2009

Finite difference approximation of a nonlinear integro-differential system

Jangveladze, Temur

þÿApplied Mathematics and Computation, Volume 215, (2009), pp. 615 628
http://hdl.handle.net/10945/39422

Applied Mathematics and Computation 215 (2009) 615–628

Contents lists available at ScienceDirect

Applied Mathematics and Computation

journal homepage: www.elsevier.com/locate/amc

Finite difference approximation of a nonlinear integro-differential system

Temur Jangveladze [a], Zurab Kiguradze [a], Beny Neta [b,*]

[a] Ivane Javakhishvili Tbilisi State University, University St. 2, 0186 Ilia Chavchavadze State University, I. Chavchavadze Av. 32, 0179 Tbilisi, Georgia
[b] Department of Applied Mathematics, Naval Postgraduate School Monterey, CA 93943, USA

A R T I C L E   I N F O

Keywords:
Nonlinear integro-differential system
Finite difference scheme

A B S T R A C T

Finite difference approximation of the nonlinear integro-differential system associated with the penetration of a magnetic field into a substance is studied. The convergence of the finite difference scheme is proved. The rate of convergence of the discrete scheme is given. The decay of the numerical solution is compared with the analytical results proven earlier.

Published by Elsevier Inc.

1. Introduction

Integro-differential equations and systems arise in the study of various problems in physics, chemistry, technology, economics, etc. One kind of integro-differential system arises in the mathematical modelling of penetration of a magnetic field into a substance. A variable magnetic field induces in the material a variable electronic field which causes the appearance of currents. The currents lead to the heating of the material and elevating its temperature. For quasistationary approximation the corresponding system of Maxwell's equations has the form [1]:

$$\frac{\partial H}{\partial t} = -rot(\nu_m rot H), \tag{1.1}$$

$$c_\nu \frac{\partial \theta}{\partial t} = \nu_m (rot H)^2, \tag{1.2}$$

where $H = (H_1, H_2, H_3)$ is the magnetic field vector, θ is temperature, c_ν and ν_m characterize the thermal heat capacity and electroconductivity of the substance.

If c_ν and ν_m depend on temperature, then the system (1.1) and (1.2) can be rewritten in the following form [2]:

$$\frac{\partial H}{\partial t} = -rot\left[a\left(\int_0^t |rot H|^2 d\tau\right) rot H\right], \tag{1.3}$$

where the function $a = a(S)$ is defined for $S \in [0, \infty)$.

If the magnetic field has the form $H = (0, U, V)$ and $U = U(x,t),\ V = V(x,t)$, then we have

$$rot(a(S) rot H) = \left(0, -\frac{\partial}{\partial x}\left(a(S)\frac{\partial U}{\partial x}\right), -\frac{\partial}{\partial x}\left(a(S)\frac{\partial V}{\partial x}\right)\right),$$

where

$$S(x,t) = \int_0^t \left[\left(\frac{\partial U}{\partial x}\right)^2 + \left(\frac{\partial V}{\partial x}\right)^2\right] d\tau. \tag{1.4}$$

* Corresponding author.
E-mail address: bneta@nps.edu (B. Neta).

0096-3003/$ - see front matter Published by Elsevier Inc.
doi:10.1016/j.amc.2009.05.061

Therefore, we obtain the following system of nonlinear integro-differential equations:

$$\frac{\partial U}{\partial t}=\frac{\partial}{\partial x}\left[a(S)\frac{\partial U}{\partial x}\right],\quad \frac{\partial V}{\partial t}=\frac{\partial}{\partial x}\left[a(S)\frac{\partial V}{\partial x}\right], \tag{1.5}$$

where S is defined by (1.4).

The model (1.3) is complex and was intensively studied by many authors. The existence and uniqueness of the global solutions of initial-boundary value problems for equations and systems of type (1.3) were studied in [2–8] and in a number of other works as well. The existence theorems, that are proved in [2,3], are based on a priori estimates, Galerkin's method and compactness arguments as in [9,10] for nonlinear parabolic equations. The asymptotic behavior as $t\to\infty$ of the solutions of such models have been the object of intensive research in recent years [11–14].

Numerous scientific works are devoted to construction and investigation of discrete analogues for integro-differential models (see, for example, [15–28]). For integro-differential models described in the paper and problems similar to them many authors study the convergence of finite difference schemes. Neta and Igwe [16] have developed a second order difference scheme for a nonlinear parabolic integro-differential model similar to (1.5). This scheme was also compared to the finite element approximation discussed in [15]. It was shown in [16] that the results of the finite difference scheme are comparable to those obtained by finite elements for the same mesh spacing using less computer storage. Iskakov et al. [17] discuss a finite volume method for the solution of an integro-differential equation in higher dimensions. They claim that "spectral elements suffer from a number of serious limitations." "Finite volume methods play a major role in the discretization of conservation laws." They "were proposed originally as a means of generating finite difference methods on general grids." see Grossmann et al. [18].

The purpose of the present work is to study the numerical solution of initial-boundary value problem for the system (1.5). The rest of the paper is organized as follows: in Section 2 we consider the finite difference scheme and prove its convergence. In the last section we conclude with numerical implementations.

2. Finite difference scheme

In the cylinder $[0,1]\times[0,T]$, where T is a positive constant, we consider finite difference scheme for the following nonlinear integro-differential problem:

$$\begin{aligned}&\tfrac{\partial U}{\partial t}-\tfrac{\partial}{\partial x}\left\{\left(1+\int_0^t\left[\left(\tfrac{\partial U}{\partial x}\right)^2+\left(\tfrac{\partial V}{\partial x}\right)^2\right]d\tau\right)\tfrac{\partial U}{\partial x}\right\}=f_1(x,t),\\&\tfrac{\partial V}{\partial t}-\tfrac{\partial}{\partial x}\left\{\left(1+\int_0^t\left[\left(\tfrac{\partial U}{\partial x}\right)^2+\left(\tfrac{\partial V}{\partial x}\right)^2\right]d\tau\right)\tfrac{\partial V}{\partial x}\right\}=f_2(x,t),\end{aligned} \tag{2.1}$$

$$U(0,t)=U(1,t)=V(0,t)=V(1,t)=0, \tag{2.2}$$

$$U(x,0)=U_0(x),\quad V(x,0)=V_0(x). \tag{2.3}$$

Here f_1,f_2,U_0 and V_0 are given functions of their arguments.

Note that the finite difference scheme for the scalar problem of (2.1)–(2.3) type was first studied in [27]. The present work can be extended to a system with an arbitrary number of unknown functions.

On $[0,1]\times[0,T]$ let us introduce a net whose mesh points denoted by $(x_i,t_j)=(ih,j\tau)$, where $i=0,1,\dots,M; j=0,1,\dots,N$ with $h=1/M,\tau=T/N$. The initial line is denoted by $j=0$. The discrete approximation at (x_i,t_j) is denoted by u_i^j, v_i^j and the exact solution to the problem (2.1)–(2.3) by U_i^j, V_i^j. We will use the following notations:

$$\Delta_x r_i^j=\frac{r_{i+1}^j-r_i^j}{h},\quad \nabla_x r_i^j=\frac{r_i^j-r_{i-1}^j}{h},\quad \Delta_t r_i^j=\frac{r_i^{j+1}-r_i^j}{\tau},\quad \nabla_t r_i^j=\Delta_t r_i^{j-1}=\frac{r_i^j-r_i^{j-1}}{\tau},\quad \|r\|=\left(\sum_{i=1}^{M-1}r_ig_ih\right)^{1/2},$$

$$\|r]|=\left(\sum_{i=1}^{M}r_ig_ih\right)^{1/2}.$$

Let us consider the finite difference scheme

$$\begin{aligned}&\Delta_t u_i^j-\Delta_x\left\{\left(1+\tau\sum_{k=1}^{j+1}\left[(\nabla_x u_i^k)^2+(\nabla_x v_i^k)^2\right]\right)\nabla_x u_i^{j+1}\right\}=f_{1,i}^j,\\&\Delta_t v_i^j-\Delta_x\left\{\left(1+\tau\sum_{k=1}^{j+1}\left[(\nabla_x u_i^k)^2+(\nabla_x v_i^k)^2\right]\right)\nabla_x v_i^{j+1}\right\}=f_{2,i}^j,\\&\qquad i=1,2,\dots,M-1,\quad j=0,1,\dots,N-1,\end{aligned} \tag{2.4}$$

$$u_0^j=u_M^j=v_0^j=v_M^j=0,\quad j=0,1,\dots,N, \tag{2.5}$$

$$u_i^0=U_{0,i},\quad v_i^0=V_{0,i},\quad i=0,1,\dots,M. \tag{2.6}$$

Multiplying the first equation of the (2.4) by τu_i^{j+1}, summing for each i from 1 to $M-1$ and using the discrete analogue of the integration by parts we get

$$\|u^{j+1}\|^2 - h\sum_{i=1}^{M-1} u_i^j u_i^{j+1} + h\tau\sum_{i=1}^{M}\left\{\left(1+\tau\sum_{k=1}^{j+1}\left[(\nabla_x u_i^k)^2 + (\nabla_x v_i^k)^2\right]\right)\left(\nabla_x u_i^{j+1}\right)^2\right\} = \tau h\sum_{i=1}^{M-1} f_{1,i}^j u_i^{j+1}. \tag{2.7}$$

