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2008-03

High order nonlinear solver

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High order nonlinear solver

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March 11, 2008

Abstract

An eighth order method for finding simple zeros of nonlinear functions is developed. The method requires two function- and three derivative-evaluation per step. If we define informational efficiency of a method as the order per function evaluation, we find that our method has informational efficiency of 1.6.

1 Introduction

There is a vast literature on the solution of nonlinear equations and nonlinear systems, see for example Ostrowski [1], Traub [2], Neta [3] and references there. In general, methods for the solution of polynomial equations are treated differently and will not be discussed here. The methods can be classified as bracketting or fixed point methods. The first class include methods that at every step produce an interval containing a root, whereas the other class produces a point which is hopefully closer to the root than the previous one. Here we develop a high order fixed point type method consisting of two steps. The first step is the fifth order method due to Jarratt [4] requiring one function- and three derivative-evaluation and the second step will only add one function-evaluation. We will show that the method is of order 8. We define informational efficiency, E, of the method (see Traub [2]) as

$$E = \frac{p}{d} \tag{1}$$

where p is the order of the method and d is the number of function- and derivative-evaluation per cycle. Another measure of efficiency is the efficiency index I defined as

$$I = p^{1/d} \tag{2}$$

In our case, we will show in section 3 that our method is of order $p = 8$ and it requires two function- and three derivative-evaluation per (two-step) cycle. Thus $d = 5$, the informational efficiency $E = 1.6$ and the efficiency index $I = 1.5156$.

2 Jarratt's Fifth Order Method

Jarratt's method for the solution of the nonlinear equation

$$f(x) \, = \, 0, \tag{3}$$

is given by the iteration

$$x_{n+1} \, = \, x_n - \frac{f(x_n)}{\frac{1}{6}f'(x_n) + \frac{1}{6}f'(y_n) + \frac{2}{3}f'(\eta_n)} \tag{4}$$

where

$$\begin{aligned} u_n &= \frac{f(x_n)}{f'(x_n)} \\ y_n &= x_n - u_n \\ v_n &= \frac{f(x_n)}{f'(y_n)} \\ \eta_n &= x_n - \frac{1}{8}u_n - \frac{3}{8}v_n \end{aligned} \tag{5}$$

Jarratt has shown that this method is of order 5 ([4]). It requires one function- and three derivative-evaluation per step. Thus the informational efficiency is 1.25.

3 New Higher Order Scheme

Suppose we create a two step method where the first step is as above, i.e.

$$z_n = x_n - \frac{f(x_n)}{\frac{1}{6}f'(x_n) + \frac{1}{6}f'(y_n) + \frac{2}{3}f'(\eta_n)} \tag{6}$$

and

$$x_{n+1} = z_n - \frac{f(z_n)}{f'(x_n)} \frac{f'(x_n) + a_1 f'(y_n) + a_2 f'(\eta_n)}{a_3 f'(x_n) + a_4 f'(y_n) + a_5 f'(\eta_n)} \tag{7}$$

We would like to find the parameters $a_1, \dots, a_5$ so as to maximize the order of covergence. Notice that the second step requires only one additional function evaluation.

Let ξ be a simple zero of $f(x)$ and let $e_n,\ \hat{e}_n$ be the errors at the n^{th} step, i.e.

$$\begin{aligned} e_n &= x_n - \xi \\ \hat{e}_n &= z_n - \xi \end{aligned} \tag{8}$$

then

$$\hat{e}_n = e_n - \frac{f(x_n)}{\frac{1}{6}f'(x_n) + \frac{1}{6}f'(y_n) + \frac{2}{3}f'(\eta_n)} \tag{9}$$

