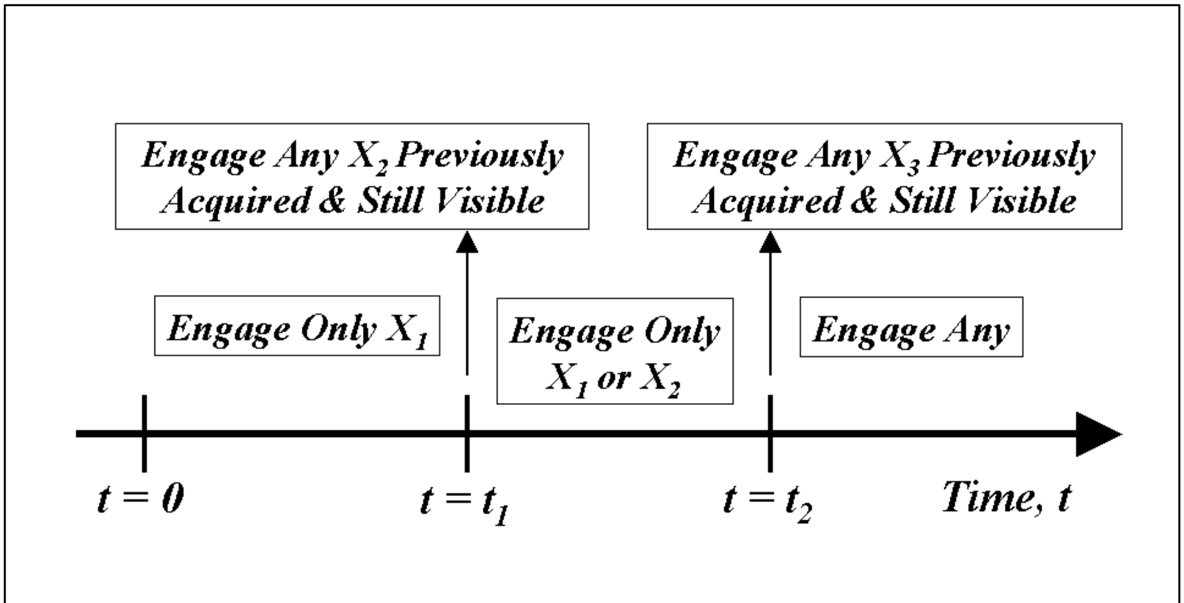

Fig. 2. Bonder and Farrell's target-engagement policy (3 periods).

5. Notation for Firer and Target Types.

It will also be convenient to establish here notation that will be used in the sequel. Without loss of generality, one can assume that (for a particular Y_j firer type) the target types have been arranged in order of decreasing priority with increasing value of the target-type index i. It will also be convenient to suppress the firer-type index j, which by our convention has been the second of the double subscripts for a Y_j firer type engaging X_i target types. Then, target type 1 (i.e. X_1) denotes the highest-priority target type, target type 2 the second-priority target type, etc., with target type m denoting the lowest-priority target type. This situation is depicted in Fig. 3 below, which also incorporates these conventions. This figure will be used in the development of the expression for the expected time to acquire a target that will be engaged.

The rules of engagement (i.e. the target-engagement policy) for the generic Y firer type against the m different X_i target types (put in order of decreasing target priority with increasing target-type index i (i.e. X_1 denotes the highest-priority X target type, while X_m denotes the lowest-priority X target type to the Y firer)) are simply the target-engagement policy for the Y firer. For Bonder and Farrell's m-period target engagement policy (see Section 8 above, especially Fig. 4), this means that m time periods are considered for the rules of engagement. During the $k^{\underline{th}}$ such time period, which extends from $\boldsymbol{t} = \boldsymbol{t}_{k-1}$ until $\boldsymbol{t} = \boldsymbol{t}_k$ (with $\boldsymbol{t}_0 = \boldsymbol{0}$ and $\boldsymbol{t}_m = +\infty$), any target of the first k priorities will be engaged immediately upon acquisition. Any other lower-priority target type (i.e. target types k+1 through m) that is acquired during this $k^{\underline{th}}$ such time period will not be engaged immediately, but will be held under surveillance as long as LOS exists to it. At the end of the $k^{\underline{th}}$ time period (that occurs at time $\boldsymbol{t} = \boldsymbol{t}_k$ (which is called the $k^{\underline{th}}$ transition time[12])), any such $(k+1)^{\underline{st}}$ priority target that is under surveillance and still visible will be engaged immediately. This ability to re-

[12] Bonder and Farrell (e.g. see Miller et al. [1978] or CCTC [1979]) use the term "search cut-off time." See also Taylor [2000d], [2000e].

member the location of a lower-priority target at the k$^{\underline{th}}$ transition time will be termed "look-back" capability. For our initial analysis here, however, it will be ignored.

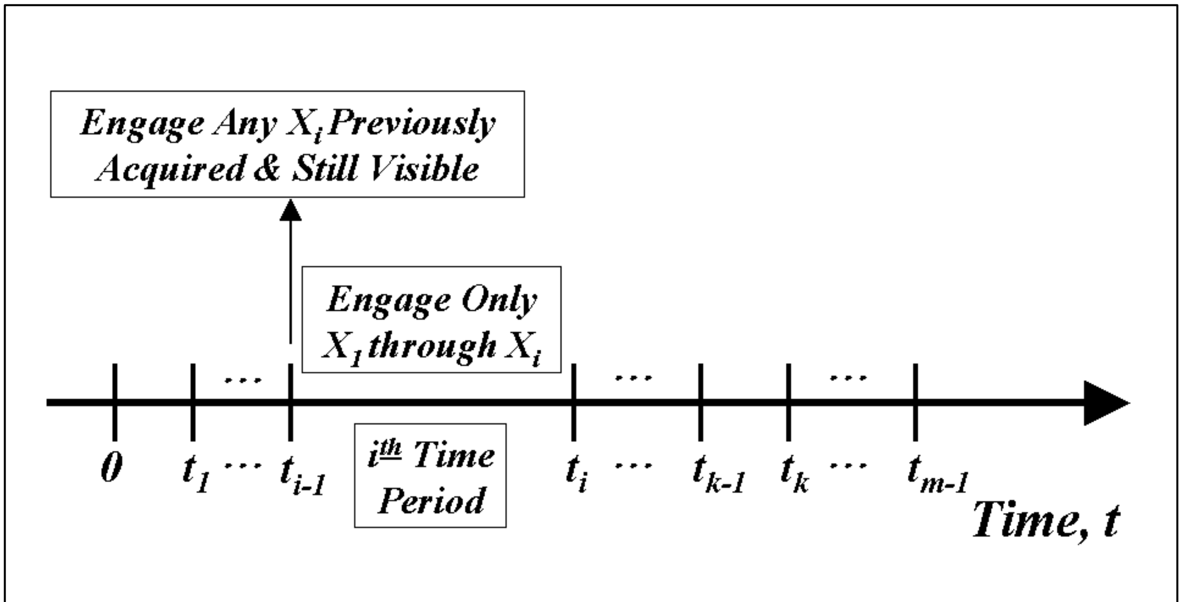

Fig. 3. Bonder-Farrell target-engagement policy (m periods).

6. Assumptions for Target Acquisition.

Concerning the target-acquisition process, it is assumed that

(I) all targets of a particular type are identical,

(II) all targets behave independently of each other,

(III) the time required to acquire a particular target type is exponentially distributed with a rate denoted as λ.

The rate at which a Y_j firer type acquires a particular X_i target (when there is only a single target present) will be denoted as $l_{X_iY_j}$ (single-target-acquisition rate for an individual Y_j firer type against X_i targets).

An important consequence of the above assumptions is that if there are $\boldsymbol{n}_{X_i}$ targets (an integer number) of type X_i present in the field of view of a Y_j observer, then the rate at which this individual Y_j firer type acquires the next such target is given by $\boldsymbol{n}_{X_i}\ l_{X_iY_j}$. A Lanchester-type model of force-on-force attrition, however, does not consider an integer number of combatants of a particular type, but uses a real (nonnegative) number (which may be thought of as an approximation to the average number of combatants (e.g. see Taylor [1983, Section 4.12])). If we let $\boldsymbol{x}_i$ denote the number[13] of X_i combatants available to be acquired, then the rate at which an individual Y_j firer type acquires the next such target is approximated by $\boldsymbol{x}_i\ l_{X_iY_j}$. Furthermore, under conditions of intermittent LOS modeled by the two-state Markov chain discussed below in Section 7, the rate at which an

[13] As just noted, such a (nonnegative real) number may be considered to be an approximation to the average number of X_i combatants.

individual Y_j firer type acquires the next such target can be written as $\boldsymbol{P_{LOS}}\ \lambda_{X_iY_j}\ \boldsymbol{\xi_i}$, where the (steady-state) probability of having LOS exist between the observer type Y_j and the X_i target type is given by (e.g. see Taylor [1982], [2000c], [2000f])

$$\boldsymbol{P_{LOS}} = \frac{\eta}{\eta + \mu}. \qquad \textbf{(3)}$$

In models like VIC, such a probability depends only on the position of observer and target on the terrain and depends on the distance (range) between them.

