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Jangveladze, Temur

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Large time behavior of solutions and finite difference scheme to a nonlinear integro-differential equation

Temur Jangveladze [a,b], Zurab Kiguradze [a,b], Beny Neta [c,*]

[a] Ivane Javakhishvili Tbilisi State University, Faculty of Exact and Natural Sciences, University Street, 2, 0186, Tbilisi, Georgia
[b] Ilia Chavchavadze State University, Faculty of Physics and Mathematics, I. Chavchavadze Av. 32, 0179, Tbilisi, Georgia
[c] Department of Applied Mathematics, Naval Postgraduate School, Monterey, CA 93943, USA

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ABSTRACT

The large-time behavior of solutions and finite difference approximations of the nonlinear integro-differential equation associated with the penetration of a magnetic field into a substance are studied. Asymptotic properties of solutions for the initial-boundary value problem with homogeneous Dirichlet boundary conditions is considered. The rates of convergence are given too. The convergence of the semidiscrete and the finite difference schemes are also proved.

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1. Introduction

A great variety of applied problems are modeled by nonlinear integro-differential equations or systems. Investigation and numerical solution of such models are the object of many scientific works, see e.g. [1–14]. Such systems arise, for instance, in the mathematical modeling of the process of penetration of an electromagnetic field into a substance. By penetrating into a material, a variable magnetic field generates a variable electronic field which causes the appearance of currents that lead to the heating of the material which in turn influence its resistance. For large oscillations of temperature the dependence should be taken into consideration. In a quasistationary case the corresponding system of Maxwell's equations has the form, see e.g. [15], p. 238:

$$\frac{\partial H}{\partial t} = -rot(\nu_m rot\, H), \tag{1.1}$$

$$c_\nu \frac{\partial \theta}{\partial t} = \nu_m \left(rot\, H\right)^2, \tag{1.2}$$

where $H = (H_1, H_2, H_3)$ is a vector of the magnetic field, θ is temperature, c_ν and ν_m characterize the thermal heat capacity and electroconductivity of the substance. The system (1.1) defines the process of diffusion of the magnetic field and (1.2) describe the change in temperature at the expense of Joule's heating without taking into account the heat conductivity.

If c_ν and ν_m depend on temperature θ, i.e. $c_\nu = c_\nu(\theta)$, $\nu_m = \nu_m(\theta)$, the system (1.1), (1.2) can be rewritten in the following form [16]:

$$\frac{\partial H}{\partial t} = -rot\left[a\left(\int_0^t |rot\, H|^2\, d\tau\right) rot\, H\right], \tag{1.3}$$

where the function $a = a(S)$ is defined for $S \in [0, \infty)$.

* Corresponding author.
E-mail address: byneta@gmail.com (B. Neta).

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Let us consider the following magnetic field H, with the form $H = (0, 0, U)$, where $U = U(x, t)$ is a scalar function of time and of one spatial variables. Then $rot\, H = \left(0, -\frac{\partial U}{\partial x}, 0\right)$ and system (1.3) will take the form

$$\frac{\partial U}{\partial t} = \frac{\partial}{\partial x}\left[a\left(\int_0^t \left(\frac{\partial U}{\partial x}\right)^2 d\tau\right)\frac{\partial U}{\partial x}\right]. \tag{1.4}$$

Study of the models of type (1.3) and (1.4) have begun in the work [16]. In this work, the existence of a generalized solution of the first boundary value problem for one-dimensional space case was proved for the case $a(S) = 1 + S$. They also proved the uniqueness for more general cases.

In the work [17] Laptev proposed some generalization of the system of type (1.3). In particular, considering the temperature of the body to be constant all along the material, i.e. depending on time, but independent of spatial coordinates, then the process of penetration of the magnetic field into the material is modeled by the averaged integro-differential system. A one-dimensional variant of this model has the form [17]:

$$\frac{\partial U}{\partial t} = a\left(\int_0^t \int_0^1 \left(\frac{\partial U}{\partial x}\right)^2 dx d\tau\right)\frac{\partial^2 U}{\partial x^2}. \tag{1.5}$$

Note that the integro-differential equations of type (1.4) and (1.5) are complex and only special cases were investigated. The existence and uniqueness of the solutions of the initial-boundary value problems for the equations of type (1.4) and (1.5) are studied, for example, in [16–21]. The existence theorems, proved in [16,18,21] are based on a-priori estimates and use Galerkin's method and compactness arguments as in [22,23] for nonlinear parabolic equations.

The purpose of this paper is to study the asymptotic behavior of solutions and semidiscrete and finite difference schemes for the Eq. (1.5). Our objective is to give large-time asymptotic behavior (as $t \to \infty$) of the solutions of the initial-boundary value problem with homogeneous Dirichlet boundary conditions for the Eq. (1.5). Here we consider the case $a(S) = 1 + S$. The asymptotic behavior of the solutions for type (1.4) models are studied in [24]. Note that in [25] difference schemes for these models were investigated. Difference schemes for a certain nonlinear parabolic integro-differential model similar to (1.4) were studied in [26]. Neta [27] also discussed the finite element approximation of that nonlinear integro-differential equation. Note also that in [28] the finite difference approximation for a linear integro-differential equations was discussed.

The rest of the paper is organized as follows. In the second section we discuss the asymptotic behavior as $t \to \infty$ of the initial-boundary value problem with zero lateral boundary data. In the Section 3 the semidiscrete and finite difference schemes for (1.5) are investigated. We conclude with some remarks on numerical implementations.

2. Large time behavior of solutions

Consider the following initial-boundary value problem:

$$\frac{\partial U}{\partial t} = (1 + S)\frac{\partial^2 U}{\partial x^2}, \quad (x, t) \in (0, 1) \times (0, \infty), \tag{2.1}$$

$$U(0, t) = U(1, t) = 0, \quad t \geq 0, \tag{2.2}$$

$$U(x, 0) = U_0(x), \quad x \in [0, 1], \tag{2.3}$$

where

$$S(t) = \int_0^t \int_0^1 \left(\frac{\partial U}{\partial x}\right)^2 dx d\tau, \tag{2.4}$$

and $U_0(x)$ is a given initial condition. We assume that $U = U(x, t)$ is a solution of the problem (2.1)–(2.4) on $[0, 1] \times [0, \infty)$ such that $U(\cdot, t)$, $\frac{\partial U(\cdot,t)}{\partial x}$, $\frac{\partial U(\cdot,t)}{\partial t}$, $\frac{\partial^2 U(\cdot,t)}{\partial x^2}$, $\frac{\partial^2 U(\cdot,t)}{\partial t\, \partial x}$ are all in $C^0([0, \infty); L_2(0, 1))$, while $\frac{\partial^2 U(\cdot,t)}{\partial t^2}$ is in $L_2((0, \infty); L_2(0, 1))$. Recall that the L_2 norm of a function u is given by:

$$\|u\| = \left[\int_0^1 u^2(x) dx\right]^{1/2}.$$

Now we estimate the solution of the problem (2.1)–(2.4) using the Sobolev spaces $H^k(0, 1)$ and $H_0^k(0, 1)$.