Taking into account the following relations:

$$h\sum_{i=1}^{M-1} u_i^j u_i^{j+1} \leqslant \frac{1}{2}\|u^j\|^2 + \frac{1}{2}\|u^{j+1}\|^2, \quad h\sum_{i=1}^{M-1} f_i^j u_i^{j+1} \leqslant \frac{1}{2}\|f^j\|^2 + \frac{1}{2}\|u^{j+1}\|^2$$

and discrete analogue of Poincare's inequality

$$\|u^{j+1}\| \leqslant \|\nabla_x u^{j+1}]| \tag{2.8}$$

from (2.7) we get

$$\|u^{j+1}\|^2 - \|u^j\|^2 + \tau\|\nabla_x u^{j+1}]|^2 \leqslant \tau\|f_1^j\|^2.$$

From this inequality it is not difficult to get the following estimation:

$$\|u^n\|^2 + \sum_{j=1}^{n}\|\nabla_x u^j]|^2\tau < C, \quad n = 1,2,\ldots,N. \tag{2.9}$$

Analogously, we can show that

$$\|v^n\|^2 + \sum_{j=1}^{n}\|\nabla_x v^j]|^2\tau < C, \quad n = 1,2,\ldots,N. \tag{2.10}$$

In (2.9) and (2.10) the constant C depends on T and on f_1 and f_2 consequently.

The a priori estimates (2.9) and (2.10) guarantee the stability and existence, see [10], of solution of the scheme (2.4)–(2.6).

The principal aim of this section is the proof of the following statement.

Theorem 2.1. *If the problem 2.1,2.2,2.3 has a sufficiently smooth solution $U = U(x,t)$, $V = V(x,t)$, then the solution $u^j = (u_1^j, u_2^j, \ldots, u_{M-1}^j)$, $v^j = (v_1^j, v_2^j, \ldots, v_{M-1}^j)$, $j = 1,2,\ldots,N$ of the difference scheme 2.4,2.5,2.6 tends to $U^j = (U_1^j, U_2^j, \ldots, U_{M-1}^j)$, $V^j = (V_1^j, V_2^j, \ldots, V_{M-1}^j)$, $j = 1,2,\ldots,N$ as $\tau \to 0$, $h \to 0$ and the following estimates are true*

$$\|u^j - U^j\| \leqslant C(\tau + h), \quad \|v^j - V^j\| \leqslant C(\tau + h), \quad j = 1,2,\ldots,N. \tag{2.11}$$

Proof. For $U = U(x,t)$ and $V = V(x,t)$ we have:

$$\begin{aligned}
&\Delta_t U_i^j - \Delta_x\left\{\left(1+\tau\sum_{k=1}^{j+1}\left[(\nabla_x U_i^k)^2 + (\nabla_x V_i^k)^2\right]\right)\nabla_x U_i^{j+1}\right\} = f_{1,i}^j - \psi_{1,i}^j,\\
&\Delta_t V_i^j - \Delta_x\left\{\left(1+\tau\sum_{k=1}^{j+1}\left[(\nabla_x U_i^k)^2 + (\nabla_x V_i^k)^2\right]\right)\nabla_x V_i^{j+1}\right\} = f_{2,i}^j - \psi_{2,i}^j,
\end{aligned} \tag{2.12}$$

$$U_0^j = U_M^j = V_0^j = V_M^j = 0, \tag{2.13}$$

$$U_i^0 = U_{0,i}, \quad V_i^0 = V_{0,i}. \tag{2.14}$$

In a usual way, it is not difficult to see that

$$\psi_{k,i}^j = O(\tau + h), \quad k = 1,2.$$

Solving (2.4)–(2.6) instead of the problem (2.1)–(2.3) we have the errors $y_i^j = u_i^j - U_i^j$ and $z_i^j = v_i^j - V_i^j$. From (2.4)–(2.6) and (2.12)–(2.14) we get

$$\begin{aligned}
&\Delta_t y_i^j - \Delta_x\left\{\left(1+\tau\sum_{k=1}^{j+1}\left[(\nabla_x u_i^k)^2 + (\nabla_x v_i^k)^2\right]\right)\nabla_x u_i^{j+1}\right.\\
&\left.-\left(1+\tau\sum_{k=1}^{j+1}\left[(\nabla_x U_i^k)^2 + (\nabla_x V_i^k)^2\right]\right)\nabla_x U_i^{j+1}\right\} = \psi_{1,i}^j,\\
&\Delta_t z_i^j - \Delta_x\left\{\left(1+\tau\sum_{k=1}^{j+1}\left[(\nabla_x u_i^k)^2 + (\nabla_x v_i^k)^2\right]\right)\nabla_x v_i^{j+1}\right.\\
&\left.-\left(1+\tau\sum_{k=1}^{j+1}\left[(\nabla_x U_i^k)^2 + (\nabla_x V_i^k)^2\right]\right)\nabla_x V_i^{j+1}\right\} = \psi_{2,i}^j,
\end{aligned} \tag{2.15}$$

$$y_0^j = y_M^j = z_0^j = z_M^j = 0, \tag{2.16}$$

$$y_i^0 = z_i^0 = 0. \tag{2.17}$$

Multiplying equations of the system (2.15) by $y^{j+1}=(y_1^{j+1},y_2^{j+1},\dots,y_{M-1}^{j+1})$ and $z^{j+1}=(z_1^{j+1},z_2^{j+1},\dots,z_{M-1}^{j+1})$, respectively, summing for each i from 1 to $M-1$, using (2.16) and the discrete analogue of formula of integration by parts we get

$$\begin{aligned}&\|y^{j+1}\|^2-h\sum_{i=1}^{M-1}y_i^{j+1}y_i^j+\tau h\sum_{i=1}^{M}\left\{\left(1+\tau\sum_{k=1}^{j+1}\left[(\nabla_x u_i^k)^2+(\nabla_x v_i^k)^2\right]\right)\nabla_x u_i^{j+1}-\left(1+\tau\sum_{k=1}^{j+1}\left[(\nabla_x U_i^k)^2+(\nabla_x V_i^k)^2\right]\right)\nabla_x U_i^{j+1}\right\}\nabla_x y_i^{j+1}\\&=\tau h\sum_{i=1}^{M-1}\psi_{1,i}^j y_i^{j+1},\quad \|z^{j+1}\|^2-h\sum_{i=1}^{M-1}z_i^{j+1}z_i^j+\tau h\sum_{i=1}^{M}\left\{\left(1+\tau\sum_{k=1}^{j+1}\left[(\nabla_x u_i^k)^2+(\nabla_x v_i^k)^2\right]\right)\nabla_x v_i^{j+1}\right.\\&\left.-\left(1+\tau\sum_{k=1}^{j+1}\left[(\nabla_x U_i^k)^2+(\nabla_x V_i^k)^2\right]\right)\nabla_x V_i^{j+1}\right\}\nabla_x z_i^{j+1}=\tau h\sum_{i=1}^{M-1}\psi_{2,i}^j z_i^{j+1}.\end{aligned}\tag{2.18}$$

Note that

$$h\sum_{i=1}^{M-1}r_i^{j+1}r_i^j=\frac{1}{2}\|r^{j+1}\|^2+\frac{1}{2}\|r^j\|^2-\frac{1}{2}\|r^{j+1}-r^j\|^2\tag{2.19}$$

and

$$\begin{aligned}&\left(\left[(\nabla_x u_i^k)^2+(\nabla_x v_i^k)^2\right]\nabla_x u_i^{j+1}-\left[(\nabla_x U_i^k)^2+(\nabla_x V_i^k)^2\right]\nabla_x U_i^{j+1}\right)(\nabla_x u_i^{j+1}-\nabla_x U_i^{j+1})\\&=\left[(\nabla_x u_i^k)^2+(\nabla_x v_i^k)^2\right](\nabla_x u_i^{j+1})^2+\left[(\nabla_x U_i^k)^2+(\nabla_x V_i^k)^2\right](\nabla_x U_i^{j+1})^2\\&\quad-\nabla_x u_i^{j+1}\nabla_x U_i^{j+1}\left[(\nabla_x U_i^k)^2+(\nabla_x V_i^k)^2+(\nabla_x u_i^k)^2+(\nabla_x v_i^k)^2\right]\\&=\frac{1}{2}\left(\nabla_x u_i^{j+1}-\nabla_x U_i^{j+1}\right)^2\left[(\nabla_x u_i^k)^2+(\nabla_x v_i^k)^2+(\nabla_x U_i^k)^2+(\nabla_x V_i^k)^2\right]-\frac{1}{2}(\nabla_x u_i^{j+1})^2\left[(\nabla_x U_i^k)^2+(\nabla_x V_i^k)^2\right]-\frac{1}{2}\\&\quad\times(\nabla_x U_i^{j+1})^2\left[(\nabla_x u_i^k)^2+(\nabla_x v_i^k)^2\right]+\frac{1}{2}(\nabla_x u_i^{j+1})^2\left[(\nabla_x u_i^k)^2+(\nabla_x v_i^k)^2\right]+\frac{1}{2}(\nabla_x U_i^{j+1})^2\left[(\nabla_x U_i^k)^2+(\nabla_x V_i^k)^2\right]\\&\geqslant\frac{1}{2}\left[(\nabla_x u_i^{j+1})^2-(\nabla_x U_i^{j+1})^2\right]\left[(\nabla_x u_i^k)^2+(\nabla_x v_i^k)^2-(\nabla_x U_i^k)^2-(\nabla_x V_i^k)^2\right].\end{aligned}\tag{2.20}$$

