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Galerkin finite element method for one nonlinear integro-differential model

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ABSTRACT

Galerkin finite element method for the approximation of a nonlinear integro-differential equation associated with the penetration of a magnetic field into a substance is studied. First type initial-boundary value problem is investigated. The convergence of the finite element scheme is proved. The rate of convergence is given too. The decay of the numerical solution is compared with the analytical results.

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1. Introduction

The goal of this paper is a study of Galerkin finite element method for approximation of a nonlinear integro-differential equation arising in mathematical modeling of the process of a magnetic field penetrating into a substance. If the coefficient of thermal heat capacity and electroconductivity of the substance highly dependent on temperature, then the Maxwell's system [1], that describe above-mentioned process, can be rewritten in the following form [2]:

$$\frac{\partial W}{\partial t} = -rot\left[a\left(\int_0^t |rotW|^2 d\tau\right) rotW\right], \tag{1.1}$$

where $W = (W_1, W_2, W_3)$ is a vector of the magnetic field and the function $a = a(\sigma)$ is defined for $\sigma \in [0, \infty)$. Let us consider magnetic field W, with the form $W = (0, 0, u)$, where $u = u(x, t)$ is a scalar function of time and of one spatial variables. Then $rotW = (0, -\frac{\partial u}{\partial x}, 0)$ and Eq. (1.1) will take the form

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left[a\left(\int_0^t \left(\frac{\partial u}{\partial x}\right)^2 d\tau\right)\frac{\partial u}{\partial x}\right]. \tag{1.2}$$

Note that (1.2) is complex, but special cases of such type models were investigated, see [2–11]. The existence of global solutions for initial-boundary value problems of such models have been proven in [2–4,10] by using the Galerkin and compactness methods [12,13]. The asymptotic behavior of the solutions of (1.2) have been the subject of intensive research in recent years (see e.g. [10,14–20]).

In [7] some generalization of equations of type (1.1) is proposed. There it was assumed that the temperature of the considered body is depending on time, but independent of the space coordinates. If the magnetic field again has the form $W = (0, 0, u)$ and $u = u(x, t)$, then the same process of penetration of the magnetic field into the material is modeled by the following integro-differential equation [7]:

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$$\frac{\partial u}{\partial t} = a\left(\int_0^t \int_0^1 \left(\frac{\partial u}{\partial x}\right)^2 dx d\tau\right) \frac{\partial^2 u}{\partial x^2}. \tag{1.3}$$

The asymptotic behavior of solutions of the initial-boundary value problem for the Eq. (1.3) and the convergence of the finite difference approximation for the case $a(\sigma) = 1 + \sigma$ were studied in [17]. The solvability and uniqueness of the solutions of (1.3) type model is studied in [10].

Note that in [17,21–24] difference schemes for (1.2), (1.3) type models were investigated. Difference schemes for one nonlinear parabolic integro-differential model similar to (1.2) were studied in [25] and [26].

The purpose of this study is to develop a Galerkin finite element method to solve (1.3). The rest of the paper is organized as follows. In the next section the variational formulation of problem is given. In the third section finite element scheme for (1.3) is described and error estimate is proven. We close with a section on numerical implementation and present numerical results comparing with the theoretical ones.

2. Variational Formulation

Consider the following initial-boundary value problem:

$$\frac{\partial u}{\partial t} = (1 + \sigma(t)) \frac{\partial^2 u}{\partial x^2} + f(x, t), \quad (x, t) \in Q = (0, 1) \times (0, T), \tag{2.1}$$

$$u(0, t) = u(1, t) = 0, \quad t \geqslant 0, \tag{2.2}$$

$$u(x, 0) = u_0(x), \quad x \in [0, 1], \tag{2.3}$$

where

$$\sigma(t) = \int_0^t \int_0^1 \left(\frac{\partial u}{\partial x}\right)^2 dx d\tau$$

and $u_0(x)$ is a given function.

One of the ingredients of finite-element method is a variational formulation of the problem. Let us denote by H the linear space of functions u satisfying (2.2) and

$$\|u(\cdot, t)\|_1 < \infty,$$

where

$$\|u(\cdot, t)\|_r = \left\{\int_0^1 \left[|u(x, t)|^2 + \sum_{i=1}^r \left|\frac{\partial^i u(x, t)}{\partial x^i}\right|^2\right] dx\right\}^{1/2}.$$

The variational formulation of problem (2.1)–(2.3) can be stated as follows: Find a function $u(x, t) \in H$ for which

$$\left\langle v, \frac{\partial u}{\partial t}\right\rangle + \left\langle (1 + \sigma(t)) \frac{\partial u}{\partial x}, \frac{\partial v}{\partial x}\right\rangle = \langle f, v\rangle, \quad \forall v \in H, \tag{2.4}$$

and

$$\langle v, u(x, 0)\rangle = \langle v, u_0(x)\rangle, \quad \forall v \in H, \tag{2.5}$$

where $\langle p(x), q(x)\rangle = \int_0^1 p(x)q(x)dx$. To approximate the solution of (2.4), (2.5) we require that u lies in a finite-dimensional subspace S_h of H for each t (see e.g. [27]). The following property concerning approximability in S_h can be readily verified for finite-element spaces (see e.g. [28]).