Theorem 2.1. *If $U_0 \in H_0^1(0, 1)$, then the solution of the problem (2.1)–(2.4) satisfies the following estimate*

$$\|U\| + \left\|\frac{\partial U}{\partial x}\right\| \leq C \exp\left(-\frac{t}{2}\right).$$

Remark. Note that here and below in this section C denote positive constants independent from t.

Proof. Let us multiply (2.1) by U and integrate over $(0, 1)$. After integrating by parts and using the boundary conditions (2.2) we get

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|U\|^2 + \int_0^1 (1+S)\left(\frac{\partial U}{\partial x}\right)^2 \mathrm{d}x = 0.$$

Since $1 + S \geq 1$ we have

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|U\|^2 + \left\|\frac{\partial U}{\partial x}\right\|^2 \leq 0. \tag{2.5}$$

Using Poincare's inequality we obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|U\|^2 + \|U\|^2 \leq 0. \tag{2.6}$$

Now multiply (2.1) by $\frac{\partial^2 U}{\partial x^2}$ and integrate over $(0, 1)$. Using again integration by parts and the boundary conditions (2.2) we get

$$\left.\frac{\partial U}{\partial t}\frac{\partial U}{\partial x}\right|_0^1 - \int_0^1 \frac{\partial^2 U}{\partial x \partial t}\frac{\partial U}{\partial x}\mathrm{d}x = \int_0^1 (1+S)\left(\frac{\partial^2 U}{\partial x^2}\right)^2 \mathrm{d}x,$$

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\|\frac{\partial U}{\partial x}\right\|^2 + (1+S)\left\|\frac{\partial^2 U}{\partial x^2}\right\|^2 = 0, \tag{2.7}$$

or

$$\frac{\mathrm{d}}{\mathrm{d}t}\left\|\frac{\partial U}{\partial x}\right\|^2 \leq 0. \tag{2.8}$$

From (2.5), (2.6) and (2.8) we find

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\exp(t)\left(\|U\|^2 + \left\|\frac{\partial U}{\partial x}\right\|^2\right)\right] \leq 0.$$

This inequality immediately proves Theorem 2.1. □

Note that Theorem 2.1 gives exponential stabilization of the solution of the problem (2.1)–(2.4) in the norm of the space $H^1(0, 1)$. Let us show that the stabilization is also achieved in the norm of the space $C^1(0, 1)$. In particular, let us show that the following estimates hold.

Theorem 2.2. *If $U_0 \in H^4(0, 1) \cap H_0^1(0, 1)$, then the solution of the problem (2.1)–(2.4) satisfies the following estimates:*

$$\left|\frac{\partial U(x,t)}{\partial x}\right| \leq C\exp\left(-\frac{t}{2}\right), \qquad \left|\frac{\partial U(x,t)}{\partial t}\right| \leq C\exp\left(-\frac{t}{2}\right).$$

To this end we need following auxiliary result.

Lemma 2.1. *For the solution of the problem (2.1)–(2.4) the following estimate holds*

$$\left\|\frac{\partial U}{\partial t}\right\| \leq C\exp\left(-\frac{t}{2}\right).$$

Proof. Let us differentiate (2.1) with respect to t,

$$\frac{\partial^2 U}{\partial t^2} = (1+S)\frac{\partial^3 U}{\partial x^2 \partial t} + \left[\int_0^1 \left(\frac{\partial U}{\partial x}\right)^2 \mathrm{d}x\right]\frac{\partial^2 U}{\partial x^2}. \tag{2.9}$$

Multiply (2.9) by $\frac{\partial U}{\partial t}$ and integrate over $(0, 1)$. Using the boundary conditions (2.2) we deduce

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1 \left(\frac{\partial U}{\partial t}\right)^2 \mathrm{d}x + (1+S)\int_0^1 \left(\frac{\partial^2 U}{\partial x \partial t}\right)^2 \mathrm{d}x + \left[\int_0^1 \left(\frac{\partial U}{\partial x}\right)^2 \mathrm{d}x\right]\int_0^1 \frac{\partial U}{\partial x}\frac{\partial^2 U}{\partial x \partial t}\mathrm{d}x = 0,$$

or

$$\frac{d}{dt}\int_0^1\left(\frac{\partial U}{\partial t}\right)^2 dx + 2(1+S)\int_0^1\left(\frac{\partial^2 U}{\partial x\partial t}\right)^2 dx = -2\left[\int_0^1\left(\frac{\partial U}{\partial x}\right)^2 dx\right]\int_0^1 \frac{\partial U}{\partial x}\frac{\partial^2 U}{\partial x\partial t}dx. \tag{2.10}$$

Let us estimate the right hand side of the last equality

$$-2\left[\int_0^1\left(\frac{\partial U}{\partial x}\right)^2 dx\right]\int_0^1 \frac{\partial U}{\partial x}\frac{\partial^2 U}{\partial x\partial t}dx = -2\int_0^1\left\{(1+S)^{-1/2}\left[\int_0^1\left(\frac{\partial U}{\partial x}\right)^2 dx\right]\frac{\partial U}{\partial x}\right\}\left\{(1+S)^{1/2}\frac{\partial^2 U}{\partial x\partial t}\right\}dx. \tag{2.11}$$

From this, using the Schwarz's inequality we get

$$-2\left[\int_0^1\left(\frac{\partial U}{\partial x}\right)^2 dx\right]\int_0^1 \frac{\partial U}{\partial x}\frac{\partial^2 U}{\partial x\partial t}dx \le (1+S)\int_0^1\left(\frac{\partial^2 U}{\partial x\partial t}\right)^2 dx$$
$$+(1+S)^{-1}\left[\int_0^1\left(\frac{\partial U}{\partial x}\right)^2 dx\right]^2\int_0^1\left(\frac{\partial U}{\partial x}\right)^2 dx. \tag{2.12}$$

Combining (2.10)–(2.12) we have

$$\frac{d}{dt}\int_0^1\left(\frac{\partial U}{\partial t}\right)^2 dx + (1+S)\int_0^1\left(\frac{\partial^2 U}{\partial x\partial t}\right)^2 dx \le (1+S)^{-1}\left[\int_0^1\left(\frac{\partial U}{\partial x}\right)^2 dx\right]^3.$$

Using Poincare's inequality and the nonnegativity of $S(t)$ we arrive at

$$\frac{d}{dt}\int_0^1\left(\frac{\partial U}{\partial t}\right)^2 dx + \int_0^1\left(\frac{\partial U}{\partial t}\right)^2 dx \le \left[\int_0^1\left(\frac{\partial U}{\partial x}\right)^2 dx\right]^3.$$

Using Theorem 2.1 to estimate the right hand side we get

$$\frac{d}{dt}\left(\exp(t)\left\|\frac{\partial U}{\partial t}\right\|^2\right) \le C\exp(-2t).$$

Therefore

$$\exp(t)\left\|\frac{\partial U}{\partial t}\right\|^2 \le C\int_0^t \exp(-2\tau)d\tau,$$

which proves the Lemma 2.1. □

Now, let us estimate $\frac{\partial^2 U}{\partial x^2}$ in the space $L_1(0,1)$. From (2.1) we have

$$\frac{\partial^2 U}{\partial x^2} = (1+S)^{-1}\frac{\partial U}{\partial t}. \tag{2.13}$$