Analogically,

$$\begin{aligned}&\left(\left[(\nabla_x u_i^k)^2+(\nabla_x v_i^k)^2\right]\nabla_x v_i^{j+1}-\left[(\nabla_x U_i^k)^2+(\nabla_x V_i^k)^2\right]\nabla_x V_i^{j+1}\right)(\nabla_x v_i^{j+1}-\nabla_x V_i^{j+1})\\&\geqslant\frac{1}{2}\left[(\nabla_x v_i^{j+1})^2-(\nabla_x V_i^{j+1})^2\right]\left[(\nabla_x u_i^k)^2+(\nabla_x v_i^k)^2-(\nabla_x U_i^k)^2-(\nabla_x V_i^k)^2\right].\end{aligned}\tag{2.21}$$

Taking into account relations (2.19)–(2.21), from (2.18) we have

$$\begin{aligned}&\|y^{j+1}\|^2+\frac{1}{2}\|y^{j+1}-y^j\|^2-\frac{1}{2}\|y^{j+1}\|^2-\frac{1}{2}\|y^j\|^2+\tau\|\nabla_x y_i^{j+1}\|^2+\|z^{j+1}\|^2+\frac{1}{2}\|z^{j+1}-z^j\|^2-\frac{1}{2}\|z^{j+1}\|^2-\frac{1}{2}\|z^j\|^2\\&\quad+\tau\|\nabla_x z_i^{j+1}\|^2+\frac{\tau^2 h}{2}\sum_{i=1}^{M}\sum_{k=1}^{j+1}\left[(\nabla_x u_i^k)^2+(\nabla_x v_i^k)^2-(\nabla_x U_i^k)^2-(\nabla_x V_i^k)^2\right]\left[(\nabla_x u_i^{j+1})^2+(\nabla_x v_i^{j+1})^2-(\nabla_x U_i^{j+1})^2-(\nabla_x V_i^{j+1})^2\right]\\&\leqslant\frac{\tau}{2}\left(\|\psi_1^j\|^2+\|\psi_2^j\|^2\right)+\frac{\tau}{2}\left(\|y^{j+1}\|^2+\|z^{j+1}\|^2\right),\quad j=0,1,\dots,N-1.\end{aligned}\tag{2.22}$$

Let us introduce the notation

$$\xi_i^j=\tau\sum_{k=0}^{j}\left[(\nabla_x u_i^k)^2+(\nabla_x v_i^k)^2-(\nabla_x U_i^k)^2-(\nabla_x V_i^k)^2\right],$$

then

$$\Delta_t\xi_i^j=(\nabla_x u_i^{j+1})^2+(\nabla_x v_i^{j+1})^2-(\nabla_x U_i^{j+1})^2-(\nabla_x V_i^{j+1})^2.$$

So, from (2.22) we get

$$\begin{aligned}&\|y^{j+1}\|^2-\|y^j\|^2+\tau^2\|\nabla_t y^{j+1}\|^2+2\tau\|\nabla_x y^{j+1}]|^2+\|z^{j+1}\|^2-\|z^j\|^2+\tau^2\|\nabla_t z^{j+1}\|^2+2\tau\|\nabla_x z^{j+1}]|^2+\tau^2\|\Delta_t\xi^j]|^2+\tau h\sum_{i=1}^{M}\xi_i^j\Delta_t\xi_i^j\\&\leqslant\tau\left(\|\psi_1^j\|^2+\|\psi_2^j\|^2\right)+\tau\left(\|y^{j+1}\|^2+\|z^{j+1}\|^2\right).\end{aligned}\tag{2.23}$$

Using (2.17), discrete analogue of Poincare's inequality (2.8) and the relation

$$\tau h \sum_{i=1}^{M} \xi_i^j \Delta_t \xi_i^j = \frac{1}{2}\|\xi^{j+1}]|^2 - \frac{1}{2}\|\xi^j]|^2 - \frac{\tau^2}{2}\|\Delta_t \xi^j]|^2,$$

from (2.23) we have

$$\|y^n\|^2 + \tau^2 \sum_{j=0}^{n-1} \|\nabla_t y^{j+1}\|^2 + \tau \sum_{j=0}^{n-1} \|\nabla_x y^{j+1}]|^2 + \|z^n\|^2 + \tau^2 \sum_{j=0}^{n-1} \|\nabla_t z^{j+1}\|^2 + \tau \sum_{j=0}^{n-1} \|\nabla_x z^{j+1}]|^2 + \frac{\tau^2}{2} \sum_{j=0}^{n-1} \|\Delta_t \xi^j]|^2 + \frac{1}{2}\|\xi^n]|^2$$
$$\leqslant \tau \sum_{j=0}^{n-1} \left(\|\psi_1^j\|^2 + \|\psi_2^j\|^2 \right), \quad n = 1, 2, \ldots, N. \tag{2.24}$$

From (2.24) we get (2.11) and thus Theorem 2.1 has been proven.

Note, that according to the scheme of proving convergence theorem, the uniqueness of the solution of the scheme (2.4)–(2.6) can be proven. In particular, if (u, v) and $(\bar{u}, \bar{v})$ are two solutions of the scheme (2.4)–(2.6), for the differences $w = u - \bar{u}$ and $\bar{w} = v - \bar{v}$ we get $\|w^n\|^2 + \|\bar{w}^n\|^2 \leqslant 0$, $n = 1, 2, \ldots, N$. So, $w = \bar{w} \equiv 0$.

3. Numerical implementation

The finite difference scheme (2.4)–(2.6) can be rewritten as follows:

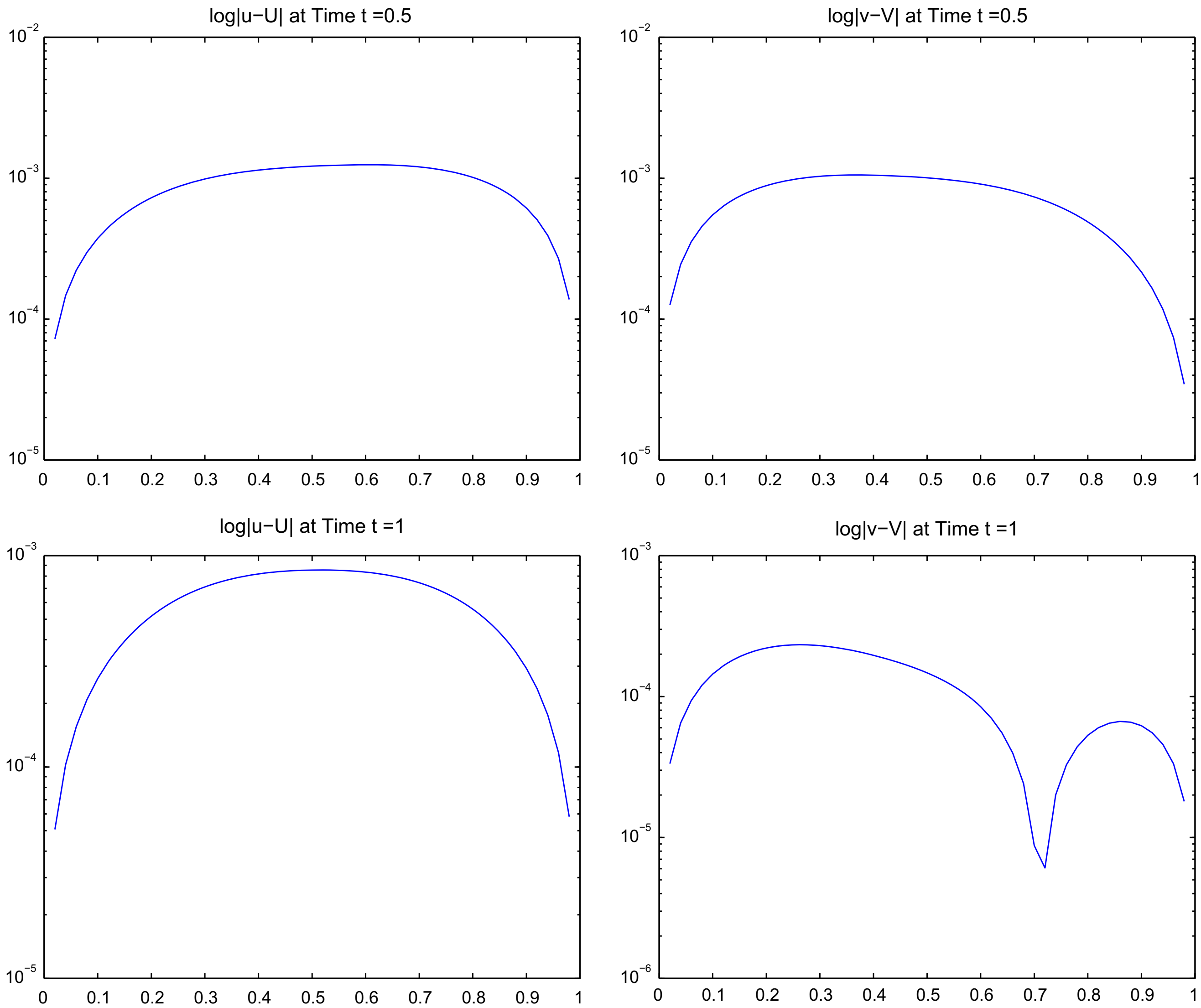

Fig. 1. The absolute value of the difference between the numerical and exact solutions for u (left) and v (right) at $t = 0.5$ (top) and $t = 1$ (bottom) on a semi-log scale.

$$\frac{u_i^{j+1}-u_i^j}{\tau}-\frac{1}{h}\left\{\left[1+\tau\sum_{k=1}^{j+1}\left(\left(\frac{u_{i+1}^k-u_i^k}{h}\right)^2+\left(\frac{v_{i+1}^k-v_i^k}{h}\right)^2\right)\right]\frac{u_{i+1}^{j+1}-u_i^{j+1}}{h}\right.$$
$$\left.-\left[1+\tau\sum_{k=1}^{j+1}\left(\left(\frac{u_i^k-u_{i-1}^k}{h}\right)^2+\left(\frac{v_i^k-v_{i-1}^k}{h}\right)^2\right)\right]\frac{u_i^{j+1}-u_{i-1}^{j+1}}{h}\right\}$$
$$=f_{1,i}^j,\quad \frac{v_i^{j+1}-v_i^j}{\tau}-\frac{1}{h}\left\{\left[1+\tau\sum_{k=1}^{j+1}\left(\left(\frac{u_{i+1}^k-u_i^k}{h}\right)^2+\left(\frac{v_{i+1}^k-v_i^k}{h}\right)^2\right)\right]\frac{v_{i+1}^{j+1}-v_i^{j+1}}{h}\right.$$
$$\left.-\left[1+\tau\sum_{k=1}^{j+1}\left(\left(\frac{u_i^k-u_{i-1}^k}{h}\right)^2+\left(\frac{v_i^k-v_{i-1}^k}{h}\right)^2\right)\right]\frac{v_i^{j+1}-v_{i-1}^{j+1}}{h}\right\}$$
$$=f_{2,i}^j,\quad i=1,2,\ldots,M-1,\quad j=0,1,\ldots,N-1. \tag{3.1}$$

Let

$$A_i^\ell=1+\tau\sum_{k=1}^{\ell}\left[\left(\frac{u_{i+1}^k-u_i^k}{h}\right)^2+\left(\frac{v_{i+1}^k-v_i^k}{h}\right)^2\right],\quad i=0,1,\ldots,M-1, \tag{3.2}$$

then (3.1) becomes

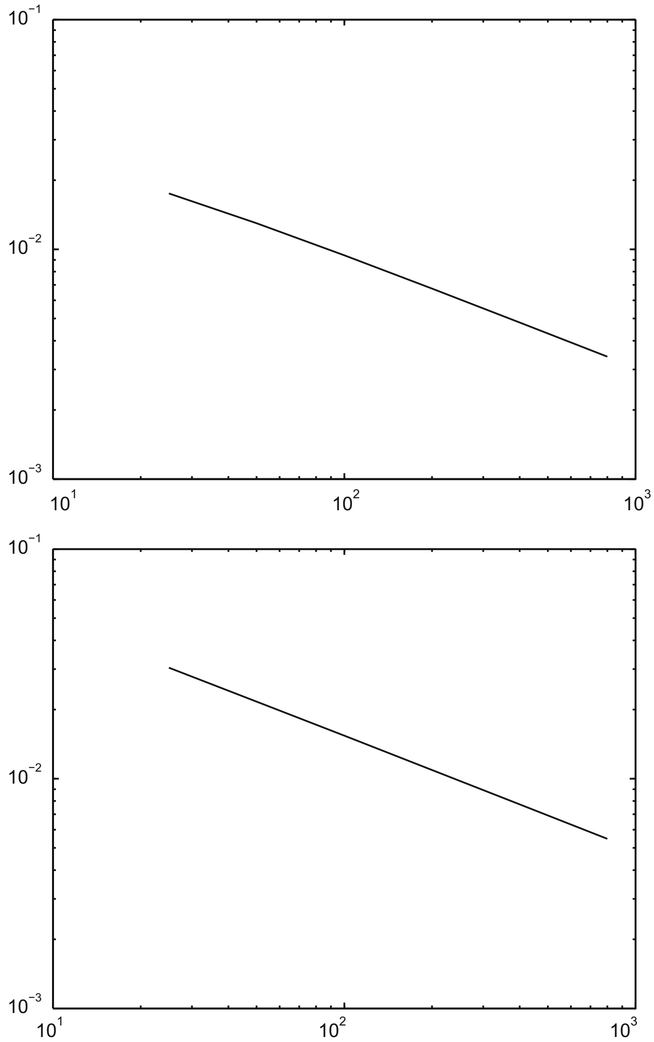

Fig. 2. The norm of the exact error as a function of the mesh size for u (top) and v (bottom) for Example 1.

$$\begin{aligned}
&\frac{u_i^{j+1}-u_i^j}{\tau}-\frac{1}{h}\left\{A_i^{j+1}\frac{u_{i+1}^{j+1}-u_i^{j+1}}{h}-A_{i-1}^{j+1}\frac{u_i^{j+1}-u_{i-1}^{j+1}}{h}\right\}=f_{1,i}^j,\\
&\frac{v_i^{j+1}-v_i^j}{\tau}-\frac{1}{h}\left\{A_i^{j+1}\frac{v_{i+1}^{j+1}-v_i^{j+1}}{h}-A_{i-1}^{j+1}\frac{v_i^{j+1}-v_{i-1}^{j+1}}{h}\right\}=f_{2,i}^j,\\
&\quad i=1,2,\ldots,M-1.
\end{aligned} \tag{3.3}$$

In order to rewrite this in matrix form, we define the vectors

$$\mathbf{u}^j=\begin{bmatrix}u_1^j\\ u_2^j\\ \vdots\\ u_{M-1}^j\end{bmatrix}$$

and similarly $\mathbf{v}^j$, $\mathbf{f}_1^j$, and $\mathbf{f}_2^j$. We also define the symmetric tridiagonal $(M-1)\times(M-1)$ matrix $\mathbf{T}$ as follows:

$$\mathbf{T}_{rs}^{\ell}=\begin{cases}-\frac{1}{h^2}A_{r-1}^{\ell}, & s=r-1,\\ \frac{1}{h^2}\left(A_r^{\ell}+A_{r-1}^{\ell}\right), & s=r,\\ -\frac{1}{h^2}A_r^{\ell}, & s=r+1,\\ 0, & \text{otherwise}.\end{cases}$$