If we expand $f(x_n)$, and $f'(x_n)$ in Taylor series (truncated after the N^{th} power) we have

$$f(x_n) = f(x_n - \xi + \xi) = f(\xi + e_n) = f'(\xi)\left(e_n + \sum_{i=2}^{N} A_i e_n^i\right) \tag{10}$$

$$f'(x_n) = f'(\xi)\left(1 + \sum_{i=2}^{N} i A_i e_n^{i-1}\right) \tag{11}$$

where

$$A_i = \frac{f^{(i)}(\xi)}{i!\,f'(\xi)} \tag{12}$$

To expand $f'(y_n)$ and $f'(\eta_n)$ we use some symbolic manipulator, such as Maple [10], we find

$$f'(y_n) = f'(\xi)\left(1 + 2A_2^2 e_n^2 + c_3 e_n^3 + c_4 e_n^4 + c_5 e_n^5 + c_6 e_n^6 + c_7 e_n^7 + +c_8 e_n^8 \ldots\right) \tag{13}$$

where

$$\begin{aligned}
c_3 &= -4A_2^3 + 4A_2A_3 \\
c_4 &= 8A_2^4 - 11A_3A_2^2 + 6A_2A_4 \\
c_5 &= 8A_2A_5 - 16A_2^5 + 28A_3A_2^3 - 20A_4A_2^2 \\
c_6 &= 12A_3^3 - 68A_3A_2^4 + 60A_4A_2^3 + 10A_2A_6 - 16A_3A_2A_4 + 32A_2^6 - 26A_5A_2^2 \\
c_7 &= 112A_4A_3A_2^2 + 36A_4A_3^2 - 168A_4A_2^4 + 72A_5A_2^3 - 20A_5A_2A_3 - 64A_2^7 \\
&\quad +160A_3A_2^5 - 32A_6A_2^2 - 24A_2A_4^2 - 84A_2A_3^3 + 12A_2A_7 \\
c_8 &= 14A_2A_8 + 27A_4^2A_3 - 38A_7A_2^2 + 88A_6A_2^3 + 110A_5A_3A_2^2 - 24A_6A_2A_3 \\
&\quad -150A_4A_2A_3^2 - 62A_5A_2A_4 + 128A_2^8 + 387A_3^3A_2^2 + 48A_5A_3^2 - 179A_5A_2^4 \\
&\quad +448A_4A_2^5 - 516A_4A_3A_2^3 - 368A_3A_2^6 + 164A_4^2A_2^2 - 72A_3^4
\end{aligned} \tag{14}$$

and

$$f'(\eta_n) = f'(\xi)\left(1 + A_2 e_n + d_2 e_n^2 + d_3 e_n^3 + d_4 e_n^4 + d_5 e_n^5 + d_6 e_n^6 + d_7 e_n^7 + d_8 e_n^8 + \ldots\right) \tag{15}$$

where

$$\begin{aligned}
d_2 &= 3A_3/4 - A_2^2/2 \\
d_3 &= A_4/2 - A_2A_3 + A_2^3 \\
d_4 &= -A_2^4/2 - 3A_3^2/8 - 3A_2A_4/4 + 47A_3A_2^2/16 + 5A_5/16 \\
d_5 &= 9A_2A_3^2/16 + 31A_4A_2^2/8 - 2A_2^5 - 3A_2A_5/8 - A_3A_2^3/4 - 3A_4A_3/8 + 3A_6/16 \\
d_6 &= -137A_3A_2^4/8 + A_2A_6/32 - 69A_4A_2^3/16 + 189A_3^2A_2^2/16 - 141A_3^3/64 + 7A_2^6 \\
&\quad +A_3A_5/16 + 7A_7/64 + A_3A_2A_4 + 143A_5A_2^2/32 \\
d_7 &= 15A_5A_2A_3/32 + 245A_3A_2^5/4 - 14A_2^7 + 993A_2A_3^3/32 + A_8/16 + 27A_2A_7/64 \\
&\quad -181A_5A_2^3/32 + 157A_6A_2^2/32 + 421A_4A_3A_2^2/32 - 147A_3^2A_2^3/2 + 33A_3A_6/64 \\
&\quad -273A_4A_3^2/32 - 75A_4A_2^4/8 + 3A_4A_5/8 \\
d_8 &= 6021A_4A_2A_3^2/64 + 9A_9/256 + 25A_2A_8/32 - 1293A_5A_3^2/128 + 983A_5A_3A_2^2/64 \\
&\quad -421A_6A_2^3/64 - 87A_4^2A_3/8 - 3297A_4A_3A_2^3/32 - 15A_6A_2A_3/64 + 22A_2^8 \\
&\quad -7A_4^2A_2^2/2 + 881A_4A_2^5/16 - 3A_5A_2A_4/2 + 2025A_3^2A_2^4/8 - 2257A_3A_2^6/16 \\
&\quad +123A_3A_7/128 - 11223A_3^3A_2^2/64 + 1361A_7A_2^2/256 + 5A_5^2/16 + 3A_4A_6/4 \\
&\quad -3643A_5A_2^4/256 + 333A_3^4/16
\end{aligned} \tag{16}$$