Thus, the (net) rate at which an individual Y_j firer type acquires X_i targets is given by $\boldsymbol{P_{LOS}}\ \lambda_{X_iY_j}\ \boldsymbol{\xi_i}$. Suppressing the designation of a specific Y firer type as discussed in Section 5 above, one will find it convenient to denote this net rate at which the Y firer acquires the $i^{\underline{th}}$ X target type simply as λ_i. In other words, for $\boldsymbol{i = 1,\ldots, m}$

$$\lambda_i = \boldsymbol{P_{LOS}}\ \lambda_{X_iY_j}\ \boldsymbol{\xi_i}. \qquad \textbf{(4)}$$

It will also be convenient to introduce the rate of acquiring any for the first i highest-priority X target types, denoted as Λ_i, which is given by

$$\Lambda_i = \sum_{k=1}^{i} \lambda_k. \qquad \textbf{(5)}$$

7. Other Assumptions and Summary.

For the reader's convenience, the assumptions that have been made to obtain the results of this paper will be summarized and presented here in an organized fashion. Some of these have already been noted above. A more detailed development of the attrition model considered here is to be found in Taylor [2001d], where model development is related to analysis of the target-engagement cycle.

Thus, because of the central role played by the target-engagement cycle in the determination of a Lanchester attrition-rate coefficient, it seems only fitting to organize this presentation around the elements of the target-engagement cycle. Accordingly, we consider the following aspects of the overall attrition process

(I) LOS-between-firer-and-target process,
(II) target-acquisition process for a firer type,
(III) target priorities for a firer type,
(IV) target-engagement policy for a firer type,
(V) target-attack process (i.e. firing-at-an-acquired-target process) for a firer type against a given target type,
(VI) termination conditions for target engagement.

The assumptions concerning the overall attrition process that have been made to obtain our new results will now be summarized and presented within the framework of the above attrition-process categories (I) through (VI).

Concerning the LOS-between-firer-and-target process (I), it is assumed that this process functions independently of the other five components of the overall attrition process. For a given firer/observer type and given target type (i.e. firer/target-type pair), it is assumed that the target can be in either one of two states: (1) invisible to the observer (i.e. LOS does not exist between the target and the observer), or (2) visible to the observer (i.e. LOS exists between the target and the observer). The time that the target spends in each of these states is exponentially distributed, with η denoting the rate at which a particular target becomes visible to the observer and μ denoting the rate at which it becomes invisible (i.e. rate of losing LOS) to the observer. Thus, $(1/\eta)$ denotes the expected time that the target is invisible, and $(1/\mu)$ denotes the expected time that the target is visible (see Taylor [2001d,Section 7.1] for further details; also Taylor [2000g]).

Concerning the target-acquisition process for a firer type (II), it is assumed that

(a) all targets of a particular type are identical,
(b) all targets behave independently of each other,
(c) the time required to acquire a particular target is exponentially distributed with a rate denoted as λ.

The rate at which a Y_j firer type acquires a particular X_i target will be denoted as $l_{X_iY_j}$. This rate can be referred to as the single-target acquisition rate for an individual Y_j firer type against X_i targets, since it is the applicable rate when there is only one X_i target type present (see Section 6 above for further details).

Concerning the target priorities for a firer type (III), it is assumed that each (and every) firer type has its own list of target priorities (see Section 5 above and Taylor [2001d, Section 9] for further details). Concerning the target-engagement policy for a firer type (IV), it is assumed that Bonder and Farrell's m-period target-engagement policy holds (see Sections 4 and 5 above).

Concerning the target-attack process (i.e. firing-at-an-acquired-target process) for a firer type against a given target type (V), it is assumed that the time to kill an acquired enemy target is exponentially distributed, with α_{ij} denoting the rate at which a Y_j firer type kills an acquired X_i target type and similarly for β_{ji} (see Taylor [2001d, Section 7.1] for further details). Concerning the termination conditions for target engagement (VI), it is assumed that an engagement ends when either one of the following two terminal states is reached

(a) target killed,
(b) LOS lost.

See Taylor [2001d, Section 7.1] for further details.

Finally, the assumptions made here differ from those made in the original work by Bonder and Farrell in the following ways

(a) conditions for engagement termination,
(b) target-engagement policy adopted.

See Taylor [2001d, Section 7] for further details.

8. Some Preliminary Mathematical Results.

The two key mathematical results that are required for the development of the next-target-type-to-be-engaged probability for Bonder and Farrell's m-period target-engagement policy are the following

(1) probability that one random variable is less than another for a finite interval of time and exponential variates,
(2) probability of engaging lower-priority target type previously acquired and still visible[14].

[14] K. Saeger has suggested taking the probability that the target is in the state of being acquired and visible (see, for example, Taylor [1982, Section 7], [2000c], [2000f] for a discussion of the system states for a typical firer).

The following result has played an essential role in the calculation of $P^{eng}_{X_iY_j}$ for any particular target-engagement policy, especially Bonder and Farrell's m-period target-engagement policy given above.

<u>Theorem 1.</u> Let S and T denote two independent random variables, exponentially distributed with rates denoted as λ_S and λ_T, respectively, e.g.

$$P\left[T \le t\right] = F_T(t) = 1 - e^{-\lambda_T t}.$$

It follows that

$$P\left[S \le T \,\middle|\, S \le t_1\right] = \left(\frac{\lambda_S}{\lambda_S + \lambda_T}\right)\left\{1 - e^{-(\lambda_S + \lambda_T)t_1}\right\}.$$

The proof of Theorem 1 is straight forward (cf. Taylor [1983, Appendix B]) and therefore omitted. It should be noted that Theorem 1 holds whether or not t_1 is finite. More general results are given in Taylor [1983, Appendix B], but always for an infinite period of time. The above result for a finite interval of time is essential for developing the author's important new result for $P^{eng}_{X_iY_j}$ for Bonder and Farrell's m-period target-engagement policy given above.

The following result is the other key result for the calculation of $P^{eng}_{X_iY_j}$ for Bonder and Farrell's m-period target-engagement policy given above.

<u>Theorem 2.</u> Consider Bonder and Farrell's m-period target-engagement policy described in Section 3 above. Assume that Bonder and Farrell's LOS and target-acquisition processes described in Taylor [2000c, Section 5] hold. Consider a population of size n_i of the $i^{\underline{th}}$ priority X target type. The probability that one or more such X_i targets that have been acquired in the interval [0, t_{i-1}] are still visible at time t_{i-1} is given by (for i = 2,…,m)

$$P^{SVX}_{i-1} = 1 - \left(1 - p^{SV1}_{i-1}\right)^{n_i}, \qquad (6)$$

where

$$p^{SV1}_{i-1} = \begin{cases} \left(\dfrac{\lambda'_i}{\lambda'_i - \mu_i}\right)\left(e^{-\mu_i t_{i-1}} - e^{-\lambda'_i t_{i-1}}\right) & \textbf{for } \lambda'_i \ne \mu_i, \\ \lambda'_i t_{i-1} e^{-\lambda'_i t_{i-1}} & \textbf{for } \lambda'_i = \mu_i. \end{cases} \qquad (7)$$

In the above

$$\mu_i = \mu_{X_iY_j} \quad \textbf{and} \quad \lambda'_i = P_{LOS}\,\lambda_{X_iY_j}. \qquad (8)$$

To prove the above theorem, one starts by considering a single target of type X_i. Denoting p^{SV1}_{i-1} simply as p_{i-1}, one easily sees that the probability that such a target that is acquired before time t_{i-1} will still be visible at time t_{i-1} is given by

$$p_{i-1} = \sum_{\substack{all \\ possibilities}} Prob\begin{pmatrix}\text{no acquisition}\\ \text{in } [0,t]\end{pmatrix} Prob\begin{pmatrix}\text{acquire between}\\ t \text{ and } t+dt\end{pmatrix} Prob\begin{pmatrix}\text{target still}\\ \text{visible at } t_{i-1}\end{pmatrix},$$

or

$$p_{i-1} = \int_0^{t_{i-1}} e^{-\lambda'_i t} e^{-\mu_i (t_{i-1} - t)} \lambda'_i \, dt,$$

whence readily follows the result for p_{i-1}^{SV1}. The result for P_{i-1}^{SVX} follows from consideration of n_i independent targets.

9. Probability of Next Target Type to Be Engaged (Serial Acquisition).

In this section the general formula for the probability of the next target type to be engaged, denoted as $P_{ij}^{eng} = P_{X_iY_j}^{eng}$, for Bonder and Farrell's m-period target-engagement policy is stated and then proved. For simplicity, it is convenient to suppress the second subscript, here j (see Section 5 above), since it will always be the same (and using it just makes formulas appear longer without any additional real information being provided). Thus, P_{ij}^{eng} becomes simply P_i^{eng}, and the acquisition rate for the i^{th} X target type becomes (see Section 6 above)

$$\lambda_i = P_{LOS_{X_iY_j}} \lambda_{X_iY_j} x_i \qquad \textit{for } i = 1, \ldots, m.$$

Also, denote P_{i-1}^{SVX} of Theorem 2 simply as P_{i-1}. Furthermore, it is convenient to adopt the following conventions

$$\prod_{k=i}^{j} (\,) = 1 \quad \textit{for } i > j,$$

and

$$\sum_{k=i}^{j} (\,) = 0 \quad \textit{for } i > j.$$

Then, one can write the final result for the next-target-type-to-be-engaged probability, denoted as P_i^{eng}, in simplest form as (for $1 \le i \le m$)

$$P_i^{eng} = \left\{ \prod_{n=1}^{i-2} (1 - P_n) \right\} \left\{ \prod_{n=1}^{i-1} e^{-\left(\sum_{j=1}^{n} \lambda_j\right)(t_n - t_{n-1})} \right\} P_{i-1}$$
$$+ \sum_{k=i}^{m} \left\{ \left[\prod_{n=1}^{k-1} (1 - P_n) \right] \left[\prod_{n=1}^{k-1} e^{-\left(\sum_{j=1}^{n} \lambda_j\right)(t_n - t_{n-1})} \right] \left(\frac{\lambda_i}{\sum_{n=1}^{k} \lambda_n} \right) \left[1 - e^{-\left(\sum_{n=1}^{k} \lambda_n\right)(t_k - t_{k-1})} \right] \right\}. \qquad (9)$$

In the above formula the following conventions have been followed: $t_0 = 0$, $t_m = +\infty$, and $P_0 = 0$. Equation (9) is the main theoretical result of the paper at hand. Since Taylor [2001e] has already developed an explicit analytical expression for the expected time to acquire the next target to be engaged, it allows one to compute an explicit analytical expression for a Lanchester attrition-rate coefficient for Bonder and Farrell's m-period target-engagement policy for direct-fire combat in a heterogeneous-target environment with serial acquisition of targets.