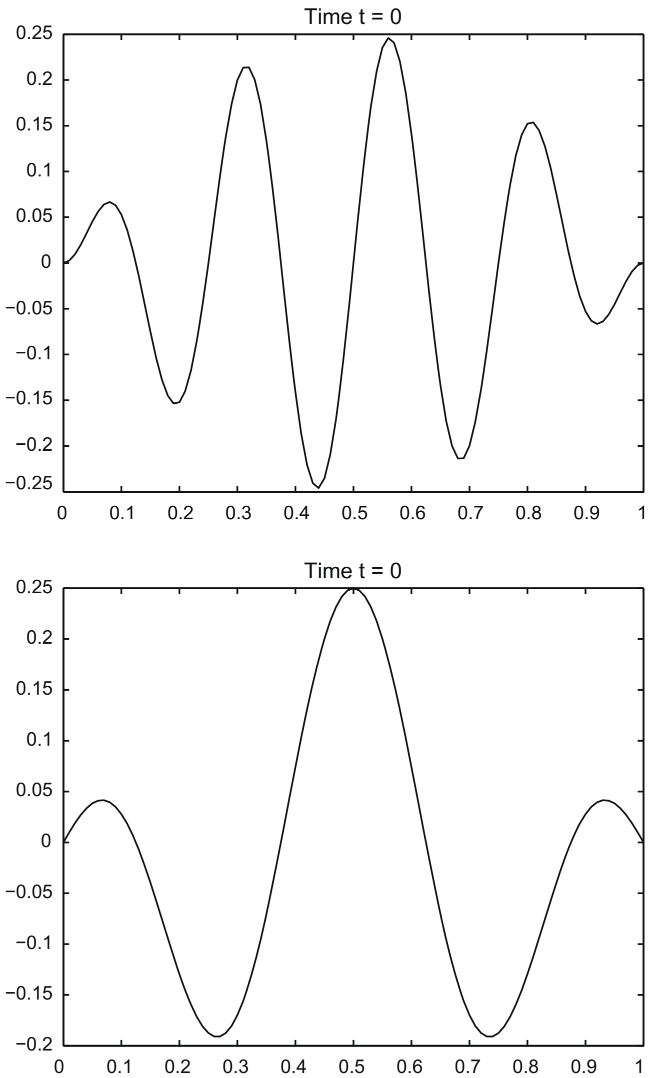

Fig. 3. The initial solution $U_0(x)=x(1-x)\sin(8\pi x)$ (top) and $V_0(x)=x(1-x)\cos(4\pi x)$ (bottom) for Example 2.

Thus the system (3.3) becomes

$$\frac{1}{\tau}\begin{bmatrix}\mathbf{u}^{j+1}\\ \mathbf{v}^{j+1}\end{bmatrix} - \frac{1}{\tau}\begin{bmatrix}\mathbf{u}^{j}\\ \mathbf{v}^{j}\end{bmatrix} + \begin{bmatrix}\mathbf{T}^{j+1} & 0\\ 0 & \mathbf{T}^{j+1}\end{bmatrix}\begin{bmatrix}\mathbf{u}^{j+1}\\ \mathbf{v}^{j+1}\end{bmatrix} - \begin{bmatrix}\mathbf{f}_1^{j}\\ \mathbf{f}_2^{j}\end{bmatrix} = 0. \tag{3.4}$$

We will use Newton's method to solve the nonlinear system (3.4). Let

$$\mathbf{P}^j = \begin{bmatrix}\mathbf{u}^{j}\\ \mathbf{v}^{j}\end{bmatrix}$$

and

$$\mathbf{F}^j = \begin{bmatrix}\mathbf{f}_1^{j}\\ \mathbf{f}_2^{j}\end{bmatrix}$$

and define

$$\mathbf{H}(\mathbf{P}^{j+1}) = \frac{1}{\tau}\mathbf{P}^{j+1} - \frac{1}{\tau}\mathbf{P}^{j} + \hat{\mathbf{T}}^{j+1}\mathbf{P}^{j+1} - \mathbf{F}^j, \tag{3.5}$$

where $\hat{\mathbf{T}}^{j+1}$ is the 2 by 2 block diagonal matrix with $\mathbf{T}^{j+1}$ on diagonal. We will now construct the gradient matrix. This matrix can be written in block form as follows:

$$\nabla\mathbf{H} = \begin{bmatrix}Q & R\\ W & Z\end{bmatrix},$$

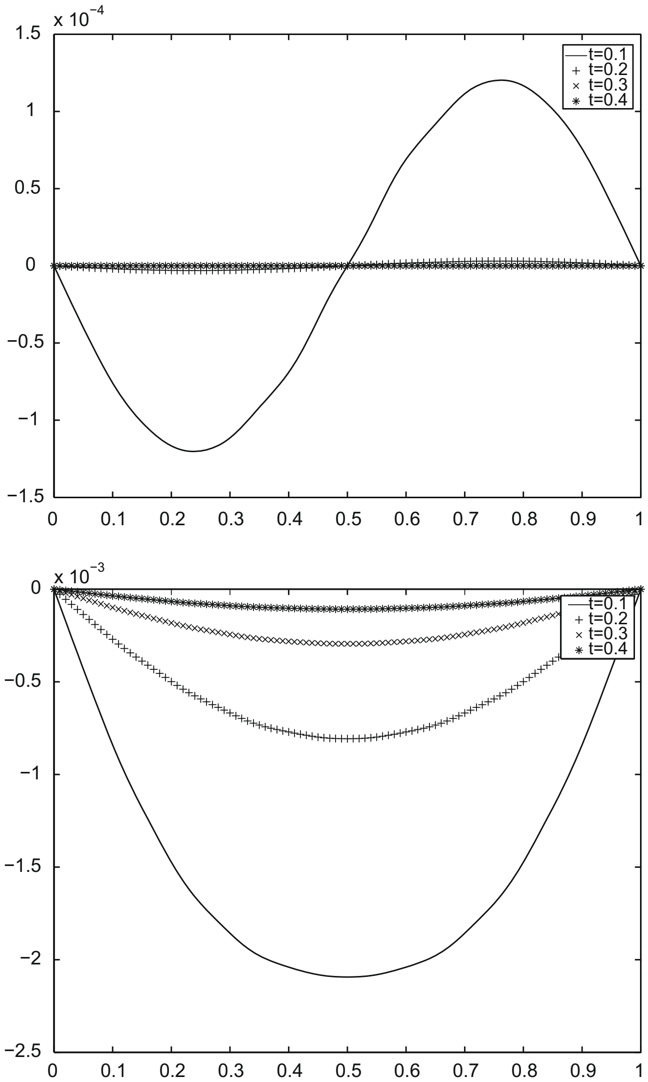

Fig. 4. The numerical solution at $t = 0.1, 0.2, 0.3, 0.4$ for u (top) and v (bottom).

where the tridiagonal matrices Q, R, W, Z are given below. Actually, the matrices R and W are identical.

$$Q_{rs} = \begin{cases} T_{rr-1}^{j+1} + \frac{\partial T_{rr-1}^{j+1}}{\partial u_{r-1}^{j+1}} u_{r-1}^{j+1} + \frac{\partial T_{rr}^{j+1}}{\partial u_{r-1}^{j+1}} u_r^{j+1}, & s = r-1, \\ \frac{1}{\tau} + T_{rr}^{j+1} + \frac{\partial T_{rr-1}^{j+1}}{\partial u_r^{j+1}} u_{r-1}^{j+1} + \frac{\partial T_{rr}^{j+1}}{\partial u_r^{j+1}} u_r^{j+1} + \frac{\partial T_{rr+1}^{j+1}}{\partial u_r^{j+1}} u_{r+1}^{j+1}, & s = r, \\ T_{rr+1}^{j+1} + \frac{\partial T_{rr}^{j+1}}{\partial u_{r+1}^{j+1}} u_r^{j+1} + \frac{\partial T_{rr+1}^{j+1}}{\partial u_{r+1}^{j+1}} u_{r+1}^{j+1}, & s = r+1, \\ 0, & \text{otherwise}, \end{cases} \tag{3.6}$$

$$R_{rs} = \begin{cases} \frac{\partial T_{rr-1}^{j+1}}{\partial v_{r-1}^{j+1}} u_{r-1}^{j+1} + \frac{\partial T_{rr}^{j+1}}{\partial v_{r-1}^{j+1}} u_r^{j+1}, & s = r-1, \\ \frac{\partial T_{rr}^{j+1}}{\partial v_r^{j+1}} u_r^{j+1} + \frac{\partial T_{rr-1}^{j+1}}{\partial v_r^{j+1}} u_{r-1}^{j+1} + \frac{\partial T_{rr+1}^{j+1}}{\partial v_r^{j+1}} u_{r+1}^{j+1}, & s = r, \\ \frac{\partial T_{rr}^{j+1}}{\partial v_{r+1}^{j+1}} u_r^{j+1} + \frac{\partial T_{rr+1}^{j+1}}{\partial v_{r+1}^{j+1}} u_{r+1}^{j+1}, & s = r+1, \\ 0, & \text{otherwise}, \end{cases} \tag{3.7}$$

Z_{rs} and W_{rs} are obtained by replacing u by v in Q_{rs} and R_{rs}, respectively.