2.1. Approximation Property

There is an integer $r \geqslant 2$ and numbers C_0, C_1 independent of h such that for any $v \in H$ there exists a $v^h \in S_h$ satisfying

$$\|v - v^h\|_\ell \leqslant C_\ell h^{j-\ell} \|v\|_j, \quad \text{for} \quad 0 \leqslant \ell \leqslant 1, \quad \ell < j \leqslant r. \tag{2.6}$$

2.2. Approximate problem

The approximation $u^h \in S_h$ to u is defined by the following variational analog of (2.4), (2.5):
Find a $u^h \in S_h$ such that

$$\left\langle v^h, \frac{\partial u^h}{\partial t}\right\rangle + \left\langle (1 + \sigma_h(t)) \frac{\partial u^h}{\partial x}, \frac{\partial v^h}{\partial x}\right\rangle = \langle f, v^h\rangle, \quad \forall v^h \in S_h, \tag{2.7}$$

and

$$\langle v^h, u^h(x,0)\rangle = \langle v^h, u_0(x)\rangle, \quad \forall v^h \in S_h, \tag{2.8}$$

where

$$\sigma_h(t) = \int_0^t \int_0^1 \left(\frac{\partial u^h}{\partial x}\right)^2 dx d\tau.$$

Once a basis has been selected for S_h, (2.7), (2.8) are equivalent to a set of N integro-differential equations, where N is dimension of S_h. The solution of such a system will be discussed in Section 4.

3. Error estimates

In this section we shall estimate the error in the finite element approximation using the norm

$$|||E|||_r = \int_0^T \int_0^1 \sum_{i=0}^r \left|\frac{\partial^i E(x,t)}{\partial x^i}\right|^2 dx dt.$$

Everywhere in the case $r = 0$ we will omitted the subscript.

Theorem. *The error in the finite element approximation u^h generated by (2.7), (2.8) satisfies the relation*

$$|||u - u^h|||_1 \leqslant h^{j-1} C\left\{\frac{1}{2}h^2\|u_0\|^2 + Ch^2|||u_t|||^2 + C^2\left[1 + h^{2(j-1)}|||u|||^2\right]|||u|||^2 + C^3\left(h^{j-1}|||u|||^2 + []u[]\right)^2\right\}^{1/2},$$

where

$$[]u[] = \int_0^T \int_0^1 |u| dx d\tau.$$

Here and below C and c_i denote various constants independent of h.

Proof. Subtracting (2.7) from (2.4) with v^h instead of v we obtain

$$\left\langle v^h, \frac{\partial u^h}{\partial t}\right\rangle + \left\langle (1+\sigma_h(t))\frac{\partial u^h}{\partial x}, \frac{\partial v^h}{\partial x}\right\rangle = \left\langle v^h, \frac{\partial u}{\partial t}\right\rangle + \left\langle (1+\sigma(t))\frac{\partial u}{\partial x}, \frac{\partial v^h}{\partial x}\right\rangle, \quad \forall v^h \in S_h.$$

Let $\tilde{u}^h$ be any function in S_h, then

$$\left\langle v^h, \frac{\partial(u^h - \tilde{u}^h)}{\partial t}\right\rangle + \left\langle \left[(1+\sigma_h(t))\frac{\partial u^h}{\partial x} - (1+\tilde{\sigma}_h(t))\frac{\partial \tilde{u}^h}{\partial x}\right], \frac{\partial v^h}{\partial x}\right\rangle$$
$$= \left\langle v^h, \frac{\partial(u - \tilde{u}^h)}{\partial t}\right\rangle + \left\langle \left[(1+\sigma(t))\frac{\partial u}{\partial x} - (1+\tilde{\sigma}_h(t))\frac{\partial \tilde{u}^h}{\partial x}\right], \frac{\partial v^h}{\partial x}\right\rangle, \quad \forall v^h \in S_h, \tag{3.1}$$

where

$$\tilde{\sigma}_h(t) = \int_0^t \int_0^1 \left(\frac{\partial \tilde{u}^h}{\partial x}\right)^2 dx d\tau.$$

Let us define the errors as follows:

$$\begin{aligned} e(x,t) &= u^h(x,t) - \tilde{u}^h(x,t), \\ E(x,t) &= u(x,t) - \tilde{u}^h(x,t). \end{aligned} \tag{3.2}$$

Since $e \in S_h$, we can let $v^h = e$ and (3.1) becomes

$$\left\langle e, \frac{\partial e}{\partial t}\right\rangle + \left\langle \left[(1+\sigma_h(t))\frac{\partial u^h}{\partial x} - (1+\tilde{\sigma}_h(t))\frac{\partial \tilde{u}^h}{\partial x}\right], \frac{\partial e}{\partial x}\right\rangle = \left\langle e, \frac{\partial E}{\partial t}\right\rangle + \left\langle \left[(1+\sigma(t))\frac{\partial u}{\partial x} - (1+\tilde{\sigma}_h(t))\frac{\partial \tilde{u}^h}{\partial x}\right], \frac{\partial e}{\partial x}\right\rangle. \tag{3.3}$$

Let us consider the second term on the left of (3.3)

$$\left\langle \left[(1+\sigma_h(t))\frac{\partial u^h}{\partial x} - (1+\tilde{\sigma}_h(t))\frac{\partial \tilde{u}^h}{\partial x}\right], \frac{\partial e}{\partial x}\right\rangle = \left\langle \frac{\partial e}{\partial x}, \frac{\partial e}{\partial x}\right\rangle + \left\langle \left[\sigma_h(t)\frac{\partial u^h}{\partial x} - \tilde{\sigma}_h(t)\frac{\partial \tilde{u}^h}{\partial x}\right], \frac{\partial e}{\partial x}\right\rangle. \tag{3.4}$$