Equation (9) may be developed by straightforward probability arguments. Obtaining motivation from Fig. 3 one can write (again adopting the conventions that $t_0 = 0$ and $t_m = +\infty$)

$$P_i^{eng} = Prob\left[X_i \textit{ engaged at } t_{i-1}\right] + \sum_{k=i}^{m} Prob\left[X_i \textit{ engaged in} \left[t_{k-1}, t_k\right]\right]. \qquad (10)$$

It is also necessary to adopt the convention Prob[X_1 engaged at $t_0 = 0$] = 0. The reader should keep in mind that in Equation (10) above the probability that X_i is engaged really means the probability that X_i is first engaged. For the development of Equation (9), one begins by considering Prob[X_i engaged at t_{i-1}].

Thus, one considers

$$\textit{Prob}\left[X_i \text{ first engaged at } t_{i-1}\right] =$$

$$\textit{Prob}\begin{bmatrix}\text{no higher - priority target}\\ \text{engaged before } t_{i-1}\end{bmatrix} \cdot \textit{Prob}\begin{bmatrix}\text{some priviously - acquired}\\ X_i \text{ still visible at } t_{i-1}\end{bmatrix}. \quad (11)$$

But the probability that some previously-acquired X_i target is still visible at t_{i-1} is simply the probability that this previously-acquired X_i target will be engaged at t_{i-1}. This latter probability is just what has been denoted as P_{i-1}. Furthermore, the probability that no higher-priority target will be engaged before t_{i-1} is given by the product of the probability that no higher-priority target type will be engaged at any earlier transition time times the probability that no higher-priority target type will be engaged in any earlier time period. The former probability is simply given by

$$\textit{Prob}\begin{bmatrix}\text{no higher - priority target engaged}\\ \text{at any earlier transition time}\end{bmatrix} = \prod_{k=1}^{i-1}\left(1 - P_k\right). \quad (12)$$

Furthermore, the probability that no higher-priority target will be engaged in any earlier time period is given by

$$\textit{Prob}\begin{bmatrix}\text{no higher - priority target engaged}\\ \text{in any earlier time period}\end{bmatrix} = \prod_{k=1}^{i-1} e^{-\left(\sum_{j=1}^{k}\lambda_j\right)\left(t_k - t_{k-1}\right)}. \quad (13)$$

In particular, in the *(i-1)*st time period no target type from X_1 through X_{i-1} can be acquired. The probability of this occurring is given by

$$\textit{Prob}\begin{bmatrix}\text{no higher - priority target acquired}\\ \text{in } (i-1)^{st} \text{ time period}\end{bmatrix} = \prod_{k=1}^{i-1} e^{-\lambda_k\left(t_{i-1} - t_{i-2}\right)} = e^{-\left(\sum_{k=1}^{i-1}\lambda_k\right)\left(t_{i-1} - t_{i-2}\right)}, \quad (14)$$

whence follows Equation (13). As stated above, the probability that no higher-priority target type will be engaged before t_{i-1} is given by

$$\textit{Prob}\begin{bmatrix}\text{no higher - priority target}\\ \text{engaged before } t_{i-1}\end{bmatrix} = \textit{Prob}\begin{bmatrix}\text{no higher - priority target engaged}\\ \text{at any earlier transition time}\end{bmatrix} \textit{Prob}\begin{bmatrix}\text{no higher - priority target engaged}\\ \text{in any earlier time period}\end{bmatrix}. \quad (15)$$

Considering the definition of P_{i-1} and Equations (11) through (15) above, one consequently finds that

$$\textit{Prob}\left[X_i \text{ first engaged at } t_{i-1}\right] = \left[\prod_{n=1}^{i-1}\left(1 - P_n\right)\right]\left[\prod_{n=1}^{i-1} e^{-\left(\sum_{j=1}^{n}\lambda_j\right)\left(t_n - t_{n-1}\right)}\right] P_{i-1}. \quad (16)$$

One next considers the probability that X_i is engaged in $[t_{k-1}, t_k]$, i.e. Prob[X_i engaged in $[t_{k-1}, t_k]$].

Thus, one considers for $k = i, \ldots, m$

$$Prob\left[X_i \text{ first engaged in } [t_{k-1}, t_k]\right] =$$

$$Prob\left[\begin{matrix}\text{no higher - priority target} \\ \text{engaged before } [t_{k-1}, t_k]\end{matrix}\right] \times Prob\left[\begin{matrix}X_i \text{ acquired before any} \\ \text{other } X \text{ target type in } [t_{k-1}, t_k]\end{matrix}\right]. \qquad (17)$$

Recalling that the time period $[t_{k-1}, t_k]$ denotes the k^{th} period of Bonder and Farrell's m-period target engagement policy during which target types X_1 through X_k are to be engaged immediately upon acquisition, one has that the probability that X_i is engaged in $[t_{k-1}, t_k]$ is given by

$$Prob\left[X_i \text{ engaged in } [t_{k-1}, t_k]\right] = Prob\left[\begin{matrix}X_i \text{ acquired before any} \\ \text{other } X \text{ target type in } [t_{k-1}, t_k]\end{matrix}\right] =$$

$$Prob\left[T_i = \min\{T_1, \ldots, T_k\} \mid 0 \le T_i \le (t_k - t_{k-1})\right]. \qquad (18)$$

Repeatedly applying Theorem 1 for the probability that one random variable is less than another in a finite interval of time, one finds that

$$Prob\left[T_i = \min\{T_1, \ldots, T_k\} \mid 0 \le T_i \le (t_k - t_{k-1})\right] = \left(\frac{\lambda_i}{\sum_{j=1}^{k} \lambda_j}\right)\left(1 - e^{-\left(\sum_{j=1}^{k} \lambda_j\right)(t_k - t_{k-1})}\right). \qquad (19)$$

Moreover, the probability that no higher-priority will be engaged before $[t_{k-1}, t_k]$ is given by the product of the probability that no higher-priority target type will be engaged at any earlier transition time times the probability that no higher-priority target type will be engaged in any earlier time period. The former probability is simply given by

$$Prob\left[\begin{matrix}\text{no higher - priority target engaged} \\ \text{at any earlier transition time}\end{matrix}\right] = \prod_{n=1}^{k-1} (1 - P_n). \qquad (20)$$

Furthermore, the probability that no higher-priority target type will be engaged in any earlier time period is given by

$$Prob\left[\begin{matrix}\text{no higher - priority target engaged} \\ \text{in any earlier time period}\end{matrix}\right] = \prod_{n=1}^{k-1} e^{-\left(\sum_{j=1}^{n} \lambda_j\right)(t_n - t_{n-1})}. \qquad (21)$$

As stated above, the probability that no higher-priority target type will be engaged before in $[t_{k-1}, t_k]$ is given by

$$Prob\left[\begin{matrix}\text{no higher - priority target} \\ \text{engaged before } [t_{k-1}, t_k]\end{matrix}\right] = Prob\left[\begin{matrix}\text{no higher - priority target engaged} \\ \text{at any earlier transition time}\end{matrix}\right] Prob\left[\begin{matrix}\text{no higher - priority target engaged} \\ \text{in any earlier time period}\end{matrix}\right]. \qquad (22)$$

Combining Equations (17) through (22), one readily finds that for $k = 1, \ldots, m$ and $i \le k$

$$\boldsymbol{Prob\left[X_i \text{ first engaged in} \left[t_{k-1}, t_k\right]\right] =}$$

$$\left[\prod_{n=1}^{k-1}\left(1 - P_n\right)\right]\left[\prod_{n=1}^{k-1} e^{-\left(\sum_{j=1}^{n}\lambda_j\right)\left(t_n - t_{n-1}\right)}\right]\left[\frac{\lambda_i}{\sum_{j=1}^{k}\lambda_j}\right]\left[1 - e^{-\left(\sum_{j=1}^{k}\lambda_j\right)\left(t_k - t_{k-1}\right)}\right]. \qquad (23)$$

Equation (9) now readily follows by substituting Equations (16) and (23) into Equation (10).

10. Summation over All Target Types.

An important cross check on the theoretical correctness of the probability distribution for the next target type to be engaged, denoted as P_i^{eng} for $i = 1,\ldots,m$, given by equation (9) above is to be able to show that its sum over all target types is equal to 1. Thus, we will show that

$$\sum_{i=1}^{m} P_i^{eng} = 1. \qquad (24)$$

However, our proof involves summing over different numbers of target types so that it is essential to explicitly denote the total number of target types present. Thus, we will henceforth in this section denote P_i^{eng} as $P_{i,m}$ to make explicit the total number of target types present. Proving (24) then amounts to proving that

$$\sum_{i=1}^{m} P_{i,m} = 1. \qquad (25)$$

Furthermore, the complexity of (9) dictates that we express (9) in a more convenient form for proving (25).