Now we compute the first partial derivative of T_{rs}^{j+1} with respect to the components of u. Taking into account (3.2) we get

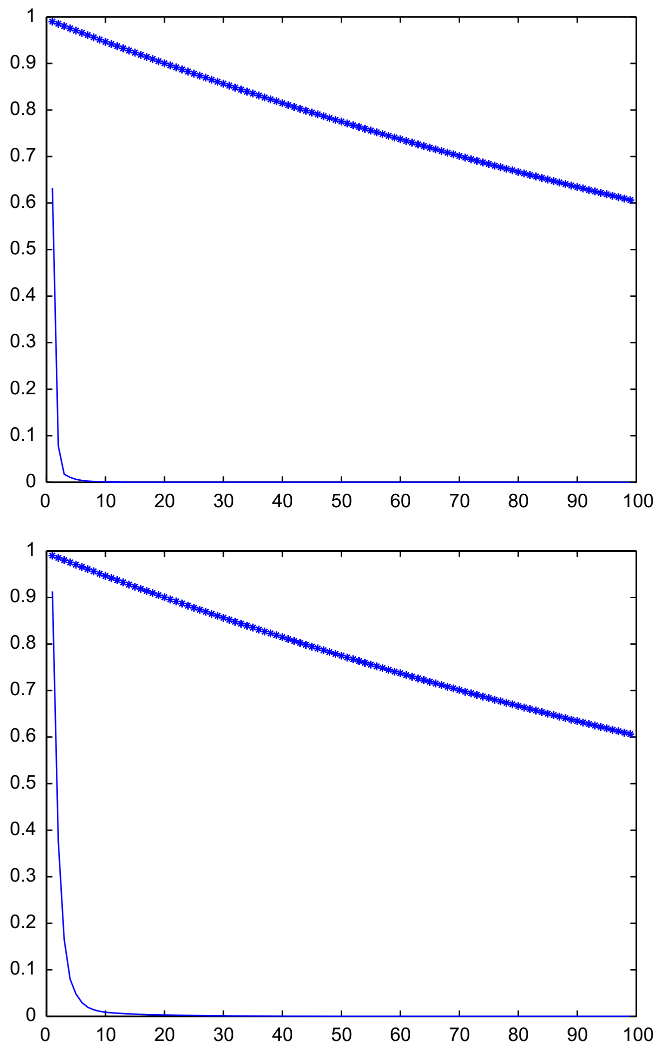

Fig. 5. The maximum norm of the numerical solution for $\frac{\partial U}{\partial x}$ (top) and $\frac{\partial V}{\partial x}$ (bottom) (Example 2) and $e^{-t/2}$. Solid line for $\frac{\partial U}{\partial x}$ and $\frac{\partial V}{\partial x}$ and line marked with * for the exponential.

$$\frac{\partial T^{j+1}_{rr-1}}{\partial u^{j+1}_s} = -\frac{1}{h^2}\frac{\partial A^{j+1}_{r-1}}{\partial u^{j+1}_s} = -\frac{\tau}{h^2}\frac{\partial}{\partial u^{j+1}_s}\left\{\left(\frac{u^{j+1}_r - u^{j+1}_{r-1}}{h}\right)^2\right\} = \begin{cases} \frac{2\tau}{h^3}\nabla_x u^{j+1}_r, & s = r-1, \\ -\frac{2\tau}{h^3}\nabla_x u^{j+1}_r, & s = r, \\ 0, & \text{otherwise}, \end{cases} \tag{3.8}$$

$$\begin{aligned} \frac{\partial T^{j+1}_{rr}}{\partial u^{j+1}_s} &= \frac{1}{h^2}\frac{\partial A^{j+1}_r}{\partial u^{j+1}_s} + \frac{1}{h^2}\frac{\partial A^{j+1}_{r-1}}{\partial u^{j+1}_s} \\ &= \frac{\tau}{h^2}\frac{\partial}{\partial u^{j+1}_s}\left\{\left(\frac{u^{j+1}_{r+1} - u^{j+1}_r}{h}\right)^2\right\} + \frac{\tau}{h^2}\frac{\partial}{\partial u^{j+1}_s}\left\{\left(\frac{u^{j+1}_r - u^{j+1}_{r-1}}{h}\right)^2\right\} \\ &= \begin{cases} -\frac{2\tau}{h^3}\Delta_x u^{j+1}_r + \frac{2\tau}{h^3}\nabla_x u^{j+1}_r, & s = r, \\ -\frac{2\tau}{h^3}\nabla_x u^{j+1}_r, & s = r-1, \\ \frac{2\tau}{h^3}\Delta_x u^{j+1}_r, & s = r+1, \\ 0, & \text{otherwise} \end{cases} \end{aligned} \tag{3.9}$$

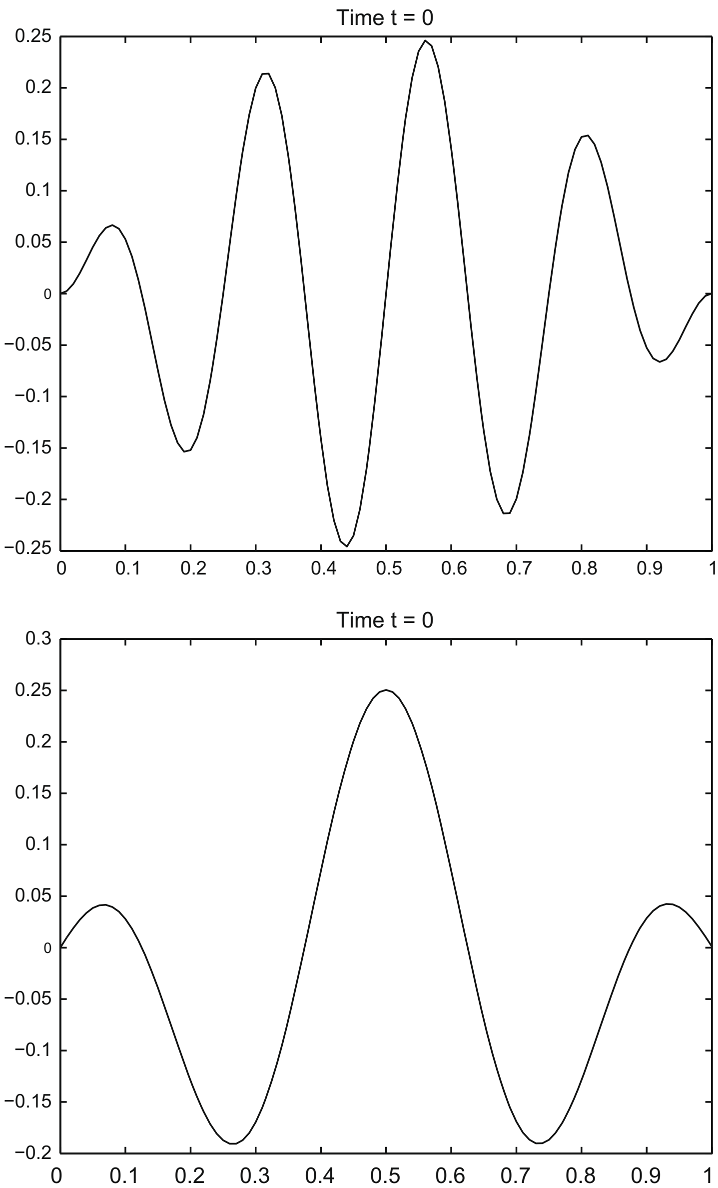

Fig. 6. The initial solution of non-homogeneous problem for $U_0(x) = x(1-x)\sin(8\pi x) + 0.0002x$ (top) and $V_0(x) = x(1-x)\cos(4\pi x) + 0.001x$ (bottom).

and

$$\frac{\partial T^{j+1}_{rr+1}}{\partial u^{j+1}_s} = -\frac{1}{h^2}\frac{\partial A^{j+1}_r}{\partial u^{j+1}_s} = -\frac{\tau}{h^2}\frac{\partial}{\partial u^{j+1}_s}\left\{\left(\frac{u^{j+1}_{r+1}-u^{j+1}_r}{h}\right)^2\right\} = \begin{cases} \frac{2\tau}{h^3}\Delta_x u^{j+1}_r, & s = r, \\ -\frac{2\tau}{h^3}\Delta_x u^{j+1}_r, & = r+1, \\ 0, & \text{otherwise.} \end{cases} \tag{3.10}$$

The partial derivatives with respect to v will have v instead of u everywhere in (3.8)–(3.10). Combining (3.6)–(3.10) we have

$$Q_{rs} = \begin{cases} -\frac{1}{h^2}A^{j+1}_{r-1} - \frac{2\tau}{h^2}\left(\nabla_x u^{j+1}_r\right)^2, & s = r-1, \\ \frac{1}{\tau} + \frac{1}{h^2}\left(A^{j+1}_r + A^{j+1}_{r-1}\right) + \frac{2\tau}{h^2}\left(\Delta_x u^{j+1}_r\right)^2 + \frac{2\tau}{h^2}\left(\nabla_x u^{j+1}_r\right)^2, & s = r, \\ -\frac{1}{h^2}A^{j+1}_r - \frac{2\tau}{h^2}\left(\Delta_x u^{j+1}_r\right)^2, & s = r+1, \\ 0, & \text{otherwise.} \end{cases} \tag{3.11}$$