Integrating on $(0,1)$ and using Schwarz's inequality we get

$$\int_0^1\left|\frac{\partial^2 U}{\partial x^2}\right|dx = \int_0^1\left|(1+S)^{-1}\frac{\partial U}{\partial t}\right|dx \le \left[\int_0^1 (1+S)^{-2}dx\right]^{1/2}\left[\int_0^1\left(\frac{\partial U}{\partial t}\right)^2 dx\right]^{1/2}.$$

Applying Lemma 2.1 and taking into account the nonnegativity of $S(t)$ we derive

$$\int_0^1\left|\frac{\partial^2 U}{\partial x^2}\right|dx \le C\exp\left(-\frac{t}{2}\right).$$

From this, taking into account the relation

$$\frac{\partial U(x,t)}{\partial x} = \int_0^1 \frac{\partial U(y,t)}{\partial y}dy + \int_0^1\int_y^x \frac{\partial^2 U(\xi,t)}{\partial \xi^2}d\xi dy$$

and the boundary conditions (2.2) it follows that

$$\left|\frac{\partial U(x,t)}{\partial x}\right| = \left|\int_0^1\int_y^x \frac{\partial^2 U(\xi,t)}{\partial \xi^2}d\xi dy\right| \le \int_0^1\left|\frac{\partial^2 U(y,t)}{\partial y^2}\right|dy \le C\exp\left(-\frac{t}{2}\right).$$

Now let us estimate $\frac{\partial U}{\partial t}$ in the norm of the space $C^1(0,1)$. Let us multiply (2.1) by $\frac{\partial^3 U}{\partial x^2 \partial t}$ and integrate over $(0,1)$. Using integration by parts we get

$$\frac{\partial U}{\partial t}\frac{\partial^2 U}{\partial x \partial t}\bigg|_0^1 - \left\|\frac{\partial^2 U}{\partial x \partial t}\right\|^2 = (1+S)\int_0^1 \frac{\partial^2 U}{\partial x^2}\frac{\partial^3 U}{\partial x^2 \partial t}\mathrm{d}x. \tag{2.14}$$

Taking into account the equality

$$\int_0^1 \frac{\partial^3 U}{\partial x^2 \partial t}\frac{\partial^2 U}{\partial x^2}\mathrm{d}x = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\|\frac{\partial^2 U}{\partial x^2}\right\|^2$$

and the boundary conditions (2.2) we arrive at

$$\frac{1+S}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\|\frac{\partial^2 U}{\partial x^2}\right\|^2 + \left\|\frac{\partial^2 U}{\partial x \partial t}\right\|^2 = 0,$$

or

$$\frac{\mathrm{d}}{\mathrm{d}t}\left\|\frac{\partial^2 U}{\partial x^2}\right\|^2 \le 0. \tag{2.15}$$

Note that from (2.14) we have

$$\left\|\frac{\partial^2 U}{\partial x \partial t}\right\|^2 \le \frac{1+S}{2}\left\|\frac{\partial^2 U}{\partial x^2}\right\|^2 + \frac{1+S}{2}\left\|\frac{\partial^3 U}{\partial x^2 \partial t}\right\|^2. \tag{2.16}$$

Now multiply (2.9) by $\frac{\partial^3 U}{\partial x^2 \partial t}$, integrate over $(0,1)$ and integrate the left hand side by parts,

$$\frac{\partial^2 U}{\partial t^2}\frac{\partial^2 U}{\partial x \partial t}\bigg|_0^1 - \int_0^1 \frac{\partial^3 U}{\partial x \partial t^2}\frac{\partial^2 U}{\partial x \partial t}\mathrm{d}x = (1+S)\left\|\frac{\partial^3 U}{\partial x^2 \partial t}\right\|^2 + \left[\int_0^1 \left(\frac{\partial U}{\partial x}\right)^2 \mathrm{d}x\right]\int_0^1 \frac{\partial^2 U}{\partial x^2}\frac{\partial^3 U}{\partial x^2 \partial t}\mathrm{d}x.$$

Now combine this with

$$\int_0^1 \frac{\partial^3 U}{\partial x \partial t^2}\frac{\partial^2 U}{\partial x \partial t}\mathrm{d}x = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\|\frac{\partial^2 U}{\partial x \partial t}\right\|^2$$

and taking into account the boundary conditions (2.2) we have

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\|\frac{\partial^2 U}{\partial x \partial t}\right\|^2 + (1+S)\left\|\frac{\partial^3 U}{\partial x^2 \partial t}\right\|^2 + \left[\int_0^1 \left(\frac{\partial U}{\partial x}\right)^2 \mathrm{d}x\right]\int_0^1 \frac{\partial^2 U}{\partial x^2}\frac{\partial^3 U}{\partial x^2 \partial t}\mathrm{d}x = 0,$$

or

$$\frac{\mathrm{d}}{\mathrm{d}t}\left\|\frac{\partial^2 U}{\partial x \partial t}\right\|^2 + 2(1+S)\left\|\frac{\partial^3 U}{\partial x^2 \partial t}\right\|^2 = -2\left[\int_0^1 \left(\frac{\partial U}{\partial x}\right)^2 \mathrm{d}x\right]\int_0^1 \frac{\partial^2 U}{\partial x^2}\frac{\partial^3 U}{\partial x^2 \partial t}\mathrm{d}x.$$

We estimate the right hand side in a similar fashion to (2.11), (2.12). It is easy to see that

$$\frac{\mathrm{d}}{\mathrm{d}t}\left\|\frac{\partial^2 U}{\partial x \partial t}\right\|^2 + (1+S)\left\|\frac{\partial^3 U}{\partial x^2 \partial t}\right\|^2 \le (1+S)^{-1}\left[\int_0^1 \left(\frac{\partial U}{\partial x}\right)^2 \mathrm{d}x\right]^2 \int_0^1 \left(\frac{\partial^2 U}{\partial x^2}\right)^2 \mathrm{d}x.$$

Using Theorem 2.1, (2.13) and Lemma 2.1 we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\left\|\frac{\partial^2 U}{\partial x \partial t}\right\|^2 + (1+S)\left\|\frac{\partial^3 U}{\partial x^2 \partial t}\right\|^2 \le C\exp(-3t). \tag{2.17}$$

Combining (2.5)–(2.7) and (2.15)–(2.17) we get

$$\begin{aligned}
&\|U\|^2 + \frac{\mathrm{d}}{\mathrm{d}t}\|U\|^2 + \left\|\frac{\partial U}{\partial x}\right\|^2 + \frac{\mathrm{d}}{\mathrm{d}t}\left\|\frac{\partial U}{\partial x}\right\|^2 + 2(1+S)\left\|\frac{\partial^2 U}{\partial x^2}\right\|^2 + \frac{\mathrm{d}}{\mathrm{d}t}\left\|\frac{\partial^2 U}{\partial x^2}\right\|^2 \\
&\quad + \left\|\frac{\partial^2 U}{\partial x \partial t}\right\|^2 + \frac{\mathrm{d}}{\mathrm{d}t}\left\|\frac{\partial^2 U}{\partial x \partial t}\right\|^2 + (1+S)\left\|\frac{\partial^3 U}{\partial x^2 \partial t}\right\|^2 \\
&\quad \le \frac{1}{2}(1+S)\left\|\frac{\partial^2 U}{\partial x^2}\right\|^2 + \frac{1}{2}(1+S)\left\|\frac{\partial^3 U}{\partial x^2 \partial t}\right\|^2 + C\exp(-3t).
\end{aligned}$$