Thus, it is now convenient to define for $i = 2,\ldots,m$

$$q_i = Prob\left[X_i \text{ engaged at time } t_{i-1}\right]. \qquad (26)$$

It follows that $q_1 = 0$, since $P_0 = 0$ at $t_0 = 0$. If we recursively define F_n for $n = 0,1,\ldots,m-1$

$$F_{n+1} = e^{-\Lambda_{n+1}\left(t_{n+1} - t_n\right)}\left(1 - P_{n+1}\right)F_n, \qquad (27)$$

with $F_0 = 1$, we can then write for $n = 2,\ldots,m$

$$q_n = F_{n-2}\, e^{-\Lambda_{n-1}\left(t_{n-1} - t_{n-2}\right)} P_{n-1}, \qquad (28)$$

where we have made use of the notation (5). Also, define $T_{i,j,m}$ for $1 \le i \le j \le m-1$ as

$$T_{i,j,m} = F_{j-1}\left(\frac{\lambda_i}{\Lambda_j}\right)\left\{1 - e^{-\Lambda_j\left(t_j - t_{j-1}\right)}\right\}, \qquad (29)$$

and for $j = m$ as (since $t_m = +\infty$)

$$T_{i,m,m} = F_{m-1}\left(\frac{\lambda_i}{\Lambda_m}\right). \qquad (30)$$

It is important to note that

$$\sum_{i=1}^{j} T_{i,j,m} = \begin{cases} F_{j-1}\left(1 - e^{-\Lambda_j (t_j - t_{j-1})}\right) & \text{for } j = 1,\dots,m-1, \\ F_{m-1} & \text{for } j = m. \end{cases} \qquad (31)$$

From the above definitions, it follows that (9) can be written as

$$P_{i,m} = q_i + \sum_{j=i}^{m} T_{i,j,m} . \qquad (32)$$

Our approach for proving (25) is motivated by first considering the sum of the first two terms. It is therefore convenient to note that

$$P_{1,m} = T_{1,1,m} + T_{1,2,m} + \sum_{j=3}^{m} T_{1,j,m} , \qquad (33)$$

and

$$P_{2,m} = q_2 + T_{2,2,m} + \sum_{j=3}^{m} T_{2,j,m} . \qquad (34)$$

Observing that $T_{1,1,m} = \left(1 - e^{\Lambda_1 t_1}\right)$ and that $q_2 = e^{-\Lambda_1 t_1}$, we may combine equations (33) and (34) to obtain

$$\sum_{i=1}^{2} P_{i,m} = \left(1 - e^{\Lambda_1 t_1}\right) + e^{-\Lambda_1 t_1} P_1 + F_1\left\{1 - e^{\Lambda_2 (t_2 - t_1)}\right\} + \sum_{i=1}^{2}\left\{\sum_{j=3}^{m} T_{i,j,m}\right\}, \qquad (35)$$

which is readily shown by the above to yield

$$\sum_{i=1}^{2} P_{i,m} = 1 - F_1 e^{\Lambda_2 (t_2 - t_1)} + \sum_{i=1}^{2}\left\{\sum_{j=3}^{m} T_{i,j,m}\right\}. \qquad (36)$$

An inductive argument then readily yields that for $2 \le n \le m-1$

$$\sum_{i=1}^{n} P_{i,m} = 1 - F_{n-1} e^{\Lambda_n (t_n - t_{n-1})} + \sum_{i=1}^{n}\left\{\sum_{j=n+1}^{m} T_{i,j,m}\right\}. \qquad (37)$$

Finally, one finds that for $n = m$

$$\sum_{i=1}^{m} P_{i,m} = \sum_{i=1}^{m-1} P_{i,m} + P_{m,m} = 1 - F_{m-2} e^{\Lambda_{m-1} (t_{m-1} - t_{m-2})} + \sum_{i=1}^{m-1}\left\{\sum_{j=m}^{m} T_{i,j,m}\right\} + q_m + T_{m,m,m} , \qquad (38)$$

or

$$\sum_{i=1}^{m} P_{i,m} = 1 - F_{m-2} e^{\Lambda_{m-1} (t_{m-1} - t_{m-2})} + \sum_{i=1}^{m} T_{i,m,m} + F_{m-2} e^{\Lambda_{m-1} (t_{m-1} - t_{m-2})} P_{m-1} , \qquad (39)$$

or

$$\sum_{i=1}^{m} P_{i,m} = 1 - F_{m-2} e^{\Lambda_{m-1} (t_{m-1} - t_{m-2})} \left(1 - P_{m-1}\right) + F_{m-1} . \qquad (40)$$

Recalling that the second term on the right-hand side of (40) is equal to $\boldsymbol{F_{m-1}}$ by (27), we see that (25) has been proven.

11. Expected Time to Acquire Target That Will Be Engaged.

Recently Taylor [2001e] has shown that the expected time to acquire a target that will be engaged according to Bonder and Farrell's m-period target-engagement policy, denoted as $E\left[T_{\alpha_Y}\right]$, is given by

$$E\left[T_{\alpha_Y}\right] = \sum_{k=1}^{m} \left\{ \prod_{j=1}^{k} e^{-\Lambda_j (t_j - t_{j-1})} (1 - P_j) \right\} \frac{1}{\Lambda_k} \left\{ 1 - e^{-\Lambda_k (t_k - t_{k-1})} \right\}, \qquad (41)$$

where $t_0 = 0$ and $t_m = +\infty$. The above expression was developed by considering a piecewise-constant acquisition rate equal to Λ_k in the k^{th} subinterval of Bonder and Farrell's m-period target-engagement policy.

For simplicity we have suppressed the firer-type index j in the writing of (41) as discussed above in Section 5. When the firer-type index j is re-introduced into (41), $E\left[T_{\alpha_Y}\right]$ becomes $E\left[T_{\alpha_{Y_j}}\right]$, which is then used in equation (2) in the computation of a numerical value for a Lanchester attrition-rate coefficient. However, in the paper at hand it will be more convenient to suppress the firer-type designation in the general expression for a Lanchester attrition-rate coefficient like (2) above.

12. Explicit Analytical Expression for Lanchester Attrition-Rate Coefficient.

Because of the complexity of the expressions for the expected time to acquire a target that will be engaged and the next-target-type-to-be-engaged probability, it is convenient to suppress the firer-type designation in (2) and write the general expression for the rate a which a single typical Y_j firer type kills X_i target types with direct fire in a heterogeneous-target environment with serial acquisition of targets as follows

$$a_i^{ser} = \frac{\left\{ \dfrac{P_i^{eng}}{\alpha_i + \mu} \right\} \alpha_i}{\left\{ E\left[T_{\alpha_Y}\right] + \sum_{k=1}^{n_X} \left\{ \dfrac{P_k^{eng}}{\alpha_k + \mu} \right\} \right\}}, \qquad (42)$$

where α_i denotes the rate at which an individual Y firer type (our generic Y firer type discussed in Section 5 above) kills acquired X_i targets. As stressed by Taylor (e.g. see Taylor [1999], [2000b], [2001d]), the two key intermediate quantities

(1) expected time to acquire a target that will be engaged,

(2) next-target-type-to-be-engaged probability,

depend on the target-engagement policy adopted, while all the other quantities in (42) do not depend on it. Our new important explicit analytical results for Bonder and Farrell's m-period target-engagement policy are obtained by specifying these two key intermediate quantities. It will be convenient to state again here these quantities given above.

Thus, in the above expression the next-target-type-to-be-engaged probability according to Bonder and Farrell's m-period target-engagement policy is given by (for $i = 1, \ldots, m$)

$$P_i^{eng} = \left\{ \prod_{n=1}^{i-2} (1 - P_n) \right\} \left\{ \prod_{n=1}^{i-1} e^{-\Lambda_n (t_n - t_{n-1})} \right\} P_{i-1} + \sum_{k=i}^{m} \left\{ \prod_{n=1}^{k-1} (1 - P_n) e^{-\Lambda_n (t_n - t_{n-1})} \right\} \left(\frac{\lambda_i}{\Lambda_k} \right) \left[1 - e^{-\Lambda_k (t_k - t_{k-1})} \right]. \qquad (43)$$

Also, the expect time to acquire a target that will be engaged according to Bonder and Farrell's target-engagement policy is given by

$$E\left[T_{a_Y} \right] = \sum_{k=1}^{m} \left\{ \prod_{j=1}^{k} e^{-\Lambda_j (t_j - t_{j-1})} \left[1 - P_j \right] \right\} \frac{1}{\Lambda_k} \left\{ 1 - e^{-\Lambda_k (t_k - t_{k-1})} \right\}. \qquad (44)$$

In both these expressions it has been convenient to use the compound acquisition rate given by (5). Also, as above the following conventions apply to these formulas: $t_0 = 0$, $t_m = +\infty$, and $P_0 = 0$. Equation (42) with the two key intermediate quantities given by (43) and (44) provides an explicit analytical expression for a Lanchester attrition-rate coefficient for Bonder and Farrell's m-period target-engagement policy.

13. Limiting-Case Analysis of Lanchester-Attrition-Rate-Coefficient Result.

A necessary (but by no means sufficient) condition for theoretical correctness of a single-weapon-system-type kill rate is that the limiting-case behavior of results be consistent with the limiting-case result obtained by other, independent means (Taylor [2000a]). For example, the author has shown that Bonder and Farrell's (e.g. see Miller et al. [1978]) attrition-rate-coefficient results for parallel acquisition do not exhibit appropriate limiting-case behavior (Taylor [2000a]; see also Shubik [1983]).