Denoting

$$\omega = \frac{\partial u^h}{\partial x}, \quad \eta = \frac{\partial \tilde{u}^h}{\partial x},$$

the last term of (3.4) can be rewritten as

$$\begin{aligned}
&\left\langle \left[\sigma_h(t)\frac{\partial u^h}{\partial x} - \tilde{\sigma}_h(t)\frac{\partial \tilde{u}^h}{\partial x}\right], \frac{\partial e}{\partial x}\right\rangle \\
&= \left\langle \omega \int_0^t \int_0^1 \omega^2 dx d\tau - \eta \int_0^t \int_0^1 \eta^2 dx d\tau, \omega - \eta \right\rangle = \frac{1}{2}\left\langle \int_0^t \int_0^1 \omega^2 dx d\tau + \int_0^t \int_0^1 \eta^2 dx d\tau, (\omega - \eta)^2 \right\rangle \\
&\quad + \frac{1}{2}\left\langle \int_0^t \int_0^1 \omega^2 dx d\tau - \int_0^t \int_0^1 \eta^2 dx d\tau, \omega^2 - \eta^2 \right\rangle \geqslant \frac{1}{2}\left\langle \int_0^t \int_0^1 \omega^2 dx d\tau - \int_0^t \int_0^1 \eta^2 dx d\tau, \omega^2 - \eta^2 \right\rangle \\
&= \frac{1}{2}\int_0^1 \int_0^t \int_0^1 (\omega^2 - \eta^2) dx d\tau (\omega^2 - \eta^2) dx = \frac{1}{2}\int_0^t \int_0^1 (\omega^2 - \eta^2) dx d\tau \int_0^1 (\omega^2 - \eta^2) dx = \frac{1}{4}\frac{d\varphi^2(t)}{dt},
\end{aligned}$$

where

$$\varphi(t) \equiv \int_0^t \int_0^1 (\omega^2 - \eta^2) dx d\tau.$$

Therefore, left hand side of (3.3) can be rewritten as follows

$$\left\langle e, \frac{\partial e}{\partial t}\right\rangle + \left\langle \left[(1+\sigma_h(t))\frac{\partial u^h}{\partial x} - (1+\tilde{\sigma}_h(t))\frac{\partial \tilde{u}^h}{\partial x}\right], \frac{\partial e}{\partial x}\right\rangle \geqslant \frac{1}{2}\frac{d}{dt}\|e\|^2 + \|e_x\|^2 + \frac{1}{4}\frac{d\varphi^2(t)}{dt}.$$

Now consider the second term on the right of (3.3)

$$\left\langle \left[(1+\sigma(t))\frac{\partial u}{\partial x} - (1+\tilde{\sigma}_h(t))\frac{\partial \tilde{u}^h}{\partial x}\right], \frac{\partial e}{\partial x}\right\rangle = \left\langle \frac{\partial E}{\partial x}, \frac{\partial e}{\partial x}\right\rangle + \left\langle \left[\sigma(t)\frac{\partial u}{\partial x} - \tilde{\sigma}_h(t)\frac{\partial \tilde{u}^h}{\partial x}\right], \frac{\partial e}{\partial x}\right\rangle. \tag{3.5}$$

Substituting for u_x from (3.2), the last equality gives

$$\begin{aligned}
\sigma(t)\frac{\partial u}{\partial x} - \tilde{\sigma}_h(t)\frac{\partial \tilde{u}^h}{\partial x} &= \left(E_x + \frac{\partial \tilde{u}^h}{\partial x}\right)\int_0^t \int_0^1 \left(E_x + \frac{\partial \tilde{u}^h}{\partial x}\right)^2 dx d\tau - \frac{\partial \tilde{u}^h}{\partial x}\int_0^t \int_0^1 \left(\frac{\partial \tilde{u}^h}{\partial x}\right)^2 dx d\tau \\
&= \left(E_x + \frac{\partial \tilde{u}^h}{\partial x}\right)\int_0^t \int_0^1 \left(E_x^2 + 2E_x\frac{\partial \tilde{u}^h}{\partial x}\right) dx d\tau + E_x \int_0^t \int_0^1 \left(\frac{\partial \tilde{u}^h}{\partial x}\right)^2 dx d\tau.
\end{aligned}$$

Taking this into account in the right hand side of (3.3) we get

$$\begin{aligned}
&\langle e, E_t\rangle + \langle E_x, e_x\rangle + \langle E_x, e_x\rangle \int_0^t \int_0^1 \left(\frac{\partial \tilde{u}^h}{\partial x}\right)^2 dx d\tau + \langle E_x, e_x\rangle \int_0^t \int_0^1 E_x^2 dx d\tau + \left\langle \frac{\partial \tilde{u}^h}{\partial x}, e_x\right\rangle \int_0^t \int_0^1 E_x^2 dx d\tau + \left\langle E_x + \frac{\partial \tilde{u}^h}{\partial x}, e_x\right\rangle \\
&\quad \times \int_0^t \int_0^1 2\frac{\partial \tilde{u}^h}{\partial x}E_x dx d\tau = \langle e, E_t\rangle + \langle E_x, e_x\rangle\left\{1 + \int_0^t \int_0^1 \left(\frac{\partial \tilde{u}^h}{\partial x}\right)^2 dx d\tau + \int_0^t \int_0^1 E_x^2 dx d\tau + \int_0^t \int_0^1 2\frac{\partial \tilde{u}^h}{\partial x}E_x dx d\tau\right\} \\
&\quad + \left\langle \frac{\partial \tilde{u}^h}{\partial x}, e_x\right\rangle\left\{\int_0^t \int_0^1 E_x^2 dx d\tau + \int_0^t \int_0^1 2\frac{\partial \tilde{u}^h}{\partial x}E_x dx d\tau\right\} \leqslant \langle e, E_t\rangle + \langle |E_x|, |e_x|\rangle\left\{1 + \int_0^t \int_0^1 \left(\frac{\partial \tilde{u}^h}{\partial x}\right)^2 dx d\tau\right. \\
&\quad \left. + \int_0^t \int_0^1 E_x^2 dx d\tau + \frac{1}{\epsilon_1}\int_0^t \int_0^1 \left(\frac{\partial \tilde{u}^h}{\partial x}\right)^2 dx d\tau + \epsilon_1 \int_0^t \int_0^1 E_x^2 dx d\tau\right\} + \left\langle \left|\frac{\partial \tilde{u}^h}{\partial x}\right|, |e_x|\right\rangle\left\{\int_0^t \int_0^1 E_x^2 dx d\tau\right. \\
&\quad \left. + 2\left|\sup_{x,t}\left(\frac{\partial \tilde{u}^h}{\partial x}\right)\right|\int_0^t \int_0^1 |E_x| dx d\tau\right\} \leqslant \langle e, E_t\rangle + \langle |E_x|, |e_x|\rangle\left\{1 + \int_0^t \int_0^1 \left(\frac{\partial \tilde{u}^h}{\partial x}\right)^2 dx d\tau\right. \\
&\quad \left. + \int_0^t \int_0^1 E_x^2 dx d\tau + \frac{1}{\epsilon_1}\int_0^t \int_0^1 \left(\frac{\partial \tilde{u}^h}{\partial x}\right)^2 dx d\tau + \epsilon_1 \int_0^t \int_0^1 E_x^2 dx d\tau\right\} + \left|\sup_{x}\left(\frac{\partial \tilde{u}^h}{\partial x}\right)\right|\int_0^1 |e_x| dx\left\{\int_0^t \int_0^1 E_x^2 dx d\tau\right. \\
&\quad \left. + 2\left|\sup_{x,t}\left(\frac{\partial \tilde{u}^h}{\partial x}\right)\right|\int_0^t \int_0^1 |E_x| dx d\tau\right\},
\end{aligned}$$