From this, keeping in mind the nonnegativity of $S(t)$, we deduce

$$\|U\|^2+\frac{\mathrm{d}}{\mathrm{d}t}\|U\|^2+\left\|\frac{\partial U}{\partial x}\right\|^2+\frac{\mathrm{d}}{\mathrm{d}t}\left\|\frac{\partial U}{\partial x}\right\|^2+\left\|\frac{\partial^2 U}{\partial x^2}\right\|^2+\frac{\mathrm{d}}{\mathrm{d}t}\left\|\frac{\partial^2 U}{\partial x^2}\right\|^2+\left\|\frac{\partial^2 U}{\partial x\partial t}\right\|^2+\frac{\mathrm{d}}{\mathrm{d}t}\left\|\frac{\partial^2 U}{\partial x\partial t}\right\|^2\le C\exp(-3t).$$

After multiplying by the function $\exp(t)$ we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\exp(t)\left(\|U\|^2+\left\|\frac{\partial U}{\partial x}\right\|^2+\left\|\frac{\partial^2 U}{\partial x^2}\right\|^2+\left\|\frac{\partial^2 U}{\partial x\partial t}\right\|^2\right)\right]\le C\exp(-2t),$$

or

$$\|U\|^2+\left\|\frac{\partial U}{\partial x}\right\|^2+\left\|\frac{\partial^2 U}{\partial x^2}\right\|^2+\left\|\frac{\partial^2 U}{\partial x\partial t}\right\|^2\le C\exp(-t).$$

From this, taking into account the relation

$$\frac{\partial U(x,t)}{\partial t}=\int_0^1\frac{\partial U(y,t)}{\partial t}\mathrm{d}y+\int_0^1\int_y^x\frac{\partial^2 U(\xi,t)}{\partial t\partial\xi}\mathrm{d}\xi\,\mathrm{d}y$$

and Lemma 2.1, we obtain

$$\left|\frac{\partial U(x,t)}{\partial t}\right|\le C\exp\left(-\frac{t}{2}\right).$$

This proves Theorem 2.2. □

Remark. The existence of globally defined solutions of the problem (2.1)–(2.4) can be obtained by a routine procedure. One first establishes the existence of local solutions on a maximal time interval and then uses the derived a-priori estimates to show that the solutions cannot escape in finite time, (see, for example, [22,23]).

3. Space discretization and finite difference scheme

Consider the problem

$$\frac{\partial U}{\partial t}-\left[1+\int_0^t\int_0^1\left(\frac{\partial U}{\partial x}\right)^2\mathrm{d}x\mathrm{d}\tau\right]\frac{\partial^2 U}{\partial x^2}=f(x,t),\tag{3.1}$$

$$U(0,t)=U(1,t)=0,\tag{3.2}$$

$$U(x,0)=U_0(x),\tag{3.3}$$

in the rectangle $Q_T=(0,1)\times(0,T)$, where T is a positive constant, $f=f(x,t)$ and $U_0=U_0(x)$ are given functions of their arguments.

We introduce a net whose mesh points are denoted by $x_i=ih$, $i=0,1,\ldots,M$, with $h=1/M$. The boundaries are specified by $i=0$ and $i=M$. Let $u_i=u_i(t)$ be the semidiscrete approximation at (x_i,t). The exact solution to the problem at (x_i,t), denoted by $U_i=U_i(t)$, is assumed to exist and be smooth enough. From the boundary conditions (3.2) we have $u_0(t)=u_M(t)=0$. At other points x_i, $i=1,2,\ldots,M-1$, the integro-differential equation will be replaced by approximating the space derivatives by a forward and backward differences. We will use the following notations for the forward and backward differences

$$u_{x,i}(t)=\frac{u_{i+1}(t)-u_i(t)}{h},\qquad u_{\bar{x},i}(t)=\frac{u_i(t)-u_{i-1}(t)}{h}.$$

Note that the values $u_i(0)$, $i=1,2,\ldots,M-1$ can be computed from the initial condition (3.3)

$$u_i(0)=U_{0,i},\quad i=1,2,\ldots,M-1.$$

Therefore the semidiscrete problem corresponding to (3.1)–(3.3) is

$$\frac{\mathrm{d}u_i}{\mathrm{d}t}-\left[1+h\sum_{l=1}^{M}\int_0^t\left(u_{\bar{x},l}\right)^2\mathrm{d}\tau\right]u_{\bar{x}x,i}=f(x_i,t),\quad i=1,2,\ldots,M-1,\tag{3.4}$$

$$u_0(t)=u_M(t)=0,\tag{3.5}$$

$$u_i(0)=U_{0,i},\quad i=0,1,\ldots,M.\tag{3.6}$$

So, we obtained a Cauchy problem (3.4)–(3.6) for a nonlinear system of ordinary integro-differential equations.

Introduce inner products and norms:

$$(u,v)_h=\sum_{i=1}^{M-1}u_iv_ih,\qquad (u,v]_h=\sum_{i=1}^{M}u_iv_ih,$$
$$\|u\|_h=(u,u)_h^{1/2},\qquad ||u]|_h=(u,u]_h^{1/2}.$$

Multiplying (3.4) by $u(t)=(u_1(t),u_2(t),\ldots,u_{M-1}(t))$, using the discrete analogue of the integration by parts and Poincare's inequality we get

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u(t)\|_h^2+||u_{\bar{x}}(t)]|_h^2\le(f(t),u(t))\le\frac{1}{2}\|f(t)\|_h^2+\frac{1}{2}\|u(t)\|_h^2\le\frac{1}{2}\|f(t)\|_h^2+\frac{1}{2}||u_{\bar{x}}(t)]|_h^2,$$

where $f(t)=(f_1(t),f_2(t),\ldots,f_{M-1}(t)),f_i(t)=f(x_i,t)$. So, we have

$$\|u(t)\|_h^2+\int_0^t||u_{\bar{x}}]|_h^2\mathrm{d}\tau\le C. \tag{3.7}$$

Remark. Here and below in the investigation of (3.4)–(3.6), C denotes a positive constant independent on h.