If one sets $t_1 = t_2 = \ldots = t_{m-1} = 0$ in Bonder and Farrell's m-period target-engagement policy discussed in Section 4 above, then their complex target-engagement policy reduces to the simple engage-any-target-that-is-acquired target-engagement policy (e.g. see Taylor [2001d]). Substituting these values into (43) and (44), one finds that (no longer suppressing the firer-type index j) equation (42) becomes

$$\alpha_{ij}^{ser} = \frac{\left\{ \dfrac{P_{LOS} \lambda_{X_i Y_j} x_i}{\alpha_{ij} + \mu} \right\} \alpha_{ij}}{\left\{ 1 + \sum_{k=1}^{n_X} \left\{ \dfrac{P_{LOS} \lambda_{X_k Y_j} x_k}{\alpha_{kj} + \mu} \right\} \right\}}, \qquad (45)$$

which is precisely the result given previously for the engage-any-target-that-is-acquired policy by Taylor [2001b, Section 16]. Thus, our new result (41) is consistent with a previously developed result in this particular limiting case[15].

14. Verification of Analytical Expressions by Simulation.

Another cross check that can be made on our new explicit analytical expression for a Lanchester attrition-rate coefficient for direct-fire combat in a heterogeneous-target environment with

[15] Taylor has stressed the importance of subjecting theoretical results to such limiting-case analysis (e.g. see Taylor [2001d, Section 10.4]).

serial acquisition of targets for Bonder and Farrell's m-period target-engagement policy is to verify its numerical value computed by (42) with (43) and (44) against Monte-Carlo simulation results. Since such a kill rate has been computed as the expected number of kills[16] in the target engagement cycle divided by its average length, it seems only appropriate to simulate the target-engagement cycle[17] and then estimate kill rates according to this principle from the output of the simulation. Since all event times are assumed to be exponentially distributed, it was particularly simple to develop such a simulation. In fact, the simulation was implemented on an Excel spreadsheet and exercised for the case of two target types with no look-back capability (see Section 5 above).

The overall flow chart upon which the Monte-Carlo simulation of the target-engagement cycle is shown in Fig. 4. Moreover, a flow chart showing the details for the determination of the target type acquired that will be engaged according to Bonder and Farrell's target-engagement policy is shown in Fig. 5, while a flow chart showing details for the determination of engagement outcome is shown in Fig. 6. This simulation was developed for the case of two target types with no look-back capability. For the determination of the target type acquired depicted in Fig. 5, a net rate of target acquisition given by, for example, $\boldsymbol{P_{LOS}}\ l_{X_1Y_j}\ \boldsymbol{x_1}$ was used. These flow charts were the basis for developing an Excel spreadsheet to implement a Monte-Carlo simulation of the target-engagement cycle. This simulation generated the occurrence of the next casualty and hence a means of verifying the analytical expression for the Lanchester attrition-rate coefficient (42) with the two key intermediate quantities given by (43) and (44)[18]. It did not settle the question, however, of verification of the attrition model, i.e. heterogeneous-force Lanchester-type equations analogous to (1) together with a model for attrition-rate coefficients yielding (42) with (43) and (44) to be used with these equations.

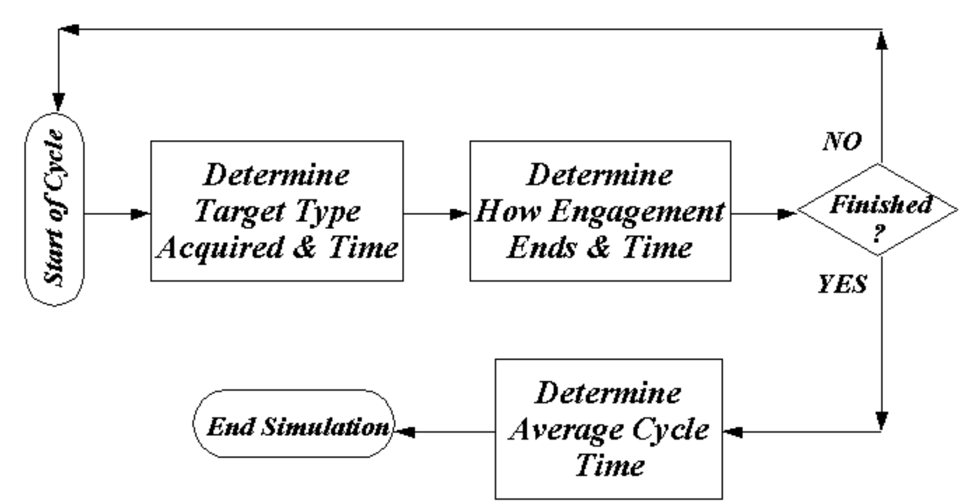

Fig. 4. Flow chart of simulation of target-engagement cycle.

[16] For the appropriate type of target, of course.
[17] For an explicit look at the target-engagement cycle for serial acquisition of targets, see Taylor [2001d].
[18] The same research that developed the analytical expression for the expected time to acquire a target that will be engaged (41), equivalently (44), also verified this analytical expression against Monte-Carlo simulation results (Taylor [2001e, Section 17]). Thus, the work at hand did not need to verify (44).

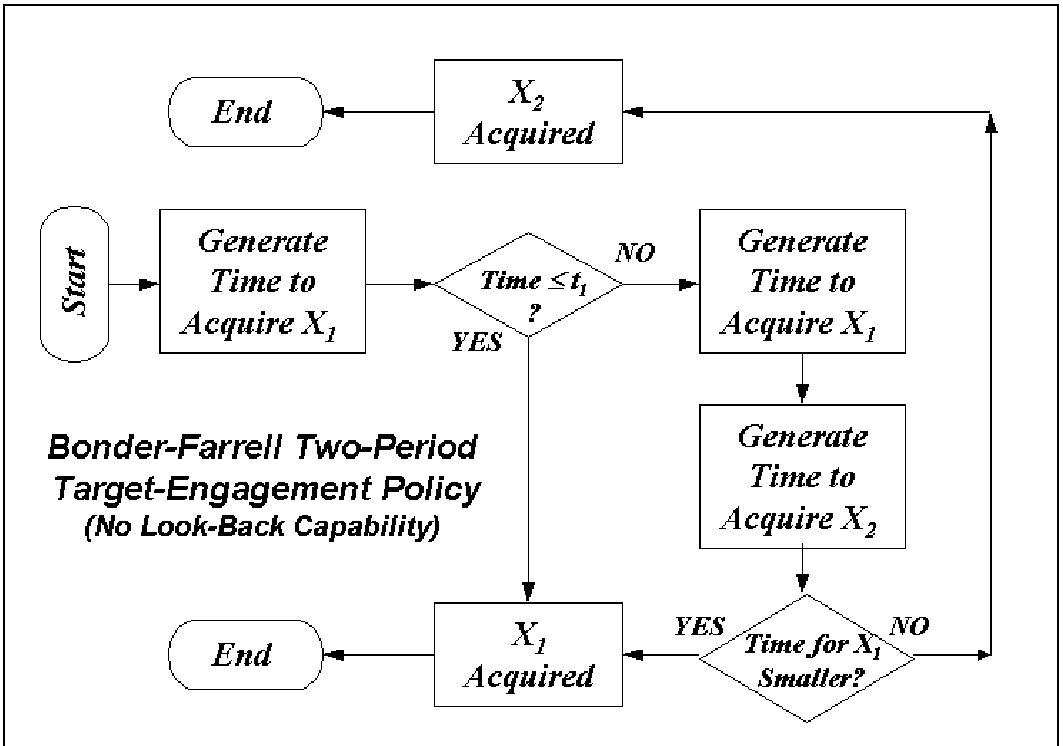

Fig. 5. Flow chart of determination of next target type to be engaged.

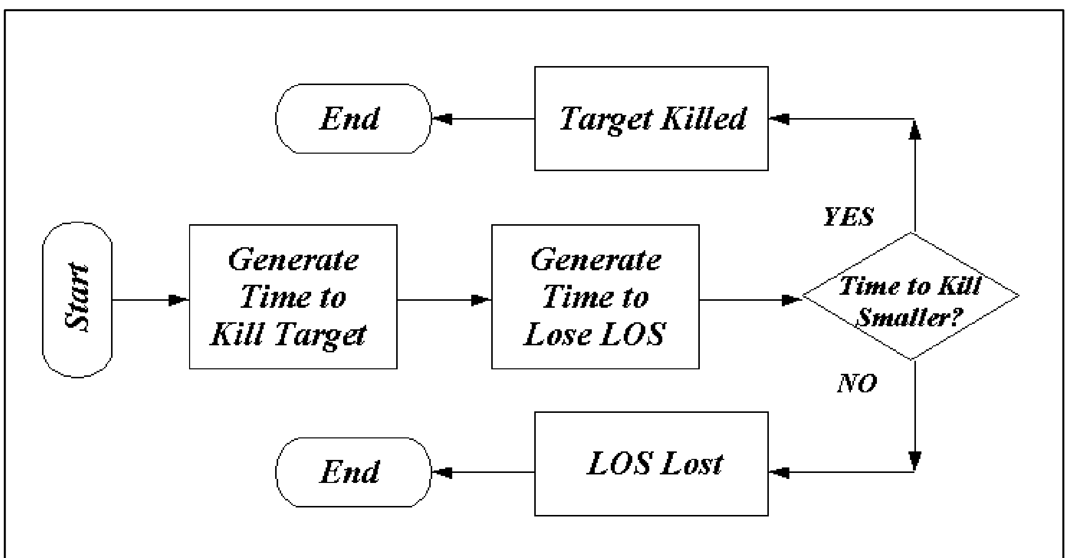

Fig. 6. Flow chart of determination of engagement outcome.