where $\epsilon_1 > 0$ comes from Schwartz inequality.

Now incorporate these estimates into (3.3) to have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|e\|^2 + \|e_x\|^2 + \frac{1}{4}\frac{d\varphi^2(t)}{dt} \leqslant < e, E_t > + < |E_x|, |e_x| > \left\{1 + \int_0^t \int_0^1 \left(\frac{\partial \tilde{u}^h}{\partial x}\right)^2 dx d\tau + \int_0^t \int_0^1 E_x^2 dx d\tau\right. \\
&\quad \left. + \frac{1}{\epsilon_1}\int_0^t \int_0^1 \left(\frac{\partial \tilde{u}^h}{\partial x}\right)^2 dx d\tau + \epsilon_1 \int_0^t \int_0^1 E_x^2 dx d\tau\right\} + \left|\sup_{x}\left(\frac{\partial \tilde{u}^h}{\partial x}\right)\right|\int_0^1 |e_x| dx\left\{\int_0^t \int_0^1 E_x^2 dx d\tau\right. \\
&\quad \left. + 2\left|\sup_{x,t}\left(\frac{\partial \tilde{u}^h}{\partial x}\right)\right|\int_0^t \int_0^1 |E_x| dx d\tau\right\}.
\end{aligned}$$

Integrate with respect to t, we have

$$\frac{1}{2}\|e(\cdot,T)\|^2+\int_0^T\|e_x\|^2dt+\varphi^2(T)+\leqslant\frac{1}{2}\|e(\cdot,0)\|^2+\int_0^T\int_0^1 eE_t dxdt$$
$$+\int_0^T\int_0^1|E_xe_x|dx\left\{1+(1+\epsilon_1)\int_0^t\int_0^1E_x^2dxd\tau+\left(1+\frac{1}{\epsilon_1}\right)\int_0^t\int_0^1\left(\frac{\partial\tilde{u}^h}{\partial x}\right)^2dxd\tau\right\}dt$$
$$+\int_0^T\left|\sup_x\left(\frac{\partial\tilde{u}^h}{\partial x}\right)\right|\int_0^1|e_x|dx\left\{\int_0^t\int_0^1E_x^2dxd\tau+2\left|\sup_{x,t}\left(\frac{\partial\tilde{u}^h}{\partial x}\right)\right|\int_0^t\int_0^1|E_x|dxd\tau\right\}dt. \tag{3.6}$$

Note that the first, third and fourth terms on the left are nonnegative and can be dropped. We also use Schwarz inequality on the right hand side for these two terms

$$\int_0^T\int_0^1 eE_t dxdt\leqslant\frac{1}{2}\left(\epsilon_2|||e|||^2+1/\epsilon_2|||E_t|||^2\right),$$
$$\int_0^T\int_0^1|E_xe_x|dx\leqslant\frac{1}{2}\left(\epsilon_3|||E_x|||^2+1/\epsilon_3|||e_x|||^2\right).$$

We can estimate the last term on the right by estimating the term in parenthesis and then take it outside the time integral

$$I\equiv\int_0^t\int_0^1E_x^2dxd\tau+2\left|\sup_{x,t}\left(\frac{\partial\tilde{u}^h}{\partial x}\right)\right|\int_0^t\int_0^1|E_x|dxd\tau\leqslant|||E_x|||^2+2\left|\sup_{x,t}\left(\frac{\partial\tilde{u}^h}{\partial x}\right)\right|\int_0^T\int_0^1|E_x|dxd\tau$$
$$=|||E_x|||^2+2\left|\sup_{x,t}\left(\frac{\partial\tilde{u}^h}{\partial x}\right)\right|\square E_x\square.$$

Therefore, the last term in (3.6) becomes

$$I\int_0^T\left|\sup_x\left(\frac{\partial\tilde{u}^h}{\partial x}\right)\right|\int_0^1|e_x|dxdt\leqslant I\sqrt{\int_0^T\left|\sup_x\left(\frac{\partial\tilde{u}^h}{\partial x}\right)\right|^2dt}\sqrt{\int_0^T\left(\int_0^1|e_x|dx\right)^2}\leqslant I\sqrt{\int_0^T\left|\sup_x\left(\frac{\partial\tilde{u}^h}{\partial x}\right)\right|^2dt}|||e_x|||$$
$$\leqslant\frac{\epsilon_4}{2}I^2\int_0^T\left|\sup_x\left(\frac{\partial\tilde{u}^h}{\partial x}\right)\right|^2dt+\frac{1}{2\epsilon_4}|||e_x|||^2.$$