The a-priori estimate (3.7) guarantees the global solvability of the problem (3.4)–(3.6).
The first result of this section is:

Theorem 3.1. *If the problem (3.1)–(3.3) has a sufficiently smooth solution $U=U(x,t)$, then the solution $u=u(t)=(u_1(t),u_2(t),\ldots,u_{M-1}(t))$ of the problem (3.4)–(3.6) tends to $U=U(t)=(U_1(t),U_2(t),\ldots,U_{M-1}(t))$ as $h\to0$ and the following estimate is true*

$$\|u(t)-U(t)\|_h\le Ch. \tag{3.8}$$

Proof. For the exact solution $U=U(x,t)$ we have

$$\frac{\mathrm{d}U_i}{\mathrm{d}t}-\left[1+h\sum_{l=1}^{M}\int_0^t(U_{\bar{x},l})^2\mathrm{d}\tau\right]U_{\bar{x}x,i}=f(x_i,t)-\psi_i(t),\quad i=1,2,\ldots,M-1, \tag{3.9}$$
$$U_0(t)=U_M(t)=0, \tag{3.10}$$
$$U_i(0)=U_{0,i},\quad i=0,1,\ldots,M, \tag{3.11}$$

where

$$\psi_i(t)=O(h).$$

Let $z_i(t)=u_i(t)-U_i(t)$ be the difference between approximate and exact solutions. From (3.4)–(3.6) and (3.9)–(3.11) we have

$$\frac{\mathrm{d}z_i}{\mathrm{d}t}-\left\{\left[1+h\sum_{l=1}^{M}\int_0^t(u_{\bar{x},l})^2\mathrm{d}\tau\right]u_{\bar{x},i}-\left[1+h\sum_{l=1}^{M}\int_0^t(U_{\bar{x},l})^2\mathrm{d}\tau\right]U_{\bar{x},i}\right\}_x=\psi_i(t), \tag{3.12}$$
$$z_0(t)=z_M(t)=0, \tag{3.13}$$
$$z_i(0)=0. \tag{3.14}$$

Multiplying (3.12) by $z(t)=(z_1(t),z_2(t),\ldots,z_{M-1}(t))$, using (3.13) and the discrete analogue of the integration by parts we get

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|z\|_h^2+\sum_{i=1}^{M}\left\{\left[1+h\sum_{l=1}^{M}\int_0^t(u_{\bar{x},l})^2\mathrm{d}\tau\right]u_{\bar{x},i}-\left[1+h\sum_{l=1}^{M}\int_0^t(U_{\bar{x},l})^2\mathrm{d}\tau\right]U_{\bar{x},i}\right\}z_{\bar{x},i}h=\sum_{i=1}^{M-1}\psi_iz_ih. \tag{3.15}$$

Note that,

$$\left\{\left[1+h\sum_{l=1}^{M}\int_0^t(u_{\bar{x},l})^2\mathrm{d}\tau\right]u_{\bar{x},i}-\left[1+h\sum_{l=1}^{M}\int_0^t(U_{\bar{x},l})^2\mathrm{d}\tau\right]U_{\bar{x},i}\right\}(u_{\bar{x},i}-U_{\bar{x},i})$$
$$=(z_{\bar{x},i})^2+h\sum_{l=1}^{M}\left[\int_0^t(u_{\bar{x},l})^2\mathrm{d}\tau u_{\bar{x},i}-\int_0^t(U_{\bar{x},l})^2\mathrm{d}\tau U_{\bar{x},i}\right](u_{\bar{x},i}-U_{\bar{x},i})$$

$$= (z_{\bar{x},i})^2 + \frac{1}{2}h\sum_{l=1}^{M}\left\{\left[\int_0^t (u_{\bar{x},l})^2 d\tau + \int_0^t (U_{\bar{x},l})^2 d\tau\right](u_{\bar{x},i} - U_{\bar{x},i})^2 + \left[\int_0^t (u_{\bar{x},l})^2 d\tau - \int_0^t (U_{\bar{x},l})^2 d\tau\right]\left[(u_{\bar{x},i})^2 - (U_{\bar{x},i})^2\right]\right\}$$
$$\geq \frac{1}{2}h\sum_{l=1}^{M}\left[\int_0^t (u_{\bar{x},l})^2 d\tau - \int_0^t (U_{\bar{x},l})^2 d\tau\right]\left[(u_{\bar{x},i})^2 - (U_{\bar{x},i})^2\right]. \tag{3.16}$$

Using (3.15) and (3.16) we have

$$\frac{1}{2}\frac{d}{dt}\|z\|_h^2 + \frac{h}{2}\sum_{l=1}^{M}\left[\int_0^t (u_{\bar{x},l})^2 d\tau - \int_0^t (U_{\bar{x},l})^2 d\tau\right] h\sum_{i=1}^{M}\left[(u_{\bar{x},i})^2 - (U_{\bar{x},i})^2\right] \leq \sum_{i=1}^{M-1}\psi_i z_i h. \tag{3.17}$$

Now introduce the notation

$$\varphi(t) = h\sum_{l=1}^{M}\int_0^t \left[(u_{\bar{x},l})^2 - (U_{\bar{x},l})^2\right] d\tau,$$

we have

$$\frac{1}{2}\frac{d}{dt}\|z\|_h^2 + \frac{1}{4}\frac{d}{dt}\varphi^2(t) \leq \sum_{i=1}^{M-1}\psi_i z_i h \leq \frac{1}{2}\|z\|_h^2 + \frac{1}{2}\|\psi\|_h^2,$$

or after integrating and using (3.14) in (3.17), we get

$$\|z(t)\|_h^2 \leq \int_0^t \|z(\tau)\|_h^2 d\tau + \int_0^t \|\psi(\tau)\|_h^2 d\tau. \tag{3.18}$$

Using Grönwall's lemma from (3.18) we get (3.8). □

Now let us consider the fully discrete scheme for the problem (3.1)–(3.3). Introduce a net whose mesh points are denoted by $(x_i, t_j) = (ih, j\tau)$, where $i = 0, 1, \ldots, M$; $j = 0, 1, \ldots, N$ with $h = 1/M$, $\tau = T/N$. The initial line is denoted by $j = 0$. The discrete approximation at (x_i, t_j) is denoted by u_i^j and the exact solution to the problem (3.1)–(3.3) at those points by U_i^j. We will use the following notations:

$$u_{t,i}^j = \frac{u_i^{j+1} - u_i^j}{\tau}, \qquad u_{\bar{t},i}^j = u_{t,i}^{j-1} = \frac{u_i^j - u_i^{j-1}}{\tau}.$$

Thus we have

$$\frac{u_i^{j+1} - u_i^j}{\tau} - \left[1 + \tau h\sum_{l=1}^{M}\sum_{k=1}^{j+1}(u_{\bar{x},l}^k)^2\right] u_{\bar{x}x,i}^{j+1} = f_i^j, \quad i = 1, 2, \ldots, M-1;\ j = 0, 1, \ldots, N-1, \tag{3.19}$$

$$u_0^j = u_M^j = 0, \quad j = 0, 1, \ldots, N, \tag{3.20}$$

$$u_i^0 = U_{0,i}, \quad i = 0, 1, \ldots, M. \tag{3.21}$$

In a similar fashion to the way we obtained (3.7), we can show that

$$\|u^n\|_h^2 + \tau\sum_{j=1}^{n} ||u_{\bar{x}}^j]|_h^2 \leq C, \quad n = 1, 2, \ldots, N. \tag{3.22}$$

Remark. Here and below C is a positive constant independent from τ and h.

The a-priori estimate (3.22) guarantees the stability of the scheme (3.19)–(3.21).