In practice, however, the authors were able to use a variant of the algorithm depicted in Fig. 5 for the determination of the time and type of the next target to be engaged for Bonder and Farrell's target-engagement policy. Although this variant (called the simple method and involving the generation of ***m*** random variables for an m-period Bonder-Farrell target-engagement policy) is extremely convenient (and for larger numbers of target types even necessary) to use for the simulation of the entire target-engagement cycle, earlier work on acquisition of targets that will be engaged by Bonder and Farrell's target-engagement policy (Taylor [2001e]) had simulated the determination of the next target type to be engaged with use of an algorithm based on Fig. 5 directly (or its 3-period extension). This method (called the m-period method), unfortunately, involves generation of ***m!*** random numbers (as opposed to ***m*** for the simple method).

What is this simple method of simulating the acquisition of the next target that will be engaged by Bonder and Farrell's target-engagement policy? For two target types it consists of generating two samples drawn from exponential distributions with rates[19] $P_{LOS}\,\lambda_{X_iY}\,\xi_i$ for $i = 1, 2$. Denote these realizations as t_{a_1} and t_{a_2}. One then computes

$$u_1 = t_{a_1} \qquad \textbf{and} \qquad u_2 = \tau_1 + t_{a_2} \tag{46}$$

where τ_1 denotes the transition time[20] for Bonder and Farrell's 2-period target-engagement policy. If $u_1 \le u_2$, then target type one is determined to have been acquired.

Extensive simulation runs revealed that the simple method yielded a satisfactory estimate for the expected time to acquire a target that will be engaged by Bonder and Farrell's target-engagement policy. This was done by comparing the average of this time to acquire generated from the simulation, denoted as $\hat{\bar{t}}_{a_Y}$ (cf. Taylor [2001e]), with the theoretical value for $\bar{t}_{a_Y}$ computed from (44) with $m = 2$, i.e.

$$\bar{t}_{a_Y} = \frac{1}{\lambda_1}\left[1 - e^{-\lambda_1 \tau_1}\right] + \frac{e^{-\lambda_1 \tau_1}}{(\lambda_1 + \lambda_2)}, \tag{47}$$

where λ_i is given by (4). An Excel spreadsheet was developed to execute the simulation of the time to acquire a target that will be engaged according to the simple method described above. The spreadsheet generated 20,000 replications of the simulation (implemented by copying the basic calculation row 20,000 times). Graphical output from a typical spreadsheet replication is shown in Fig. 7. Twenty replications of this spreadsheet yielded 400,000 replications of the time-to-acquire simulation and an estimate $\hat{\bar{t}}_{a_Y}$ = 1.72429 hours, which compared extremely favorably with the theoretical value of 1.72507. It was therefore concluded that the simple method of simulation would yield satisfactory results for the simulation of the entire target-engagement cycle for Bonder and Farrell's target-engagement policy. Furthermore, such a simulation using the simple method to generate acquisition times for targets that will be engaged would be much more efficient than the 2-period method.

Thus, an Excel spreadsheet was developed to implement the simulation of the target-engagement cycle described above (using the simple method, however, for simulating the acquisition of a target that will be engaged according to Bonder and Farrell's target-engagement policy). This spreadsheet calculated an estimate of the average time to acquire the next target to be engaged, denoted as $\hat{\bar{t}}_{a_Y}$, and also an estimate of the average target-engagement-cycle time, denoted as $\hat{\bar{t}}_{cycle_Y}$. Results for 20,000 replications of the simulation are shown in Fig. 8 below for the average cycle

[19] It is not essential for our purposes here to identify the firer type any more precisely than simply a Y firer.
[20] As pointed out above, Bonder and Farrell (e.g. see Miller et al. [1978] or CCTC [1979]) use the term "search cut-off time."

time. The simulation computed $\hat{\hat{t}}_{a_Y}$ = 1.744 hours, which corresponded very favorably with the theoretical value of 1.725. Also (as shown in Fig. 8), the simulation computed $\hat{\hat{t}}_{cycle_Y}$ = 2.047 hours, which again corresponded very favorably with the theoretical value of 2.029.

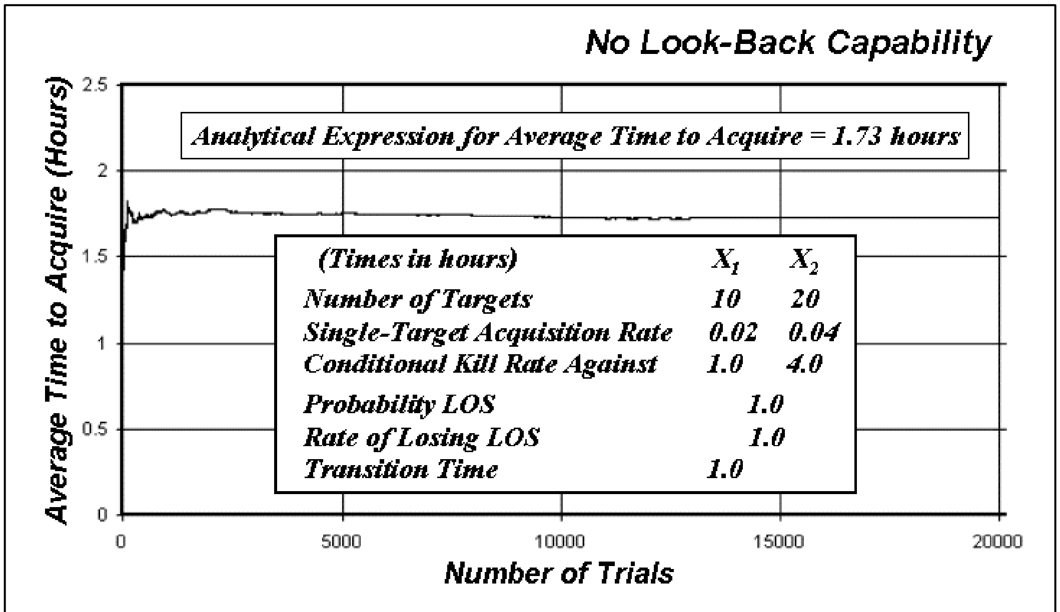

Fig. 7. Graphical simulation results for one replication of spreadsheet simulation for average time to acquire a target that will be engaged for simple method.

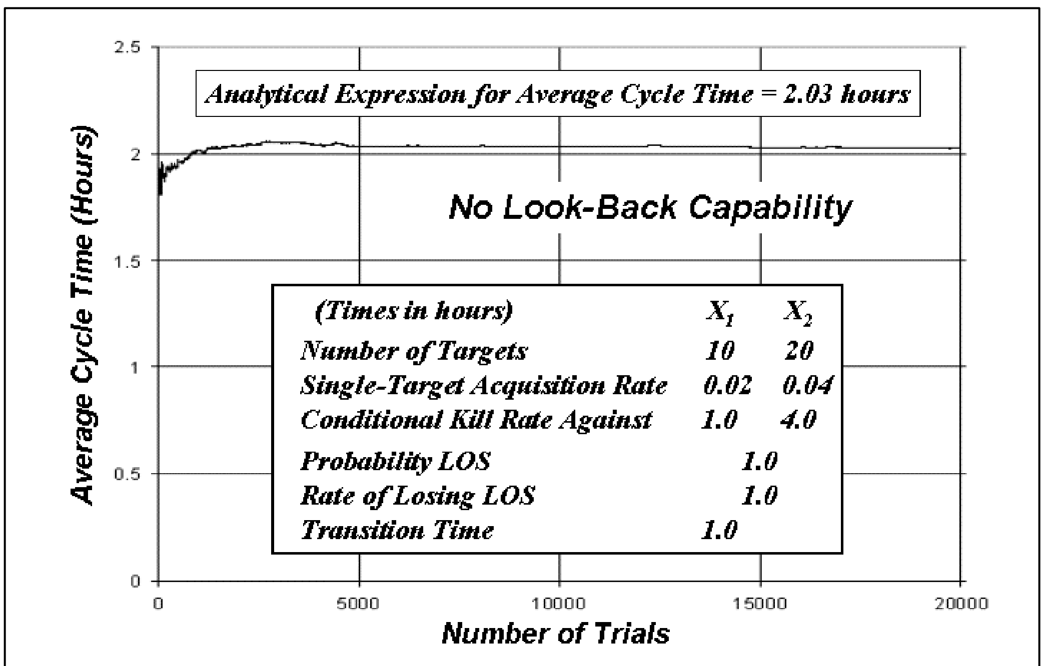

Fig. 8. Simulation results for average target-engagement-cycle time.