Therefore, (3.6) gives

$$|||e_x|||^2\leqslant\frac{1}{2}\|e(\cdot,0)\|^2+\frac{\epsilon_2}{2}|||e|||^2+\frac{1}{2\epsilon_2}|||E_t|||^2+M_1\left(\frac{\epsilon_3}{2}|||E_x|||^2+\frac{1}{2\epsilon_3}|||e_x|||^2\right)+(1+\epsilon_1)|||E_x|||^2\left(\frac{\epsilon_3}{2}|||E_x|||^2+\frac{1}{2\epsilon_3}|||e_x|||^2\right)$$
$$+\frac{\epsilon_4}{2}I^2\int_0^T\left|\sup_x\left(\frac{\partial\tilde{u}^h}{\partial x}\right)\right|^2dt+\frac{1}{2\epsilon_4}|||e_x|||^2,$$

where

$$M_1\equiv1+\left(1+\frac{1}{\epsilon_1}\right)\int_0^T\int_0^1\left(\frac{\partial\tilde{u}^h}{\partial x}\right)^2dxd\tau.$$

Combining terms we have

$$\left[1-\frac{M_1}{2\epsilon_3}-\frac{1}{2\epsilon_4}-\frac{1+\epsilon_1}{2\epsilon_3}|||E_x|||^2\right]|||e_x|||^2-\frac{\epsilon_2}{2}|||e|||^2\leqslant\frac{1}{2}\|e(\cdot,0)\|^2+\frac{1}{2\epsilon_2}|||E_t|||^2+\left[\frac{M_1\epsilon_3}{2}+\frac{(1+\epsilon_1)\epsilon_3}{2}|||E_x|||^2\right]|||E_x|||^2$$
$$+\frac{\epsilon_4}{2}\int_0^T\left|\sup_x\left(\frac{\partial\tilde{u}^h}{\partial x}\right)\right|^2dt\left(|||E_x|||^2+2\left|\sup_{x,t}\left(\frac{\partial\tilde{u}^h}{\partial x}\right)\right|\square E_x\square\right)^2.$$

Now we use Poincare' inequality

$$\|e_x\|\geqslant C_p\|e\|$$

to show that

$$|||e_x|||\geqslant C_p|||e|||$$

for possibly different constant C_p.

We have

$$\left[1-\frac{M_1}{2\epsilon_3}-\frac{1}{2\epsilon_4}-\frac{1+\epsilon_1}{2\epsilon_3}|||E_x|||^2-\frac{\epsilon_2}{2C_p^2}\right]|||e_x|||^2\leqslant\frac{1}{2}\|e(\cdot,0)\|^2+\frac{1}{2\epsilon_2}|||E_t|||^2+\left[\frac{M_1\epsilon_3}{2}+\frac{(1+\epsilon_1)\epsilon_3}{2}|||E_x|||^2\right]|||E_x|||^2$$
$$+\frac{\epsilon_4}{2}\int_0^T\left|\sup_x\left(\frac{\partial\tilde{u}^h}{\partial x}\right)\right|^2dt\left(|||E_x|||^2+2\left|\sup_{x,t}\left(\frac{\partial\tilde{u}^h}{\partial x}\right)\right|\square E_x\square\right)^2. \tag{3.7}$$

Note that we can choose ϵ_i, $i = 1,\ldots,4$ so that the coefficient of the term on the left hand side of (3.7)

$$M_2 \equiv 1 - \frac{M_1}{2\epsilon_3} - \frac{1}{2\epsilon_4} - \frac{1+\epsilon_1}{2\epsilon_3}|||E_x|||^2 - \frac{\epsilon_2}{2C_p^2}$$

is a positive constant. Recall that

$$|||e|||_1^2 = |||e|||^2 + |||e_x|||^2 \leqslant \left(1 + \frac{1}{C_p^2}\right)|||e_x|||^2,$$

so using (3.7) we get

$$|||e|||_1^2 \leqslant c_1^2\left\{\frac{1}{2}\|e(\cdot,0)\|^2 + \frac{1}{2\epsilon_2}|||E_t|||^2 + \left[\frac{M_1\epsilon_3}{2} + \frac{(1+\epsilon_1)\epsilon_3}{2}|||E_x|||^2\right]|||E_x|||^2 + c_2\left(|||E_x|||^2 + 2\left|\sup_{x,t}\left(\frac{\partial\tilde{u}^h}{\partial x}\right)\right|[]E_x[]\right)^2\right\},$$

where

$$c_1 = \sqrt{\frac{\left(1+\frac{1}{C_p^2}\right)}{M_2}}$$

and

$$c_2 = \frac{\epsilon_4}{2}\int_0^T \left|\sup_x\left(\frac{\partial\tilde{u}^h}{\partial x}\right)\right|^2 dt.$$

Since E is the interpolation error, from (2.6) we have

$$|||E||| \leqslant C_0 h^j |||u|||,$$
$$|||E_x||| \leqslant C_1 h^{j-1}|||u|||,$$
$$|||E_t||| \leqslant C_2 h^j |||u_t|||,$$
$$[]E_x[] \leqslant C_3 h^{j-1}[]u[],$$

which yield

$$|||e|||_1 \leqslant h^{j-1}c_1\left\{\frac{1}{2}h^2\|u_0\|^2 + c_3h^2|||u_t|||^2 + \left[c_4 + c_5h^{2(j-1)}|||u|||^2\right]c_7|||u|||^2 + c_2\left(c_7h^{j-1}|||u|||^2 + c_6[]u[]\right)^2\right\}^{1/2}, \tag{3.8}$$

where

$$c_3 = \frac{1}{2\epsilon_2}C_2^2, \quad c_4 = \frac{M_1\epsilon_3}{2}, \quad c_5 = \frac{(1+\epsilon_1)\epsilon_3}{2}C_1^2, \quad c_6 = 2\left|\sup_{x,t}\left(\frac{\partial\tilde{u}^h}{\partial x}\right)\right|C_3, \quad c_7 = C_1^2. \tag{3.9}$$