Z_{rs} is obtained by replacing u by v in Q_{rs} and

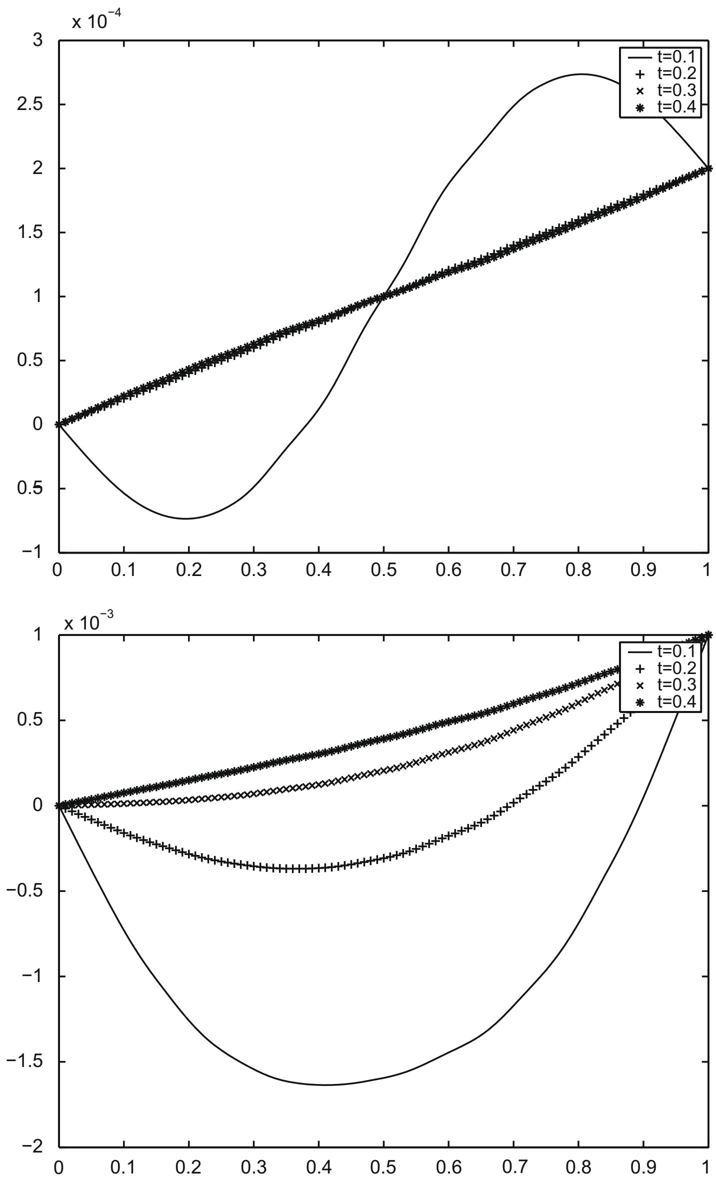

Fig. 7. The numerical solution of non-homogeneous problem at $t = 0.1, 0.2, 0.3, 0.4$ for u (top) and v (bottom).

$$R_{rs} = W_{rs} = \begin{cases} -\frac{2\tau}{h^2}\nabla_x u_r^{j+1}\nabla_x v_r^{j+1}, & s = r-1, \\ \frac{2\tau}{h^2}\Delta_x u_r^{j+1}\Delta_x v_r^{j+1} + \frac{2\tau}{h^2}\nabla_x u_r^{j+1}\nabla_x v_r^{j+1}, & s = r, \\ -\frac{2\tau}{h^2}\Delta_x u_r^{j+1}\Delta_x v_r^{j+1}, & s = r+1, \\ 0, & \text{otherwise}. \end{cases} \tag{3.12}$$

Using definition (3.5) Newton's method for the system (3.4) is given by

$$\nabla \mathbf{H}\left(\mathbf{P}^{j+1}\right)|^{(n)}\left(\mathbf{P}^{j+1}|^{(n+1)} - \mathbf{P}^{j+1}|^{(n)}\right) = -\mathbf{H}\left(\mathbf{P}^{j+1}\right)|^{(n)}.$$

Theorem 3.1. *Given the nonlinear system of equations*

$$H_i(P_1,\ldots,P_{2M-2}) = 0, \quad i = 1,2,\ldots,2M-2.$$

If H_i are three times continuously differentiable in a region containing the solution $\xi_1,\ldots,\xi_{2M-2}$ and the Jacobian does not vanish in that region, then Newton's method converges at least quadratically (see [29]).

The Jacobian is the matrix ∇H computed above. The term $\frac{1}{\tau}$ on diagonal ensures that the Jacobian does not vanish. The differentiability is guaranteed, since ∇H is quadratic.

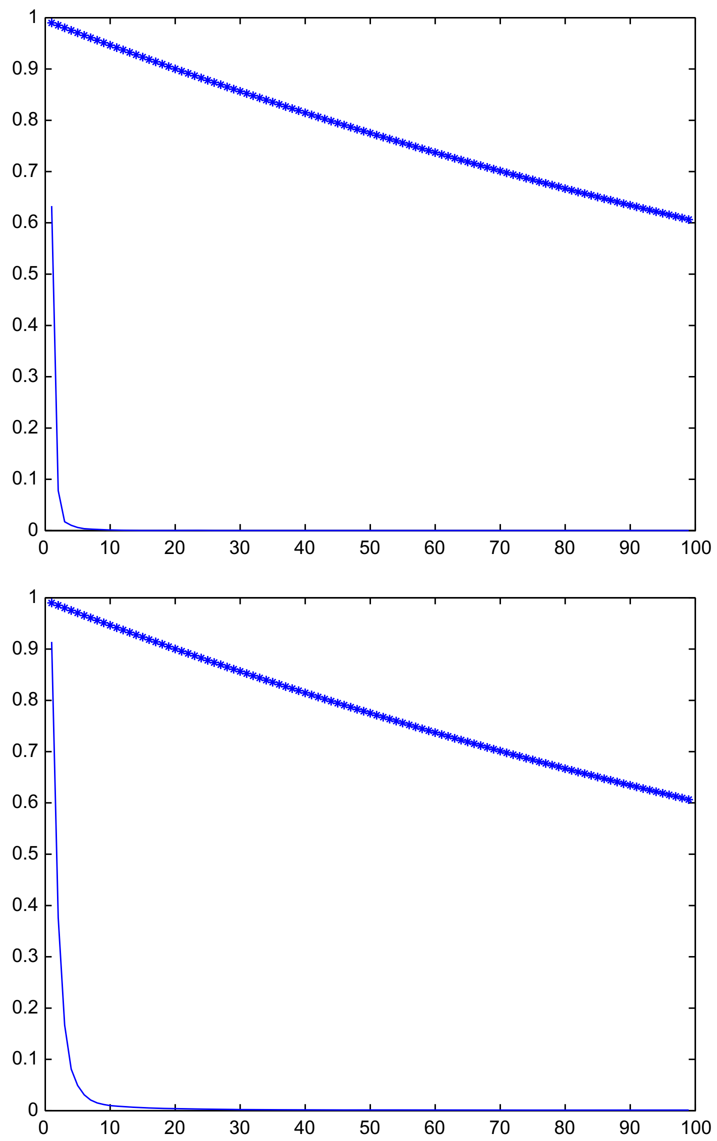

Fig. 8. The maximum norm of the numerical solution for $\frac{\partial U}{\partial x}$ (top) and $\frac{\partial V}{\partial x}$ (bottom) (Example 3) and $e^{-t/2}$. Solid line for $\frac{\partial U}{\partial x}$ and $\frac{\partial V}{\partial x}$ and line marked with * for the exponential.

In our first numerical experiment (Example 1) we have chosen the right-hand side so that the exact solution is given by

$$U(x,t) = x(1-x)\sin(x+t), \quad V(x,t) = x(1-x)\cos(x+t).$$

In this case the right-hand side is

$$\begin{aligned} f_1(x,t) &= x(1-x)\cos(x+t) - \alpha((1-2x)\sin(x+t) + x(1-x)\cos(x+t)) \\ &\quad -\beta(-2\sin(x+t) + 2(1-2x)\cos(x+t) - x(1-x)\sin(x+t)), \\ f_2(x,t) &= -x(1-x)\sin(x+t) - \alpha((1-2x)\cos(x+t) - x(1-x)\sin(x+t)) \\ &\quad -\beta(-2\cos(x+t) - 2(1-2x)\sin(x+t) - x(1-x)\cos(x+t)), \end{aligned}$$

where

$$\alpha = 10xt - 4t + 4x^3t - 6x^2t, \quad \beta = 1 + t + 5x^2t - 4xt + x^4t - 2x^3t.$$

The parameters used are $M = 100$ which dictates $h = 0.01$. Since the method is implicit we can use $\tau = h$ and we took 100 time steps. In the next four subplots we plotted the absolute value of the difference between the numerical and exact solutions on a semi-log axis at $t = 0.5$ and $t = 1$ (Fig. 1) and it is clear that the two solutions are almost identical.

In order to check the rate of convergence of the numerical scheme, we have ran the same example with $h = 1/25, 1/50, 1/100, 1/200, 1/400$ and $h = 1/800$. We have computed the error for each h for every time step. The norm is computed and plotted on a log–log scale to show that the finite difference scheme is first order in space, see Fig. 2.