The second result of this section is the following:

Theorem 3.2. *If the problem (3.1)–(3.3) has a sufficiently smooth solution $U = U(x,t)$, then the solution $u^j = (u_1^j, u_2^j, \ldots, u_{M-1}^j)$, $j = 1, 2, \ldots, N$ of the finite difference scheme (3.19)–(3.21) tends to the $U^j = (U_1^j, U_2^j, \ldots, U_{M-1}^j)$ for $j = 1, 2, \ldots, N$ as $\tau \to 0$, $h \to 0$ and the following estimate is true*

$$\|u^j - U^j\|_h \leq C(\tau + h), \quad j = 1, 2, \ldots, N. \tag{3.23}$$

Proof. For the exact solution $U = U(x, t)$ of the problem (3.1)–(3.3) we have

$$\frac{U_i^{j+1} - U_i^j}{\tau} - \left[1 + \tau h \sum_{l=1}^{M} \sum_{k=1}^{j+1} (U_{\bar{x},l}^k)^2\right] U_{\bar{x}x,i}^{j+1} = f_i^j - \psi_i^j, \tag{3.24}$$

$$U_0^j = U_M^j = 0, \tag{3.25}$$

$$U_i^0 = U_{0,i}, \tag{3.26}$$

where

$$\psi_i^j = O(\tau + h).$$

Solving (3.19)–(3.21) instead of the problem (3.1)–(3.3) we have the error $z_i^j = u_i^j - U_i^j$. From (3.19)–(3.21) and (3.24)–(3.26) we get

$$\frac{z_i^{j+1} - z_i^j}{\tau} - \left\{\left[1 + \tau h \sum_{l=1}^{M} \sum_{k=1}^{j+1} (u_{\bar{x},l}^k)^2\right] u_{\bar{x},i}^{j+1} - \left[1 + \tau h \sum_{l=1}^{M} \sum_{k=1}^{j+1} (U_{\bar{x},i}^k)^2\right] U_{\bar{x},i}^{j+1}\right\}_x = \psi_i^j, \tag{3.27}$$

$$z_0^j = z_M^j = 0, \tag{3.28}$$

$$z_i^0 = 0. \tag{3.29}$$

Multiplying (3.27) by $z^{j+1} = (z_1^{j+1}, z_2^{j+1}, \ldots, z_{M-1}^{j+1})$, using (3.28), and the discrete analogue of integration by parts we get

$$\begin{aligned} &\|z^{j+1}\|_h^2 - (z^{j+1}, z^j)_h + \tau h \sum_{i=1}^{M} \left\{\left[1 + \tau h \sum_{l=1}^{M} \sum_{k=1}^{j+1} (u_{\bar{x},l}^k)^2\right] u_{\bar{x},i}^{j+1}\right. \\ &\quad \left. - \left[1 + \tau h \sum_{l=1}^{M} \sum_{k=1}^{j+1} (U_{\bar{x},l}^k)^2\right] U_{\bar{x},i}^{j+1}\right\} z_{\bar{x},i}^{j+1} = \tau (\psi^j, z^{j+1})_h. \end{aligned} \tag{3.30}$$

Taking into account the relations:

$$(z^{j+1}, z^j)_h = \frac{1}{2}\|z^{j+1}\|_h^2 + \frac{1}{2}\|z^j\|_h^2 - \frac{1}{2}\|z^{j+1} - z^j\|_h^2,$$

$$\begin{aligned} &\left[(u_{\bar{x},l}^k)^2 u_{\bar{x},i}^{j+1} - (U_{\bar{x},l}^k)^2 U_{\bar{x},i}^{j+1}\right](u_{\bar{x},i}^{j+1} - U_{\bar{x},i}^{j+1}) = (u_{\bar{x},l}^k)^2 (u_{\bar{x},i}^{j+1})^2 + (U_{\bar{x},l}^k)^2 (U_{\bar{x},i}^{j+1})^2 - (u_{\bar{x},l}^k)^2 u_{\bar{x},i}^{j+1} U_{\bar{x},i}^{j+1} - (U_{\bar{x},l}^k)^2 U_{\bar{x},i}^{j+1} u_{\bar{x},i}^{j+1} \\ &\quad = (u_{\bar{x},l}^k)^2 (u_{\bar{x},i}^{j+1})^2 + (U_{\bar{x},l}^k)^2 (U_{\bar{x},i}^{j+1})^2 - \left[(u_{\bar{x},l}^k)^2 + (U_{\bar{x},l}^k)^2\right] u_{\bar{x},i}^{j+1} U_{\bar{x},i}^{j+1} \\ &\quad \geq (u_{\bar{x},l}^k)^2 (u_{\bar{x},i}^{j+1})^2 + (U_{\bar{x},l}^k)^2 (U_{\bar{x},i}^{j+1})^2 - \frac{1}{2}\left[(u_{\bar{x},l}^k)^2 + (U_{\bar{x},l}^k)^2\right]\left[(u_{\bar{x},i}^{j+1})^2 + (U_{\bar{x},i}^{j+1})^2\right] \\ &\quad = \frac{1}{2}(u_{\bar{x},l}^k)^2 \left[(u_{\bar{x},i}^{j+1})^2 - (U_{\bar{x},i}^{j+1})^2\right] - \frac{1}{2}(U_{\bar{x},l}^k)^2 \left[(u_{\bar{x},i}^{j+1})^2 - (U_{\bar{x},i}^{j+1})^2\right] \\ &\quad = \frac{1}{2}\left[(u_{\bar{x},l}^k)^2 - (U_{\bar{x},l}^k)^2\right]\left[(u_{\bar{x},i}^{j+1})^2 - (U_{\bar{x},i}^{j+1})^2\right], \end{aligned}$$

from (3.30) we have

$$\begin{aligned} &\|z^{j+1}\|_h^2 + \frac{1}{2}\|z^{j+1} - z^j\|_h^2 - \frac{1}{2}\|z^{j+1}\|_h^2 - \frac{1}{2}\|z^j\|_h^2 + \tau || z_{\bar{x}}^{j+1}]|_h^2 \\ &\quad + \frac{\tau^2 h^2}{2} \sum_{i=1}^{M} \sum_{l=1}^{M} \sum_{k=1}^{j+1} \left[(u_{\bar{x},l}^k)^2 - (U_{\bar{x},l}^k)^2\right]\left[(u_{\bar{x},i}^{j+1})^2 - (U_{\bar{x},i}^{j+1})^2\right] \\ &\leq \frac{\tau}{2\varepsilon}\|\psi^j\|_h^2 + 2\varepsilon\tau\|z^{j+1}\|_h^2, \quad \forall \varepsilon > 0, j = 0, 1, \ldots, N-1. \end{aligned} \tag{3.31}$$

Introduce the notations

$$\xi^j = \tau h \sum_{k=1}^{j} \sum_{l=1}^{M} \left[(u_{\bar{x},l}^k)^2 - (U_{\bar{x},l}^k)^2\right], \qquad \xi^0 = 0,$$

then

$$\xi_t^j = h \sum_{i=1}^{M} \left[(u_{\bar{x},i}^{j+1})^2 - (U_{\bar{x},i}^{j+1})^2 \right].$$

So, from (3.31) we get

$$\|z^{j+1}\|_h^2 - \|z^j\|_h^2 + \tau^2 \|z_{\bar{t}}^{j+1}\|_h^2 + \tau ||z_{\bar{x}}^{j+1}]|_h^2 + \tau^2 \left(\xi_t^j\right)^2 + \tau \xi^j \xi_t^j \leq \frac{\tau}{\varepsilon} \|\psi^j\|_h^2 + 4\varepsilon\tau \|z^{j+1}\|_h^2. \tag{3.32}$$