Using the triangle inequality we get

$$|||u-u^h|||_1^2 = |||u - \bar{u}^h + \bar{u}^h - u^h|||_1^2 \leqslant |||E|||_1^2 + |||e|||_1^2.$$

Therefore,

$$|||u-u^h|||_1^2 \leqslant c_8^2h^{2(j-1)}|||u|||^2 + h^{2(j-1)}c_1^2\left\{\frac{1}{2}h^2\|u_0\|^2 + c_3h^2|||u_t|||^2 + \left[c_4 + c_5h^{2(j-1)}|||u|||^2\right]c_7|||u|||^2 + c_2\left(c_7h^{j-1}|||u|||^2 + c_6[]u[]\right)^2\right\},$$

or

$$|||u-u^h|||_1^2 \leqslant h^{2(j-1)}c_1^2\left\{\frac{1}{2}h^2\|u_0\|^2 + c_3h^2|||u_t|||^2 + \left[c_9 + c_5h^{2(j-1)}|||u|||^2\right]c_7|||u|||^2 + c_2\left(c_7h^{j-1}|||u|||^2 + c_6[]u[]\right)^2\right\}, \tag{3.10}$$

where $c_9 = c_4 + \frac{c_8^2}{c_1^2c_7}$. From (3.10), by denoting $C = \max_{i=1,2,\ldots,9}\{c_i\}$ we complete proof of theorem.

4. Numerical Solution

For the numerical solution of (2.7), (2.8) we let $\phi_1(x),\ldots,\phi_N(x)$ be a basis for S_h. Therefore $u^h \in S_h$ can be represented by

$$u^h(x,t) = \sum_{j=1}^N u_j(t)\phi_j(x). \tag{4.1}$$

Since (2.7), (2.8) are valid for all $v^h \in S_h$, one can let $v^h = \phi_k$. This yields the following system for the weights $\mathbf{u}(t)$:

$$M\dot{\mathbf{u}} + K(u)\mathbf{u} = \mathbf{F}, \tag{4.2}$$

$$M\mathbf{u}(0) = \mathbf{U}, \tag{4.3}$$

where

$$M_{jk} = \langle \phi_k, \phi_j \rangle, \tag{4.4}$$

$$K(u)_{jk} = \langle (1 + \sigma_h(t))\phi'_k, \phi'_j \rangle, \tag{4.5}$$

$$\mathbf{F}_j = \langle \phi_j, f \rangle, \quad \mathbf{U}_j = \langle \phi_j, u_0 \rangle. \tag{4.6}$$

Now we can evaluate $\sigma_h(t)$ as follows

$$\begin{aligned}\sigma_h(t) &= \int_0^t \int_0^1 \left(\sum_{\ell=1}^N u_\ell(\tau)\phi'_\ell(x) \right)^2 dx d\tau = \int_0^t \int_0^1 \sum_{\ell=1}^N \sum_{m=1}^N u_\ell(\tau) u_m(\tau) \phi'_\ell(x) \phi'_m(x) dx d\tau \\ &= \sum_{\ell=1}^N \sum_{m=1}^N \int_0^t u_\ell(\tau) u_m(\tau) \underbrace{\left[\int_0^1 \phi'_\ell(x) \phi_m\prime(x) dx \right]}_{=\widetilde{K}_{\ell m}} d\tau = \sum_{\ell=1}^N \sum_{m=1}^N \widetilde{K}_{\ell m} \int_0^t u_\ell(\tau) u_m(\tau) d\tau. \end{aligned} \tag{4.7}$$

The time integral can be approximated by the trapezoidal rule ($0 = t_0 < t_1 < t_2 < \cdots < t_n = t$) as follows

$$\int_0^t u_\ell(\tau) u_m(\tau) d\tau = \sum_{p=0}^n \Delta t \zeta_p u_\ell(t_p) u_m(t_p), \tag{4.8}$$

where $\zeta_p = 1/2$ for $p = 0, n$ and $\zeta_p = 1$ for $p = 1, \ldots, n-1$. Combining (4.8) and (4.7) with (4.5), we get

$$\begin{aligned} K(u)_{jk} &= \langle \left(1 + \sum_{\ell=1}^N \sum_{m=1}^N \widetilde{K}_{\ell m} \sum_{p=0}^n \Delta t \zeta_p u_\ell(t_p) u_m(t_p) \right) \phi'_k, \phi'_j \rangle = \left(1 + \sum_{\ell=1}^N \sum_{m=1}^N \widetilde{K}_{\ell m} \sum_{p=0}^n \Delta t \zeta_p u_\ell(t_p) u_m(t_p) \right) \widetilde{K}_{jk} \\ &= \left(1 + \sum_{p=0}^n \Delta t \zeta_p v(t_p) \right) \widetilde{K}_{jk}, \end{aligned} \tag{4.9}$$

where $v(t) = \mathbf{u}^T(t)\widetilde{K}\mathbf{u}(t)$. To solve the system (4.2) and (4.3), we use Taylor's series. Let

$$\mathbf{u}(t + \Delta t) = \mathbf{u}(t) + (\Delta t)\dot{\mathbf{u}}(t) + \frac{1}{2}(\Delta t)^2 \ddot{\mathbf{u}}(t) + O\left((\Delta t)^3\right). \tag{4.10}$$

