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Aliquot sums of Fibonacci numbers

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Abstract

Here, we investigate the Fibonacci numbers whose sum of aliquot divisors is also a Fibonacci number (the prime Fibonacci numbers have this property).

1 Introduction

Let $(F_n)_{n\geq 1}$ be the sequence of Fibonacci numbers. For a positive integer n we write $\sigma(n)$ for the sum of divisors function of n. Recall that a number n is called *multiply perfect* if $n \mid \sigma(n)$. If $\sigma(n) = 2n$, then n is called *perfect*. In [2], it was shown that there are only finitely many multiply perfect Fibonacci

*This paper was written while at Auburn University Montgomery, Department of Mathematics, Montgomery, AL 36124-4023, USA numbers, and in [3], it was shown that no Fibonacci number is perfect. For a positive integer n, the value $\varphi(n)$ of the Euler function is defined to be the number of natural numbers less than or equal to n and coprime to n.

Let $s(n) = \sigma(n) - n$. The number $s(n)$ is sometimes called *the sum of aliquot divisors* of n. Two positive integers m and n (with $m \neq n$) are called *amicable* if $s(m) = n$ and $s(n) = m$. It is not known if there exist infinitely many amicable pairs, but Pomerance [5] showed that the sum of the reciprocals of all the members of all amicable pairs is convergent.

Here, we search for Fibonacci numbers F_n such that $s(F_n)$ is a Fibonacci number. In particular, prime Fibonacci numbers have the above property. We put

$$\mathcal{A} = \{n : s(F_n) = F_m \text{ for some positive integer } m\}.$$

In this paper, we give an upper bound on the counting function of $\mathcal{A}$.

Theorem 1. *There exists a positive constant c_0 such that the inequality*

$$\#\mathcal{A}(x) < c_0 \frac{x}{\log\log\log x}$$

holds for all $x > e^{e^e}$.

Throughout this paper, we use the Vinogradov symbols $\gg$, $\ll$ and the Landau symbols O, $\asymp$ and o with their usual meanings. We recall that $A \ll B$, $B \gg A$ and $A = O(B)$ are all equivalent and mean that $|A| < cB$ holds with some constant c, while $A \asymp B$ means that both $A \ll B$ and $B \ll A$ hold. For a positive real number x we write $\log x$ for the maximum between 2 and the natural logarithm of x. Note that with this convention, the function $\log x$ is sub-multiplicative; i.e., the inequality $\log(xy) \leq \log x \log y$ holds for all positive numbers x and y. For a positive real number t and a subset $\mathcal{B}$ of the positive integers, we write $\mathcal{B}(t) = \mathcal{B} \cap [1, t]$. We use p, q, P and Q with or without subscripts to denote prime numbers.

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2 The Proof of Theorem 1

Let x be a large positive real number.

2.1 Some sieving

Let $\omega(n)$ and $\Omega(n)$ be the number of prime divisors of n and the number of prime power divisors of n (> 1), respectively. Let

$$\mathcal{A}_1(x) = \{n \le x : \omega(n) < 0.9 \log\log x \text{ or } \Omega(n) > 1.1 \log\log x\}. \tag{1}$$

By the Turán-Kubilius inequalities (see [8])

$$\sum_{n \le x} (f(n) - \log\log x)^2 = O(x \log\log x) \qquad \text{for both } f \in \{\omega, \Omega\},$$

we infer that

$$\#\mathcal{A}_1(x) \ll \frac{x}{\log\log x}. \tag{2}$$

Let $y = (\log\log x)^{1/3}$ and let

$$\mathcal{A}_2(x) = \{n \le x : p \nmid n \text{ for all primes } p < y\}. \tag{3}$$

By Brun's sieve,

$$\#\mathcal{A}_2(x) \ll x \prod_{p<y} \left(1 - \frac{1}{p}\right) \ll \frac{x}{\log y} \ll \frac{x}{\log\log\log x}. \tag{4}$$

We now write

$$\sigma(F_n) = F_n + F_m,$$

and we look at bounds for m in terms of n, where $n \le x$ does not belong to $\mathcal{A}_1(x) \cup \mathcal{A}_2(x)$.