Using (3.29) and the discrete analogue of Poincare's inequality

$$\|z^{j+1}\|_h^2 \leq \frac{1}{8} ||z_{\bar{x}}^{j+1}]|_h^2$$

and the relation

$$\tau \xi^j \xi_t^j = \frac{1}{2} \left(\xi^{j+1}\right)^2 - \frac{1}{2} \left(\xi^j\right)^2 - \frac{\tau^2}{2} \left(\xi_t^j\right)^2,$$

we have from (3.32)

$$\|z^n\|_h^2 + \tau^2 \sum_{j=0}^{n-1} \|z_{\bar{t}}^{j+1}\|_h^2 + \frac{\tau}{2} \sum_{j=0}^{n-1} ||z_{\bar{x}}^{j+1}]|_h^2 + \frac{\tau^2}{2} \sum_{j=0}^{n-1} \left(\xi_t^j\right)^2 + \frac{1}{2} \left(\xi^n\right)^2 \leq C \sum_{j=0}^{n-1} \|\psi^j\|_h^2 \tau, \quad n = 1, 2, \ldots, N. \tag{3.33}$$

From (3.33) we get (3.23) and thus Theorem 3.2 has been proven. □

Remark. Note, that according to the scheme of proving convergence theorem, the uniqueness of the solution of the scheme (3.19)–(3.21) can be proven. In particular, assuming existence of two solutions u and $\bar{u}$ of the scheme (3.19)–(3.21), for the difference $\bar{z} = u - \bar{u}$ we get $\|\bar{z}^n\|_h^2 \leq 0$, $n = 1, 2, \ldots, N$. So, $\bar{z} \equiv 0$.

4. Numerical implementation remarks

We now comment on the numerical implementation of the discrete problem (3.19)–(3.21). Note that (3.19) can be rewritten as:

$$\frac{1}{\tau} u_i^{j+1} - A\left(\mathbf{u}^{j+1}\right) \frac{u_{i+1}^{j+1} - 2u_i^{j+1} + u_{i-1}^{j+1}}{h^2} - f_i^j - \frac{1}{\tau} u_i^j = 0, \quad i = 1, \ldots, M-1.$$

where

$$A\left(\mathbf{u}^{j+1}\right) = 1 + \tau h \sum_{\ell=1}^{M} \sum_{k=1}^{j+1} \left(\frac{u_\ell^k - u_{\ell-1}^k}{h} \right)^2.$$

This system can be written in matrix form

$$\mathbf{H}\left(\mathbf{u}^{j+1}\right) \equiv \mathbf{G}\left(\mathbf{u}^{j+1}\right) - \frac{1}{\tau} \mathbf{u}^j - \mathbf{f}^j = 0.$$

The vector $\mathbf{u}$ containing all the unknowns $u_1, \ldots, u_{M-1}$ at the level indicated. The vector $\mathbf{G}$ is given by

$$\mathbf{G}\left(\mathbf{u}^{j+1}\right) = \mathbf{T}\left(\mathbf{u}^{j+1}\right) \mathbf{u}^{j+1},$$

where the matrix $\mathbf{T}$ is symmetric and tridiagonal with elements

$$\mathbf{T}_{ir} = \begin{cases} \dfrac{1}{\tau} + 2\dfrac{A}{h^2}, & r = i, \\ -\dfrac{A}{h^2}, & r = i \pm 1. \end{cases}$$

Newton's method for the system is given by

$$\nabla \mathbf{H}\left(\mathbf{u}^{j+1}\right)|^{(n)} \left(\mathbf{u}^{j+1}|^{(n+1)} - \mathbf{u}^{j+1}|^{(n)}\right) = -\mathbf{H}\left(\mathbf{u}^{j+1}\right)|^{(n)}.$$

The elements of the matrix $\nabla \mathbf{H}\left(\mathbf{u}^{j+1}\right)$ require the derivative of A. The elements are:

$$\nabla \mathbf{H}\left(\mathbf{u}^{j+1}\right)|_{ir} = \begin{cases} \dfrac{1}{\tau} + \dfrac{2}{h^2} A\left(\mathbf{u}^{j+1}\right) - \dfrac{\partial A\left(\mathbf{u}^{j+1}\right)}{\partial u_i^{j+1}} \delta_i^{j+1}, & r = i, \\ -\delta_i^{j+1} \dfrac{\partial A\left(\mathbf{u}^{j+1}\right)}{\partial u_r^{j+1}} - \dfrac{1}{h^2} A\left(\mathbf{u}^{j+1}\right), & r = i \pm 1, \\ -\delta_i^{j+1} \dfrac{\partial A\left(\mathbf{u}^{j+1}\right)}{\partial u_r^{j+1}}, & \text{otherwise,} \end{cases}$$

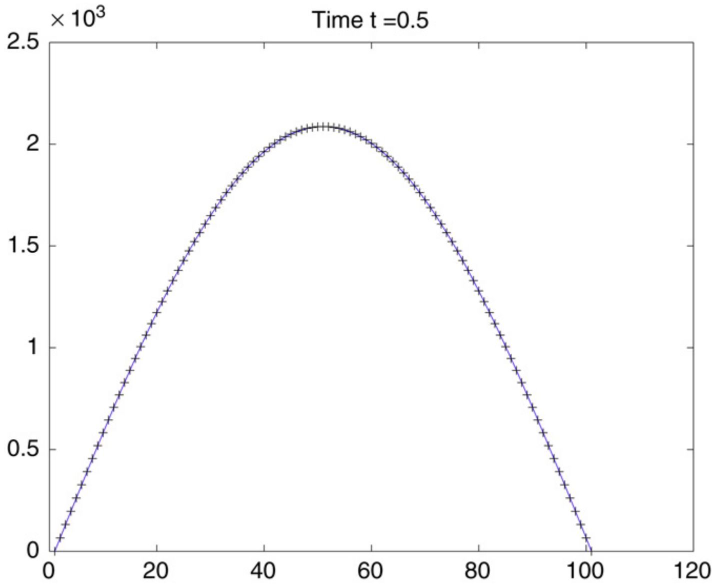

Fig. 1. The solution at $t = 0.5$. The exact solution is the solid line and the numerical solution is marked by $+$.

where

$$\delta_i^{j+1} = \frac{u_{i+1}^{j+1} - 2u_i^{j+1} + u_{i-1}^{j+1}}{h^2}.$$

To evaluate the partial derivatives, we use

$$\begin{aligned}\frac{\partial A}{\partial u_r^{j+1}} &= \frac{\partial}{\partial u_r^{j+1}}\left[1 + \tau h \sum_{\ell=1}^{M}\sum_{k=1}^{j+1}\left(\frac{u_\ell^k - u_{\ell-1}^k}{h}\right)^2\right] \\ &= \frac{\partial}{\partial u_r^{j+1}}\left[C + \tau h\left(\frac{u_r^{j+1} - u_{r-1}^{j+1}}{h}\right)^2 + \tau h\left(\frac{u_{r+1}^{j+1} - u_r^{j+1}}{h}\right)^2\right] \\ &= 2\tau h \frac{u_r^{j+1} - u_{r-1}^{j+1}}{h}\cdot\frac{1}{h} + 2\tau h \frac{u_{r+1}^{j+1} - u_r^{j+1}}{h}\cdot\left(-\frac{1}{h}\right) \\ &= -2\tau h \frac{u_{r+1}^{j+1} - 2u_r^{j+1} + u_{r-1}^{j+1}}{h^2}.\end{aligned}$$

Note that we incorporated into the constant C all the terms that are independent of u_r^{j+1}.