Differentiating (4.2) with respect to t, one has

$$M\ddot{\mathbf{u}} + K(u)\dot{\mathbf{u}} + \dot{K}\mathbf{u} = \dot{\mathbf{F}}, \tag{4.11}$$

where

$$\dot{K}_{kj} = \langle \dot{\sigma}_h \phi'_j, \phi'_k \rangle = \left\langle \int_0^1 \left(\sum_{\ell=1}^N u_\ell(t) \phi'_\ell \right)^2 dx \phi'_j, \phi'_k \right\rangle = \sum_{\ell=1}^N \sum_{m=1}^N u_\ell(t) u_m(t) \widetilde{K}_{\ell m} \widetilde{K}_{kj} = \mathbf{u}^T(t) \widetilde{K} \mathbf{u}(t) \widetilde{K}_{kj} = v(t) \widetilde{K}_{kj}. \tag{4.12}$$

Now multiply (4.10) by M and using (4.2), (4.11) and (4.12), we have after dropping terms of order higher than second

$$\begin{aligned} M(\mathbf{u}(t+\Delta t) - \mathbf{u}(t)) &= (\Delta t) M \dot{\mathbf{u}}(t) + \frac{1}{2}(\Delta t)^2 M \ddot{\mathbf{u}}(t) = (\Delta t)[\mathbf{F} - K(u)\mathbf{u}] + \frac{1}{2}(\Delta t)^2 \left[\dot{\mathbf{F}} - K(u)\dot{\mathbf{u}} - \dot{K}\mathbf{u} \right] \\ &= (\Delta t)[\mathbf{F} - K(u)\mathbf{u}] + \frac{1}{2}(\Delta t)^2 \left[\dot{\mathbf{F}} - K(u) M^{-1} (\mathbf{F} - K(u)\mathbf{u}) - \dot{K}\mathbf{u} \right] \\ &= (\Delta t) \left[\mathbf{F} + \frac{1}{2}(\Delta t)\dot{\mathbf{F}} - \frac{1}{2}(\Delta t) K(u) M^{-1} \mathbf{F} \right] - (\Delta t) K(u) \left[\mathbf{u} - \frac{1}{2} \Delta t M^{-1} K(u) \mathbf{u} \right] - \frac{1}{2}(\Delta t)^2 \dot{K} \mathbf{u}. \end{aligned} \tag{4.13}$$

If we take $t = t_n$ as in (4.8) and we denote $\mathbf{u}^n = \mathbf{u}(t_n)$ then substituting for K and $\dot{K}$ from (4.9) and (4.12), we get

$$\begin{aligned} M\left(\mathbf{u}^{n+1} - \mathbf{u}^n\right) = (\Delta t) &\left[\mathbf{F}^n + \frac{1}{2}(\Delta t)\dot{\mathbf{F}}^n - \frac{1}{2}(\Delta t) \left(1 + \sum_{p=0}^n \Delta t \zeta_p v(t_p) \right) \widetilde{K} M^{-1} \mathbf{F}^n \right] \\ &- (\Delta t) \left(1 + \sum_{p=0}^n \Delta t \zeta_p v(t_p) \right) \widetilde{K} \left[\mathbf{u}^n - \frac{1}{2} \Delta t M^{-1} (1 + \sum_{p=0}^n \Delta t \zeta_p v(t_p) \right) \widetilde{K} \mathbf{u}^n \right] - \frac{1}{2}(\Delta t)^2 v(t) \widetilde{K} \mathbf{u}^n. \end{aligned} \tag{4.14}$$

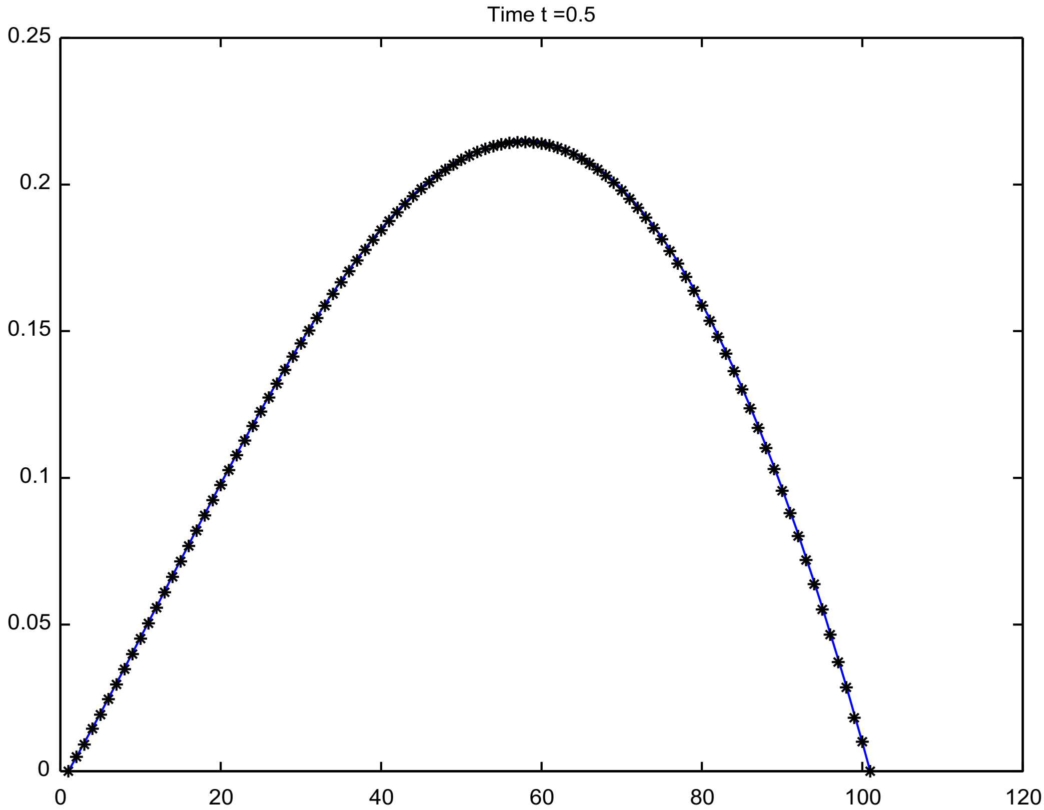

Fig. 1. The solution at $t = 0.5$. The exact solution is solid line and the numerical solution is marked by $*$.