The spreadsheet also computed estimates of the kill rates of the Y firer against the two X target types. It computed an estimate of the average number of X_i casualties per engagement (i.e. per replication of the target-engagement cycle), denoted as $\hat{\bar{n}}_{k_{X_iY}}$. Estimates of the kill rates for an individual Y firer against each of the two X target types, denoted as $\hat{\alpha}_i^{ser}$ for $i = 1,2$, were computed by Taylor's Principle as the ratio of the expected number of kills against the particular target type per target-engagement cycle divided by its expected length, namely

$$\hat{\alpha}_i^{ser} = \frac{\hat{\bar{n}}_{k_{X_iY}}}{\hat{\bar{t}}_{k_Y}}. \qquad (48)$$

Theoretical values for these single-weapon-system-type kill rates were computed by (42) with $n_X = 2$, namely

$$\alpha_i^{ser} = \frac{\left(\frac{P_i^{eng}}{\alpha_i + \mu}\right)\alpha_i}{\left\{\frac{1}{\lambda_1}\left(1 - e^{-\lambda_1\tau_1}\right) + \frac{e^{-\lambda_1\tau_1}}{(\lambda_1 + \lambda_2)} + \sum_{k=1}^{n_X}\left(\frac{P_k^{eng}}{\alpha_k + \mu}\right)\right\}}, \qquad (49)$$

where $i = 1,2$ and

$$P_1^{eng} = 1 - e^{-\lambda_1\tau_1} + e^{-\lambda_1\tau_1}(1 - P_1)\left[\frac{\lambda_1}{\lambda_1 + \lambda_2}\right], \qquad (50)$$

$$P_2^{eng} = e^{-\lambda_1\tau_1}P_1 + e^{-\lambda_1\tau_1}(1 - P_1)\left[\frac{\lambda_2}{\lambda_1 + \lambda_2}\right]. \qquad (51)$$

Results for 20,000 replications of the simulation are shown in Fig. 9 below. The simulation computed $\hat{\alpha}_1^{ser}$ = 0.0830 X_1 casualties per hour per Y firer, which corresponded extremely favorably with a theoretical value of 0.0850. Also, it computed $\hat{\alpha}_2^{ser}$ = 0.2561 X_2 casualties per hour per Y firer, which again corresponded extremely favorably with a theoretical value of 0.2583.

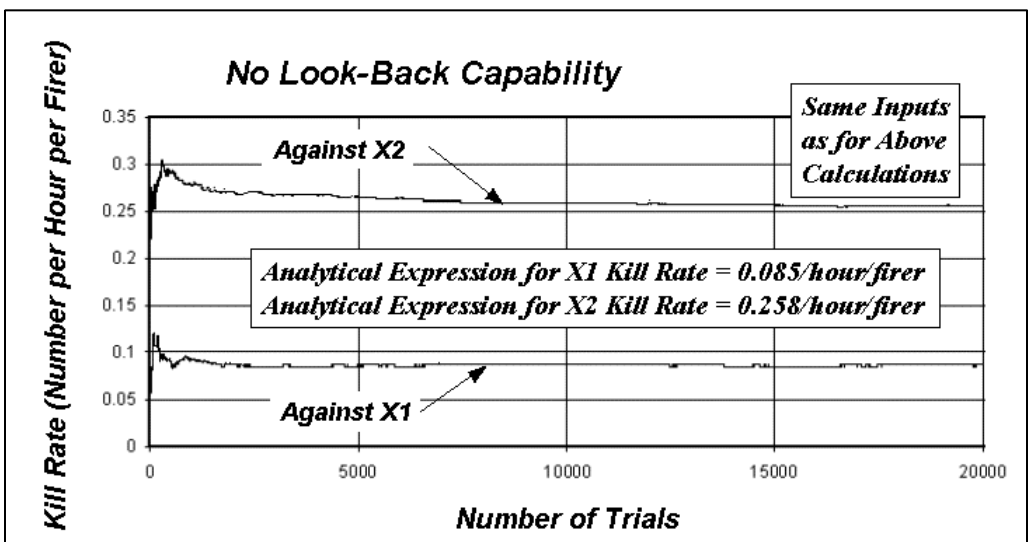

Fig. 9. Simulation results for Y kill rates against X_i target types.

15. Final Comments.

This working paper has developed an explicit analytical expression (42) for a Lanchester attrition-rate coefficient for direct-fire combat in a heterogeneous-target environment with serial acquisition of targets with the two key intermediate quantities (namely, the next-target-type-to-be-engaged probability and the expected time to acquire a target that will be engaged according to Bonder and Farrell's m-period target-engagement policy) given by (43) and (44). The contribution of this paper and its companion (Taylor [2001e]) has been to provide explicit analytical expressions for these two key intermediate quantities for Bonder and Farrell's m-period target-engagement policy, while previous work by Taylor [2001d] had developed the general expression for a Lanchester attrition-rate coefficient for serial acquisition of targets. Remarkably, the authors do not know of any such previously existing explicit analytical results for direct-fire combat for Bonder and Farrell's target-engagement policy, even in the Vector-in-Commander (VIC) documentation (e.g. see TRAC-FLVN [1992]), which does give a somewhat different general expression for a Lanchester attrition-rate coefficient (i.e. one at variance with Taylor's general expression). Thus, some very interesting questions are raised by the work at hand concerning what is actually used for direct-fire attrition in VIC.

The development of explicit analytical expressions for the two key intermediate quantities for Bonder and Farrell's target-engagement policy was made possible by new mathematical-modeling discoveries by Taylor. In particular, Taylor's investigation of target acquisition for a target ensemble (Taylor [2001b, Sections 11 and 12]) made possible the development of an expression for the expected time to acquire a target that will be engaged according to Bonder and Farrell's target-engagement policy. Moreover, Taylor's new results for the probability of one random variable being less than another for a finite interval of time (see Theorem 1 above in Section 8) made possible the development of an expression for the next-target-type-to-be-engaged probability. Taylor and Neta jointly worked out the proof that the sum of this probability over all target types is equal to 1 (see Section 10 above).

As Taylor has emphasized[21], for the purposes of military modeling and analysis it is important to understand the dependence of a Lanchester attrition-rate coefficient on a firer type's target-engagement policy (equivalently, rules of engagement for enemy target types) in a heterogeneous-target environment. He has stressed the importance of expressing a Lanchester attrition-rate coefficient in a form like (2), which depends explicitly on the two key intermediate quantities

(a) expected time to acquire target that will be engaged,

(b) next-target-type-to-be-engaged probability.

Furthermore, these two key intermediate quantities carry all the dependence on the firer type's target-engagement policy, with the rest of the general expression for the Lanchester attrition-rate coefficient [for example (2)] being independent of it. In other words, the expression (2) gives the general form for a Lanchester attrition-rate coefficient for serial acquisition of targets in a heterogeneous-target environment before any target-engagement policy has been specified for a firer type.

Conversely, once one has developed analytical expressions for these two key intermediate quantities for a particular target-engagement policy[22], one can simply compute with (2) numerical values for the corresponding Lanchester attrition-rate coefficients in a heterogeneous-target environment. Moreover, the following are the only such target-engagement policies that have been considered in military operations research that are known to these authors

(1) engage-any-target-that-you-acquire target-engagement policy,

[21] E.g. see Taylor [1999], [2000b], [2000c], [2001a], [2001b].

[22] Providing, of course, that all other aspects of the attrition process (cf. Section 7 above) have remained the same.

(2) Taylor's constant-probability-of-engagement-for-a-given-target-type target-engagement policy[23],

(3) Bonder and Farrell's m-period target-engagement policy.

Remarkably, the only known instances in which explicit analytical expressions for the two key intermediate quantities have been developed (and hence explicit Lanchester-attrition-rate-coefficient results obtained for a heterogeneous-target environment) are those contained in the work of Taylor [2001d] and the paper at hand. In fact, there has not even previously existed any general principle for computing a numerical value for a Lanchester attrition-rate coefficient in a heterogeneous-target environment[24] before Taylor's [2001b] recent work (see also Taylor [2001c]).

This is a field of study in which some amazing claims exist. For example, some claim that the only theoretically correct results that exist are contained in the VIC documentation[25] (e.g. see TRAC-FLVN [1992]). Others claim that it simply does not make any difference for combat models whether or not theoretically correct results for attrition are used. The latter claim can only be scientifically substantiated by computational investigations and these involve having theoretically correct results for Lanchester attrition-rate coefficients. Thus, one is led one way or the other to the following important question. **How does one go about verifying that an explicit analytical expression for a Lanchester attrition-rate coefficient is theoretically correct?**

Thus, to these authors there appear to be two general approaches to verifying whether or not an explicit analytical expression for a Lanchester attrition-rate coefficient is theoretically correct. These are the following:

(1) further independent theoretical justification,

(2) simulation of the target-engagement cycle and comparison of statistical estimates computed from such simulation results (i.e. output) with theoretical values.

Moreover, further independent theoretical justification includes both investigation of limiting-case behavior and also theoretical consistency checks that are context dependent. We will now consider further these two sub cases of further independent theoretical justification.

In Taylor [2001d] one of the authors investigated the limiting-case behavior of a Lanchester attrition-rate coefficient for a heterogeneous-target environment for the special case of homogeneous forces, i.e. $\boldsymbol{n_X = 1}$ in equation (19) of Taylor [2001d]. The expression (19) readily passed this limiting-case-analysis test[26]. Thus, it is essential that any attrition-rate-coefficient expression for a heterogeneous-target environment still hold when there is only a single enemy target type. Furthermore, the engage-any-target-that-you-acquire target-engagement policy may be considered to be a limiting case of Bonder and Farrell's m-period target-engagement policy (see Section 13 above). This observation was used to run a further cross check on our results in Section 13 above (see also Taylor [2001e, Section 16]). Additionally, the pursuit of the work reported here has led to the discovery of several context-dependent checks that could be applied to our analytical results. For ex-

[23] See, for example, the references contained in Footnote 21 above.

[24] Or even a homogeneous-target environment with intermittent line of sight (LOS) between groups of firers and targets, i.e. with a stochastic model of the LOS process in the homogeneous-target environment.