Theorem 4.1. *Given the nonlinear system of equations*

$$g_i\left(x_1, \ldots, x_{M-1}\right) = 0, \quad i = 1, 2, \ldots, M-1.$$

If g_i are three times continuously differentiable in a region containing the solution $\xi_1, \ldots, \xi_{M-1}$ and the Jacobian does not vanish in that region, then Newton's method converges at least quadratically. See [29].

In our case we can write

$$g_i = u_i^{j+1} - \tau A\left(\mathbf{u}^{j+1}\right)\frac{u_{i+1}^{j+1} - 2u_i^{j+1} + u_{i-1}^{j+1}}{h^2} - \tau f_i^j - u_i^j = 0, \quad i = 1, \ldots, M-1.$$

The Jacobian is the matrix ∇H computed above. The term $\frac{1}{\tau}$ on the diagonal ensures that the Jacobian does not vanish. The differentiability is guaranteed, since ∇H is quadratic. Newton's method is costly, because the matrix changes at every step of the iteration. One can use a modified Newton (keep the same matrix for several iterations) but the rate of convergence will be slower.

In our first numerical experiment we have chosen the right hand side so that the exact solution is given by

$$u(x, t) = x(1-x)\mathrm{e}^{-x-t}.$$

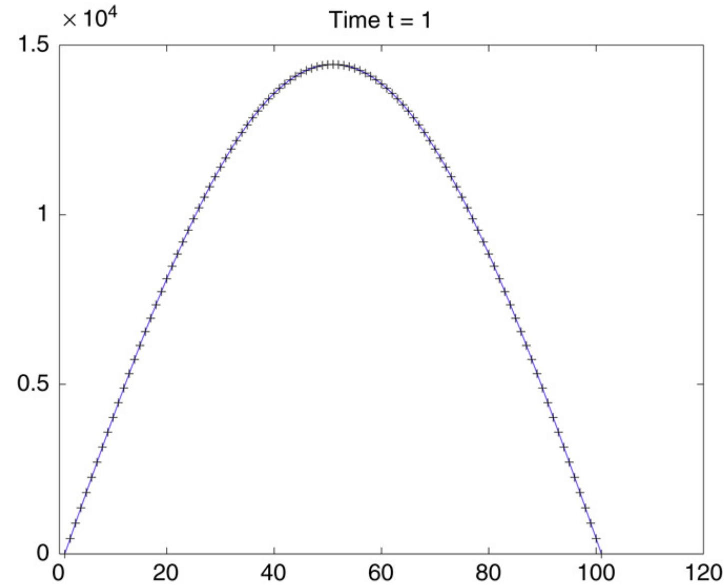

Fig. 2. The solution at $t = 1.0$. The exact solution is the solid line and the numerical solution is marked by $+$.

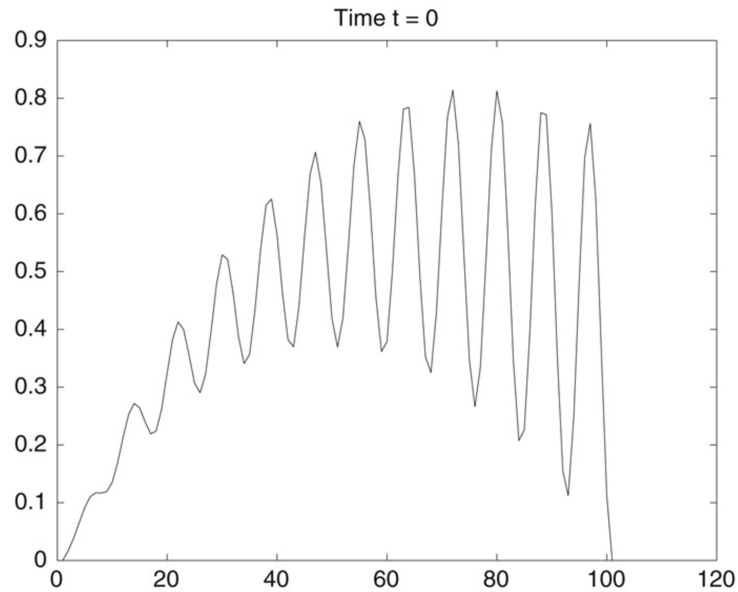

Fig. 3. The initial solution.

In this case the right hand side is

$$f(x,t) = -\left[\frac{9}{8} - \frac{3}{8}\mathrm{e}^{-2} - \left(\frac{1}{8} - \frac{3}{8}\mathrm{e}^{-2}\right)\mathrm{e}^{-2t}\right]\left(-4 + 5x - x^2\right)\mathrm{e}^{-x-t} - x(1-x)\mathrm{e}^{-x-t}.$$

The parameters used are $M = 100$ which dictates $h = 0.01$. Since the method is implicit we can use $\tau = h$ and we took 100 time steps. In the next two figures we plotted the numerical solution (marked with $+$) and the exact solution at $t = 0.5$ (Fig. 1) and $t = 1.0$ (Fig. 2) and it is clear that the two solutions are identical.

In our next experiment we have taken zero right hand side and initial solution given by

$$u(x,0) = x(1-x) + x\left(\mathrm{e}^{-x} - \mathrm{e}^{-1}\cos(24\pi x)\right).$$

In this case, we know that the solution will decay in time. The parameters M, h, τ are as before. In Fig. 3, we plotted the initial solution and in Fig. 4, we have the numerical solution at four different times. It is clear that the numerical solution is approaching zero for all x. Therefore the numerical solution of our experiment fully agrees with the theoretical results.

We have experimented with several other initial solutions, and in all cases we noticed the decay of the numerical solution as expected.

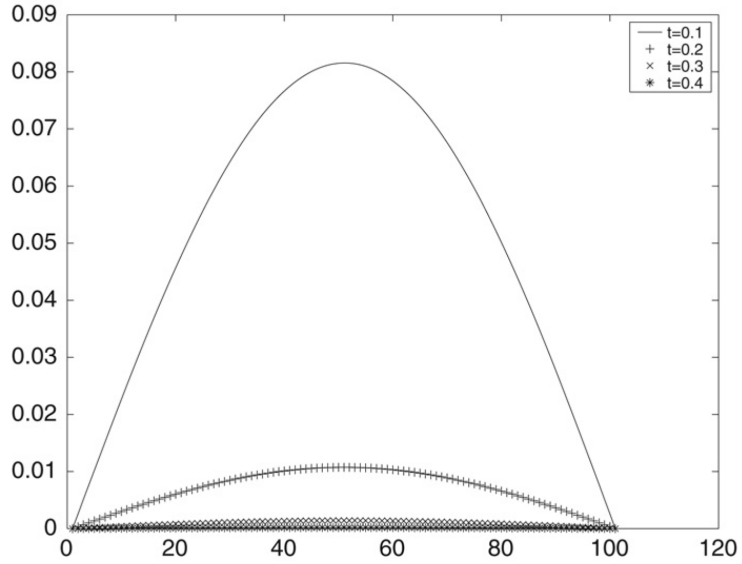

Fig. 4. The numerical solution at $t = 0.1, 0.2, 0.3, 0.4$.

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