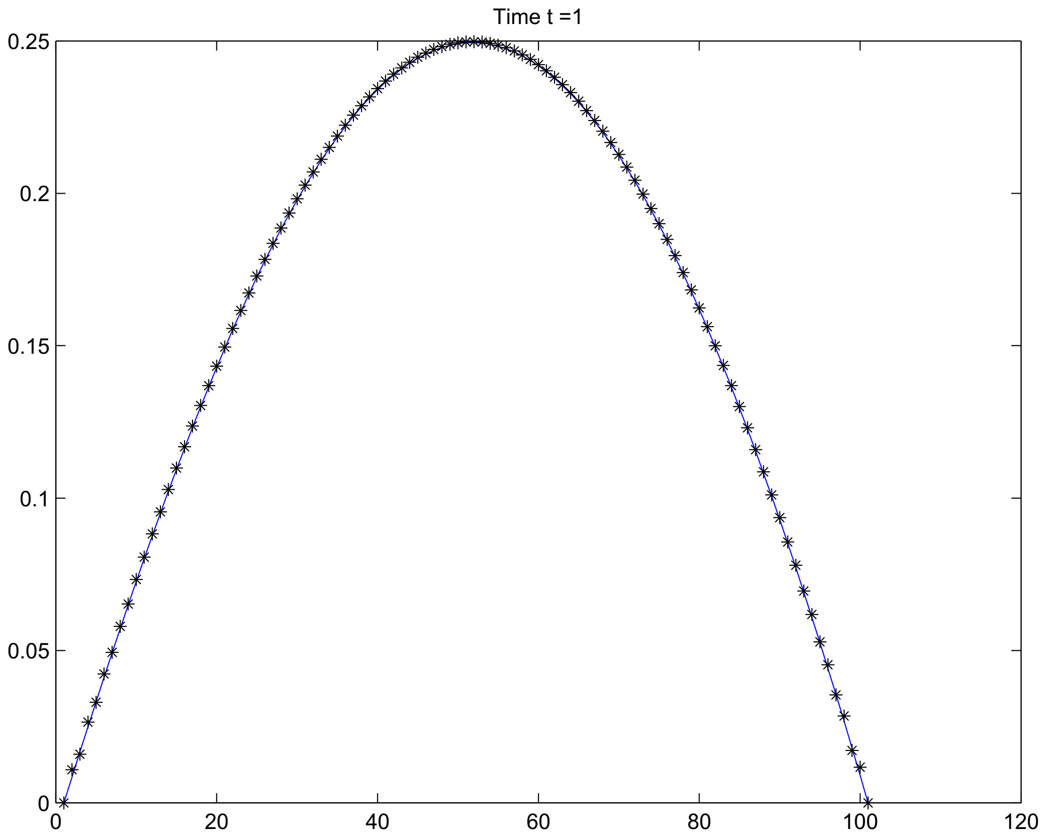

Fig. 2. The solution at $t = 1.0$. The exact solution is solid line and the numerical solution is marked by $*$.

Table 1
The errors for various values of the grid spacing and the approximate rate of convergence.

h	Error	Rate
.2	.0071411	.964499
.04	.0015122	.978861
.02	.00076726	.988986
.01	.00038657	.994375
.005	.00019404	

Now let us denote

$$\psi^n = 1 + \sum_{p=0}^{n} \Delta t \zeta_p v^p, \tag{4.15}$$

then

$$M\mathbf{u}^{n+1} = M\mathbf{u}^n - (\Delta t)\psi^n \widetilde{K}\left[\mathbf{u}^n - \frac{1}{2}\Delta t M^{-1}\psi^n \widetilde{K}\mathbf{u}^n\right] - \frac{1}{2}(\Delta t)^2 v^n \widetilde{K}\mathbf{u}^n + (\Delta t)\left[\mathbf{F}^n + \frac{1}{2}(\Delta t)\dot{\mathbf{F}}^n - \frac{1}{2}(\Delta t)\psi^n \widetilde{K} M^{-1}\mathbf{F}^n\right]. \tag{4.16}$$

In our first numerical experiment we have chosen the right hand side so that the exact solution is given by

$$u(x,t) = x(1-x)\sin(x+t).$$

In this case the right hand side is

$$f(x,t) = x(1-x)\cos(x+t) - \left(1 + \frac{11}{60}t - \frac{1}{8}\cos(t)\sin(t) - \frac{1}{8}\cos(1+t)\sin(1+t) + \frac{1}{8}\cos(1)\sin(1)\right) \times (-2\sin(x+t) + 2(1-x)\cos(x+t) - 2x\cos(x+t) - x(1-x)\sin(x+t)).$$

The parameters used are $M = 100$ which dictates $h = 0.01$. In the next two figures we plotted the numerical solution (marked with $*$) and the exact solution at $t = 0.5$ (Fig. 1) and $t = 1.0$ (Fig. 2) and it is clear that the two solutions are almost identical.

We have ran the same example with various values of h and measured the error $|||u - u_h|||_1$. The results are given in Table 1.

We have experimented with several other initial solutions, and in all cases we noticed the agreement with the exact solution.

Remark. Clearly, in general, the finite element approximation requires larger storage. In this case, the nonlinear system was solved using Taylor series approximation and one has to solve a banded system at each time step. No iteration is required. On the other hand, in the finite difference case, we have solved the nonlinear system using Newton's method which required iterating with dense matrices. Both methods yield comparable numerical results.

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