2.2 Bounds for m in terms of n

We start with a lower bound for m. Let $\gamma = (1+\sqrt{5})/2$ be the golden section. Let $n \le x$ not in $\mathcal{A}_1(x) \cup \mathcal{A}_2(x)$. Then, there exists $p < y$ such that $p \mid n$. Thus, $F_p \mid F_n$, therefore

$$\gamma^m > F_m = s(F_n) \ge \frac{F_n}{F_p} \gg \gamma^{n-p} \ge \gamma^{n-y},$$

where we used the fact that $F_n \asymp \gamma^n$. Hence,

$$m \ge n - y + O(1),$$

therefore

$$m \geq n - 2y,$$

once x is sufficiently large. We now look at an upper bound for m. Note that

$$\gamma^{m-n} \ll \frac{F_m}{F_n} \leq \frac{\sigma(F_n)}{F_n} \leq \frac{F_n}{\varphi(F_n)} \leq \prod_{p \mid F_n} \left(1 + \frac{1}{p-1}\right). \tag{5}$$

For every prime number p let $z(p)$ be its order of apparition in the Fibonacci sequence, and for a positive integer d let $\mathcal{P}_d = \{p : z(p) = d\}$. It is known that $p \equiv \pm 1 \pmod{z(p)}$ holds for all primes $p > 5$ and it is clear that

$$F_d \geq \prod_{p \in \mathcal{P}_d} p \gg (d-1)^{\#\mathcal{P}_d},$$

therefore

$$\#\mathcal{P}_d \ll \frac{d}{\log d}. \tag{6}$$

Furthermore, $z(p) \gg \log p$. We now get by taking logarithms in (5) that

$$m - n \leq \sum_{p \mid F_n} \frac{1}{p-1} + O(1) \leq \sum_{d \mid n} \sum_{p \in \mathcal{P}_d} \frac{1}{p-1} + O(1).$$

Obviously,

$$\sum_{p \in \mathcal{P}_d} \frac{1}{p-1} \leq \sum_{\substack{p \equiv \pm 1 \pmod{d} \\ p < d^2}} \frac{1}{p-1} + \frac{\#\mathcal{P}_d}{d^2 - 2} \ll \frac{\log \log d}{\varphi(d)},$$

where in the above inequality we have used estimate (6) as well as the known fact that the inequality

$$\sum_{\substack{p \equiv a \pmod{b} \\ p < t}} \frac{1}{p-1} \leq \frac{1}{p_1(a,b) - 1} + O\left(\frac{\log \log t}{\varphi(b)}\right), \tag{7}$$

holds uniformly in coprime positive integers $a < b$ and positive real numbers t, where $p_1(a,b)$ is the first prime in the arithmetic progression $a \pmod{b}$

(see, for example, [4]). Since the function $\log\log d$ is sub-multiplicative, we get that

$$\begin{aligned} m-n &\le \prod_{p^\mu\,||\,n}\left(1+O\left(\sum_{\nu=1}^{\mu}\frac{\log\log(p^\nu)}{p^\nu}\right)\right)\\ &\le \exp\left(O\left(\sum_{p\,|\,n}\frac{\log\log p}{p}+\sum_{p\ge 2}\sum_{\nu\ge 2}\frac{\log\log(p^\nu)}{p^\nu}\right)\right)\\ &= \exp\left(O\left(\sum_{p\,|\,n}\frac{\log\log p}{p}+1\right)\right). \end{aligned}$$

Since $n\notin\mathcal{A}_1(x)$, it follows that $\omega(n)<1.1\log\log x$. Thus, if we write $p_1<p_2<\ldots$ for the increasing sequence of all the prime numbers, then

$$\begin{aligned} \sum_{p\,|\,n}\frac{\log\log p}{p} &\le \sum_{i=1}^{\omega(n)}\frac{\log\log p}{p}\le\int_2^{p_{\omega(n)}}\frac{\log\log t}{t}d\pi(t)\\ &\ll (\log\log p_{\omega(n)})^2\ll(\log\log\log\log x)^2. \end{aligned}$$

Hence,

$$m-n\le\exp\big(O((\log\log\log\log x)^2)\big)<2y,$$

where the last inequality holds if x is large. In conclusion, if $n\le x$ is not in $\mathcal{A}_1(x)\cup\mathcal{A}_2(x)$, then $m\in[n-2y,n+2y]$.

2.3 More sieving

Let $\mathcal{Q}=\{q: z(q)<q^{1/3}\}$. Note that uniformly in $t>1$,

$$2^{\#\mathcal{Q}(t)}\le\prod_{\substack{q\in\mathcal{Q}\\ q<t}}q\le\prod_{n<t^{1/3}}F_n<\gamma^{\sum_{n<t^{1/3}}n}<\gamma^{t^{2/3}},$$

therefore

$$\#\mathcal{Q}(t)\ll t^{2/3},$$

which shows that

$$\begin{aligned}
\sum_{\substack{q\in\mathcal{Q}\\ q>s}} \frac{1}{q} &\le \int_s^\infty \frac{1}{t} d\#\mathcal{Q}(t) \\
&\le \frac{\#\mathcal{Q}(t)}{t}\Big|_{t=s}^{t=\infty} + \int_s^\infty \frac{\#\mathcal{Q}(t)}{t^2} \\
&\ll \frac{1}{s^{1/3}} + \int_s^\infty \frac{dt}{t^{4/3}} \ll \frac{1}{s^{1/3}}. \qquad (8)
\end{aligned}$$