In our next experiment (Example 2) we have taken zero right-hand side and initial condition given by

$$U_0(x) = U(x,0) = x(1-x)\sin(8\pi x), \quad V_0(x) = V(x,0) = x(1-x)\cos(4\pi x).$$

In this case, we know that the solution will decay in time [14]. The parameters M, h, τ are as before. In Fig. 3, we plotted the initial solution and in Fig. 4, we have the numerical solution at four different times. In both figures the top subplot is for u and the bottom subplot is for v. It is clear that the numerical solution is approaching zero for all x. We have also plotted the maximum norm of the partial derivatives $\frac{\partial U}{\partial x}$ and $\frac{\partial V}{\partial x}$ versus the exponential $e^{-t/2}$. Fig. 5 shows that the maximum norm of $\frac{\partial U}{\partial x}$ (top) and $\frac{\partial V}{\partial x}$ (bottom) decays faster than the exponential. Therefore, the numerical approximation of the x-derivative of the solution of our experiment fully agrees with the theoretical results given in [14].

We have experimented with several other initial solutions, and in all cases we noticed the decay of the numerical solution as expected [14].

We have experimented problem with nonhomogeneous boundary conditions on one side of lateral boundary as well (Example 3). In this case we have taken following initial conditions:

$$U_0(x) = U(x,0) = x(1-x)\sin(8\pi x) + 0.0002x, \quad V_0(x) = V(x,0) = x(1-x)\cos(4\pi x) + 0.001x.$$

We plotted the initial solution in Fig. 6 and the numerical solution at various times in Fig. 7. Now the solution approaches the steady state solution $U(x) = 0.0002x$ and $V(x) = 0.001x$, respectively.

We have also plotted the maximum norm of the partial derivatives $\frac{\partial U}{\partial x}$ and $\frac{\partial V}{\partial x}$ versus the exponential $e^{-t/2}$. Fig. 8 shows that the maximum norm of $\frac{\partial U}{\partial x}$ (top) and $\frac{\partial V}{\partial x}$ (bottom) decays faster than the exponential. Therefore, the numerical approximation of the x-derivative of the solution of our experiment shows exponential decay as in the homogeneous case. Theoretically we could not prove better than polynomial decay [14]. It is possible that this faster decay happens only under special circumstances.

Acknowledgement

The research described in this publication was made possible in part by Award No. NSS #17-07 of the US Civilian Research & Development Foundation (CRDF), the Georgia National Science Foundation (GNSF) and the Georgian Research and Development Foundation (GRDF). The designated project has been fulfilled by financial support of the Georgia National Science Foundation (Grant #GNSF/ST07/3-176). Any idea in this publication is possessed by the authors and may not represent the opinion of the Georgia National Science Foundation itself. The second author thanks the Naval Postgraduate School for hosting him. The authors thank the referees for their valuable comments.

References

[1] L. Landau, E. Lifschitz, Electrodynamics of Continuous Media. Course of Theoretical Physics, vol. 8 (Translated from the Russian), Pergamon Press, Oxford–London–New York–Paris; Addision-Wesley Publishing Co. Inc., Reading, MA, 1960; Russian original: Gosudarstv. Izdat. Tehn-Teor. Lit., Moscow, 1957.

[2] D. Gordeziani, T. Dzhangveladze(Jangveladze), T. Korshia, Existence and uniqueness of the solution of a class of nonlinear parabolic problems, Differ. Uravn. 19 (1983) 1197–1207 (Russian). English Translation: Diff. Eq. 19 (1983) 887–895.

[3] T. Dzhangveladze(Jangveladze), The first boundary value problem for a nonlinear equation of parabolic type, Dokl. Akad. Nauk. SSSR 269 (1983) 839–842 (Russian). English Translation: Soviet Phys. Dokl. 28 (1983) 323–324.

[4] T. Dzhangveladze(Jangveladze), A nonlinear integro-differential equation of parabolic type, Differ. Uravn. 21 (1985) 41–46 (Russian). English Translation: Diff. Eq. 21 (1985) 32–36.

[5] G. Laptev, Mathematical singularities of a problem on the penetration of a magnetic field into a substance, Zh. Vychisl. Mat. Mat. Fiz. 28 (1988) 1332–1345 (Russian). English Translation: USSR Comput. Math. Math. Phys. 28 (1990) 35–45.
[6] G. Laptev, Quasilinear parabolic equations which contains in coefficients Volterra's operator, Math. Sbornik 136 (1988) 530–545 (Russian). English Translation: Sbornik Math. 64 (1989) 527–542.
[7] N. Long, A. Dinh, Nonlinear parabolic problem associated with the penetration of a magnetic field into a substance, Math. Mech. Appl. Sci. 16 (1993) 281–295.
[8] N. Long, A. Dinh, Periodic solutions of a nonlinear parabolic equation associated with the penetration of a magnetic field into a substance, Comput. Math. Appl. 30 (1995) 63–78.
[9] M. Vishik, On solvability of the boundary value problems for higher order quasilinear parabolic equations, Math. Sbornik 59 (1962) 289–325 (Russian).
[10] J. Lions, Quelques methodes de resolution des problemes aux limites non-lineaires, Dunod, Gauthier-Villars, Paris, 1969.
[11] T. Dzhangveladze(Jangveladze), Z. Kiguradze, Asymptotics of a solution of a nonlinear system of diffusion of a magnetic field into a substance, Sibirsk. Mat. Zh. 47 (2006) 1058–1070 (Russian). English Translation: Siberian Math. J. 47 (2006) 867–878.
[12] T. Dzhangveladze(Jangveladze), Z. Kiguradze, On the stabilization of solutions of an initial-boundary value problem for a nonlinear integro-differential equation, Differ. Uravn. 43 (2007) 833–840 (Russian). English Translation: Diff. Eq. 43 (2007) 854-861.
[13] T. Dzhangveladze(Jangveladze), Z. Kiguradze, Asymptotic behavior of the solution of a nonlinear integro-differential diffusion equation, Differ. Uravn. 44 (2008) 517–529 (Russian). English Translation: Diff. Eq. 44 (2008) 538–550.
[14] T. Jangveladze, Z. Kiguradze, B. Neta, Large time behavior of solutions to a nonlinear integro-differential system, J. Math. Anal. Appl. 351 (2009) 382–391, doi:10.1016/j.jmaa.2008.10.016.
[15] B. Neta, Numerical solution of a nonlinear integro-differential equation, J. Math. Anal. Appl. 89 (1982) 598–611.
[16] B. Neta, J.O. Igwe, Finite differences versus finite elements for solving nonlinear integro-differential equations, J. Math. Anal. Appl. 112 (1985) 607–618.
[17] A.B. Iskakov, S. Descombes, E. Dormy, An integro-differential formulation for magnetic induction in bounded domains: boundary element-finite volume method, J. Comput. Phys. 197 (2004) 540–554.
[18] C. Grossmann, H.-G. Roos, M. Stynes, Numerical Treatment of Partial Differential Equations, Springer-Verlag, Berlin, 2007.
[19] I. Sloan, V. Thomee, Time discretization of an integro-differential equation of parabolic type, SIAM J. Numer. Anal. 23 (1986) 1052–1061.
[20] E. Yanik, G. Fairweather, Finite element methods for parabolic and hyperbolic partial integro-differential equations, Nonlinear Anal. 12 (1988) 785–809.
[21] V. Thomee, N. Zhang, Error estimates for semidiscrete finite element methods for parabolic integro-differential equations, Math. Comput. 53 (1989) 121–139.
[22] C. Chen, V. Thomee, L. Wahlbin, Finite element approximation of a parabolic integro-differential equation with a weakly singular kernel, Math. Comput. 58 (1992) 587–602.
[23] A. Pani, V. Thomee, L. Wahlbin, Numerical methods for hyperbolic and parabolic integro-differential equations, J. Integral Equat. Appl. 4 (1992) 533–584.
[24] V. Thomee, N. Zhang, Backward Euler type methods for parabolic integro-differential equations with non-smooth data, in: R.P. Agarwal (Ed.), Contributions in Numerical Mathematics (Singapore), World Scientific Series in Applicable Analysis, vol. 2, World Scientific Publishing Co., Singapore, 1993, pp. 373–388.
[25] N. Zhang, On fully discrete Galerkin approximations for partial integro-differential equations of parabolic type, Math. Comput. 60 (1993) 133–166.
[26] V. Thomee, L. Wahlbin, Long time numerical solution of a parabolic equation with memory, Math. Comput. 62 (1994) 477–496.
[27] T. Jangveladze, Convergence of a difference scheme for a nonlinear integro-differential equation, Proc. I Vekua Inst. Appl. Math. 48 (1998) 38–43.
[28] Z. Kiguradze, Finite difference scheme for a nonlinear integro-differential system, Proc. I Vekua Inst. Appl. Math. 50–51 (2000–2001) 65–72.
[29] W.C. Rheinboldt, Methods for Solving Systems of Nonlinear Equations, SIAM, Philadelphia, 1970.