[25] Interestingly enough, the VIC documentation (which is cloned from Miller et al. [1978], equivalently CCTC [1979]) does not contain enough technical details to resolve this issue. Since no explicit analytical results for the two key intermediate quantities are given in it, the theoretical correctness of its attrition-rate-coefficient results cannot be checked against those given in the paper at hand. Moreover, there are not even any algorithmic results leading to the explicit computation of numerical values for attrition-rate coefficients given in it so that the issue cannot even be resolved by numerical investigations.

[26] Unfortunately, such limiting-case analysis did not detect the flaw concerning target acquisition for an ensemble of targets that was present in the author's earlier work for a heterogeneous-target environment.

ample, since the next-target-type-to-be-engaged probability is actually a component of a probability mass function, summing over the target-type index for the full range of target types must yield the value 1.0. This theoretical-cross check was verified in Section 10 above. Furthermore, since the expected time to acquire a target that will be engaged was developed by direct computation of an expected value involving a probability density function, the integral of this probability density function over all time must also be equal to 1.0. This fact was verified (both for the case of no look-back capability and also for the case of look-back capability) in Taylor [2001e].

A Monte-Carlo simulation of the target-engagement cycle (upon which all our derivations have been based) was developed (see Section 14 above) and used to generate stochastic realizations of it from which statistical estimates of kill rates and related quantities of interest (e.g. length of the target-engagement cycle) were computed and compared with theoretical values. This computational investigation produced a remarkable agreement between kill rates (and some other related quantities) computed from the simulation results with theoretical values calculated from analytical formulas (cf. Fig. 7 through 9 above). In other words, this simulation investigation confirmed the theoretical soundness of the analytical results developed for Lanchester attrition-rate coefficients by the work at hand. Of course, one must be very careful to use an appropriate Monte-Carlo simulation for such work.

The reader should bear in mind that all the attrition-rate-coefficient results given in the paper at hand have been developed by the so-called freestanding-analytical-model approach[27]. Moreover, such a relatively simple analytical expression for a Lanchester attrition-rate coefficient for direct-fire ground combat in a heterogeneous-target environment (i.e. one applicable to combined-arms or joint warfare) has simply not existed before for this independent-analytical-model approach. Such developments should be of particular interest to DTRA because of the importance of having campaign-analysis tools available for investigating issues concerning weapons of mass destruction (WMD). In particular, data support does not currently exist for the hierarchy-of-models approach (i.e. the so-called attrition-calibration (ATCAL) approach) for computing such kill rates for the ground-combat model in ITEM. Therefore, one is forced to consider using kill rates developed according to the freestanding-analytical-model approach. Thus, the developments of the paper at hand (and also Taylor [2001b], [2001d], [2001e]) can be used as part of a basis for upgrading the direct-fire ground-combat attrition algorithms in joint campaign models of interest to DTRA.

Finally, after doing the research reported here, the authors have severe concerns about the theoretical correctness of the formulas for the direct-fire Lanchester attrition-rate coefficients used in VIC (and to be used in AWARS). In fact, no explicit formulas for such kill rates (or the key intermediate quantities [namely, the next-target-type-to-engaged probability and the expected time for a firer type to acquire the next target to be engaged]) could be found in the VIC documentation (e.g. see TRAC-FLVN [1992]). The VIC model (as well as the Vector series of models), of course, uses Bonder and Farrell's m-period target-engagement policy. It is of great theoretical importance to have attrition-rate-coefficient methodology that applies to all target-engagement policies. The work at hand has shown how Taylor's revolutionary new attrition-rate-coefficient methodology (e.g. see Taylor [2001b], [2001d]) can be used to develop Lanchester attrition-rate coefficients for Bonder and Farrell's m-period target-engagement policy. Moreover, this work has generated theoretical results that bring into serious question whether any of the attrition-rate-coefficient results reported in the VIC documentation are theoretically correct at all.

[27] See Section 2 above (also Taylor [2000f, Section 2]).

References.

C.B. Barfoot, “The Lanchester Attrition-Rate Coefficient: Some Comments on Seth Bonder’s Paper and a Suggested Alternate Method,” Opns. Res. 17, 888-894 (1969).

S. Bonder, “The Lanchester Attrition-Rate Coefficient,” Operations Research 15, 221-232 (1967).

S. Bonder, “The Mean Lanchester Attrition Rate,” Operations Research 18, 179-181 (1970).

S. Bonder and R.L. Farrell (eds.), "Development of Models for Defense Systems Planning," Report No. SRL 2147 TR 70-2, Systems Research Laboratory, The University of Michigan, Ann Arbor, MI, September 1970 (also available from Nat. Tech. Info. Service, U.S. Department of Commerce, 5285 Port Royal Road, Springfield, VA as AD 714 677).

Command and Control Technical Center (CCTC), “Vector-2 System for Simulation of Theater-Level Combat, TM 201-79, Washington, DC, January 1979.

D.G. Luenberger, Introduction to Dynamic Systems: Theory, Models, and Applications, John Wiley & Sons, New York, 1979.

G. Miller, W. White, and D. Thompson, “VECTOR-2 System for Simulation of Theater-Level Combat,” Report No. VRI-CCTC-2 FR 78-1, Vector Research, Inc., Ann Arbor, MI, October 1978.

L. Padulo and M.A. Arbib, System Theory, W.B. Saunders Company, Philadelphia, 1974.

M. Shubik (Ed.), Mathematics of Conflict, North-Holland Publishing Company, Amsterdam, 1983.

J.G. Taylor, "A Tutorial on the Determination of Single-Weapon-System-Type Kill Rates for Use in Lanchester-Type Combat Models," NPS 55-82-021, Naval Postgraduate School, Monterey, CA, August 1982 (AD A124 937).

J.G. Taylor, Lanchester Models of Warfare, Vol. I & II, Military Applications Section, Operations Research Society of America, Alexandria, VA, 1983.

J.G. Taylor, “New Results for Single-Weapon-System-Type Kill Rates in Lanchester-Type Combat Models,” PowerPoint Presentation for Paper given at Second International MAS Meeting, Military Applications Society (MAS), El Paso, TX, September 21, 1999.

J.G. Taylor, “Taylor’s New Kill Rates: Comparison with Bonder-Farrell Kill Rates in VIC,” Unpublished Working Paper, MOVES Academic Group, Naval Postgraduate School, Monterey, CA, April 2000. (a)

J.G. Taylor, “New Developments for Lanchester Attrition-Rate Coefficients,” PowerPoint Presentation for Paper given at Spring InfORMS Meeting, Session sponsored by Military Applications Society (MAS), Salt Lake City, UT, May 9, 2000. (b)

J.G. Taylor, "Thumbnail Sketch of Comparison of Bonder-Farrell Results in VIC and Taylor’s New Kill-Rate Results, With Implications for JWARS," Working Paper #1, OSD PA&E JWARS Project, MOVES Academic Group, Naval Postgraduate School, Monterey, CA, August 2000. (c)

J.G. Taylor, "Computation of Kill Rates Corresponding to Bonder and Farrell's m-Period Target-Engagement Policy," Working Paper #2, OSD PA&E JWARS Project, MOVES Academic Group, Naval Postgraduate School, Monterey, CA, August 2000. (d)

J.G. Taylor, "Probability of Next Target Type to Be Engaged for Bonder and Farrell's m-Period Target-Engagement Policy," Working Paper #4, OSD PA&E JWARS Project, MOVES Academic Group, Naval Postgraduate School, Monterey, CA, August 2000. (e)

J.G. Taylor, "Overview of Research on Attrition Models to Support MACE," Working Paper #1, National Simulation Center (NSC) Project, MOVES Academic Group, Naval Postgraduate School, Monterey, CA, September 2000. (f)

J.G. Taylor, "Representation of Line-of-Sight Process in Lanchester Attrition-Rate Coefficients," in Proceedings for the Third International MAS Meeting, Military Applications Society, San Antonio, TX, November 2000. (g)

J.G. Taylor, "Taylor's New Kill-Rate Results for Serial Acquisition: Comparison with Bonder-Farrell Results in VIC," Working Paper #1, Defense Threat Reduction Agency (DTRA) Project, MOVES Academic Group, Naval Postgraduate School, Monterey, CA, January 2001. (a)

J.G. Taylor, "Conceptual Basis for Calculating Kill Rates (Serial Acquisition of Targets)," Working Paper #2, Defense Threat Reduction Agency (DTRA) Project, MOVES Academic Group, Naval Postgraduate School, Monterey, CA, May 2001. (b)

J.G. Taylor, "Further New Results for Lanchester Attrition-Rate Coefficients," PowerPoint Presentation for Paper given at Fourth International MAS Meeting, Military Applications Society (MAS), Quantico, VA, May 22, 2001. (c)

J.G. Taylor, "General Expression for a Lanchester Attrition-Rate Coefficient (Serial Acquisition of Targets)," Working Paper #3, Defense Threat Reduction Agency (DTRA) Project, MOVES Academic Group, Naval Postgraduate School, Monterey, CA, June 2001. (d)

J.G. Taylor, "Expected Time to Acquire Target That Is to Be Engaged for Bonder and Farrell's m-Period Target-Engagement Policy," Working Paper #4, Defense Threat Reduction Agency (DTRA) Project, MOVES Academic Group, Naval Postgraduate School, Monterey, CA, June 2001. (e)

U.S. Army TRADOC Analysis Center – Fort Leavenworth (TRAC-FLVN), "Vector-In-Commander (VIC) Version 5.0 Documentation," disk version, Fort Leavenworth, KS, 1992 (available on the Internet at www.trac.army.mil/vic/docs.htm).