The error at the end of the first substep of the n^{th} iteration is not in Jarratt [4] and thus we give it here

$$\begin{aligned}
\hat{e}_n &= \left(\frac{1}{24}A_5 - \frac{1}{4}A_3^2 + A_2^4 + \frac{1}{8}A_3A_2^2 + \frac{1}{2}A_2A_4\right)e_n^5 \\
&+ \left(-5A_2^5 - \frac{1}{4}A_3A_4 - \frac{5}{4}A_2^2A_4 + \frac{25}{24}A_2A_5 + \frac{5}{8}A_2A_3^2 + \frac{35}{8}A_3A_2^3 + \frac{1}{8}A_6\right)e_n^6 + \ldots
\end{aligned} \tag{17}$$

Notice that

$$f(z_n) = f'(\xi)\left(\hat{e}_n + \ldots\right) = f'(\xi)\left[\left(\frac{1}{24}A_5 - \frac{1}{4}A_3^2 + A_2^4 + \frac{1}{8}A_3A_2^2 + \frac{1}{2}A_2A_4\right)e_n^5 + \ldots\right] \tag{18}$$

Now substitute (11), (13), (15) and (18) into (7) and expand the quotient in Taylor series, we get

$$e_{n+1} = \sum_{i=0}^{M} B_i e_n^{i+5} \tag{19}$$

where the coefficients B_i depend on the parameters $a_1, \ldots, a_5$.

$$B_0 = -\frac{1 + a_1 + a_2 - a_3 - a_4 - a_5}{a_3 + a_4 + a_5} \tag{20}$$

By choosing $a_5 = 1 + a_1 + a_2 - a_3 - a_4$, we annihilate the first coefficient.

$$B_1 = A_2 \frac{a_5(1 + 3a_1 + 2a_2) + a_4(2a_1 + a_2) + a_3(2 + 4a_1 + 3a_2)}{(a_3 + a_4 + a_5)^2} \tag{21}$$

Upon using the value of a_5 above, we get

$$B_1 = \frac{A_2(3a_1 + 2a_2 + a_3 - a_4 + 1)}{1 + a_1 + a_2} \tag{22}$$

Choosing $a_4 = 1 + 3a_1 + 2a_2 + a_3$ annihilates the second coefficient. We now substitute these choices for a_5 and a_4 in B_2

$$B_2 = A_3 \frac{3(a_1 + a_2 + a_3)}{2(1 + a_1 + a_2)} + A_2^2 \frac{2 + a_1 + 3a_2 + 3a_3}{1 + a_1 + a_2} \tag{23}$$

To ensure that $B_2 = 0$ we have to take $a_1 = 1$, and $a_3 = -1 - a_2$. Thus $a_4 = 3 + a_2$ and $a_5 = a_2$. Thus we have a one-parameter family of methods of order 8 with error constant

$$B_3 = A_2^3 \frac{a_2 - 7}{2 + a_2} - A_4 \frac{3}{2 + a_2} - A_2 A_3 \frac{7a_2 + 23}{2(2 + a_2)} \tag{24}$$

The method is then

$$\begin{aligned} z_n &= x_n - \frac{f(x_n)}{\frac{1}{6}f'(x_n) + \frac{1}{6}f'(y_n) + \frac{2}{3}f'(\eta_n)} \\ x_{n+1} &= z_n - \frac{f(z_n)}{f'(x_n)} \frac{f'(x_n) + f'(y_n) + a_2 f'(\eta_n)}{(-1 - a_2)f'(x_n) + (3 + a_2)f'(y_n) + a_2 f'(\eta_n)} \end{aligned} \tag{25}$$

The choice $a_2 = -2$ is not allowed (denominator of B_0 will vanish.) There is no way to annihilate this coefficient, and thus the method is of order 8. The informational efficiency of the method is then $E = 1.6$ and the efficiency index (see [2]) is $I = p^{1/d} = 1.5156$.

4 Numerical Experiments

We have experimented with our method and compared it to the fifth order Jarratt's method. In our first experiment we took the function

$$f(x) = x^2 - (1 - x)^n, \qquad n = 1, 5, 10$$

		Jarratt		Our Method	
Function	Initial Guess	# of Iterations	Abs. Error	# of Iterations	Abs. Error
$x^2-(1-x)$	1	2	1(-18)	2	1(-18)
$x^2-(1-x)^5$	1	2	1.96(-8)	2	1.11(-16)
$x^2-(1-x)^{10}$	1	2	5.18(-6)	2	2.00(-11)

Table 1: Number of iterations and absolute error for Jarratt's fifth order method and Ours

For $n = 1$ the problem is easy and both methods perform extremely well. Starting with $x_0 = 1$ both methods converged and the absolute error in 2 iterations is 10^{-18}. For $n = 5, 10$ our method gave better accuracy than Jarratt's after 2 iterations, see Table 1.

For the next examples we took the functions listed in Table 2. As can be seen in this table

		Jarratt		Our Method	
Function	Initial Guess	# of Iterations	Abs. Error	# of Iterations	Abs. Error
$xe^{x^2}-\sin^2 x+3\cos x+5$	-1	2	1.89(-18)	2	1(-18)
$\sin x-0.5$	1	2	1.11(-16)	2	1(-18)
$\sin^2 x-x^2+1$	1	3	1.(-18)	2	1.(-18)
$\sin^2 x-x^2+1$	3	3	1(-18)	2	1(-18)
$2xe^{-1}+1-2e^{-x}$	1	2	1(-18)	2	1(-18)
$2xe^{-2}+1-2e^{-2x}$	1	3	1.66(-16)	3	1.11(-16)
$2xe^{-3}+1-2e^{-3x}$	1	11	5.18(-11)	4	8.33(-16)

Table 2: Number of iterations and absolute error for Jarratt's fifth order method and Ours

the distinction between the methods in noticeable in the last two cases. For the case shown on the last row in Table 2 our method requires about a third of the number of iterations for much smaller absolute error.

We now turn to the last five examples listed in Table 3 along with the initial guess used. In all these cases our method outperformed Jarratt's as can be seen in Table 4. In the last case Jarratt's method didn't converge after 51 iterations.

Number	Function	Initial Guess x_0
1	$3x + \sin x - e^{-x}$	0
2	$e^x - 4x^2$	0.75
3	$x - 3\ln x$	2
4	$e^{x^2+7x-30} - 1$	3.5
5	$x^2 \sin^2 x + e^{x^2 \cos x \sin x} - 28$	5

Table 3: List of Experiments with Initial Guesses

	Jarratt		Our Method	
Number	# of Iterations	Abs. Error	# of Ietrations Iterations	Abs. Error
1	1	2.02(-7)	1	1.17(-11)
2	1	1.89(-8)	1	2.37(-12)
3	1	2.21(-5)	1	1.46(-7)
4	4	9.58(-5)	4	2.64(-9)
5	51	.32	5	1.(-18)

Table 4: Number of iterations and absolute error for Jarratt's fifth order method and Ours

Conclusions

Here we developed an eighth-order method to obtain simple zeroes of nonlinear equations. The method requires two function- and three derivative-evaluation per (two-step) cycle. Numerical experiments demonstrate the efficiency of our method as compared to Jarratt's fifth-order scheme.

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