We now put $u = (\log x)^3$ and let

$$\mathcal{A}_3(x) = \{n \le x, z(p)p \mid n \text{ for some } p > u\}. \qquad (9)$$

For every fixed prime $p > u$, the number of $n \le x$ which are multiples of $pz(p)$ is $\lfloor x/pz(p) \rfloor \le x/pz(p)$. So,

$$\begin{aligned}
\#\mathcal{A}_3(x) &\le \sum_{p>u} \frac{x}{pz(p)} \le \sum_{\substack{p>u\\ p\notin\mathcal{Q}}} \frac{x}{pz(p)} + \sum_{\substack{p>u\\ p\in\mathcal{Q}}} \frac{x}{z(p)p} \\
&\ll \sum_{u^{1/3}<d\le x} \sum_{\substack{p\equiv\pm1 \pmod d\\ p<d^3}} \frac{x}{dp} + \frac{x}{u^{1/3}} \\
&\ll x \sum_{u^{1/3}<d\le x} \frac{\log\log d}{d\varphi(d)} + \frac{x}{u^{1/3}} \\
&\ll x \sum_{u^{1/3}<d\le x} \frac{(\log\log d)^2}{d^2} + \frac{x}{u^{1/3}} \\
&\ll x(\log\log x)^2 \sum_{d>u^{1/3}} \frac{1}{d^2} + \frac{x}{u^{1/3}} \ll \frac{x(\log\log x)^2}{(\log x)^{1/3}}, \qquad (10)
\end{aligned}$$

where in the above estimates we used (8) with $s = u^{1/3}$, the fact that $\phi(d) \gg d/\log\log d$ for all d, as well as estimate (7) with $b = d$ and $a = 1$ and $d - 1$, respectively.

We finally put $\omega_u(n)$ for the number of prime factors $p \le u$ of n, $v = 2\log\log\log x$ and let

$$\mathcal{A}_4(x) = \{n \le x : \omega_u(n) > v\}. \qquad (11)$$

Again by Turán-Kubilius inequality,

$$\sum_{n<x}(\omega_u(n) - \log\log u)^2 = O(x \log\log u),$$

and since $\log\log u = (1 + o(1)) \log\log\log x$, we get easily that

$$\#\mathcal{A}_4(x) \ll \frac{x}{\log\log\log x}. \tag{12}$$

From now on, we deal only with numbers $n \le x$ which are not in $\mathcal{A}_1(x) \cup \mathcal{A}_2(x) \cup \mathcal{A}_3(x) \cup \mathcal{A}_4(x)$.

2.4 The 2-adic order of $\sigma(F_n)$

Let $K = \lfloor 0.8 \log\log x \rfloor$. Since $n \notin \cup_{i=1}^4 \mathcal{A}_i(x)$, we get that n has $\omega(n) - \omega_u(n) > 0.9 \log\log x - 2 \log\log\log x > K$ prime factors $P > u$, once x is sufficiently large. Let $P_1 > P_2 > \ldots > P_K$ be the first (largest) prime factors of n. Then $P_K > u$. Note that

$$F_n = \left(\prod_{i=0}^{K-1} \frac{F_{n/P_1 \ldots P_i}}{F_{n/P_1 \ldots P_{i+1}}} \right) F_{n/P_1 \ldots P_K},$$

where by convention we take $P_0 = 1$. Let

$$L_i = \frac{F_{n/P_1 \ldots P_i}}{F_{n/P_1 \ldots P_{i+1}}} \qquad \text{for } i = 0, \ \ldots, \ K-1 \qquad \text{and} \qquad L_K = F_{n/P_1 \ldots P_K}.$$

We next observe that L_i and L_j are coprime for all $0 \le i < j \le K$. Indeed, assume that $i < j \le K$ and Q are such that $Q \mid \gcd(L_i, L_j)$. Then

$$Q \,\Big|\, \gcd\left(F_{n/P_1 \ldots P_{i+1}}, \frac{F_{n/P_1 \ldots P_i}}{F_{n/P_1 \ldots P_{i+1}}} \right).$$

However, it is well-known that the greatest common divisor appearing above divides P_{i+1}. Hence, $Q = P_{i+1}$, and $Q \mid F_n$, therefore $z(Q) \mid n$. Since $Q > u > 5$ for large x, we get that $Qz(Q) \mid n$ contradicting the fact that $n \notin \mathcal{A}_3(x)$. Thus, L_i and L_j are indeed coprime for all $i < j$.

In [6], Ribenboim and McDaniel studied square-classes of Fibonacci numbers. Given two integers m and n, they are in the same square-class if $F_m F_n$

is a square. It follows from their results that if $m > 12n$ and n is sufficiently large, then m and n are not in the same square-class. In particular, if x is large, then none of the numbers L_i is a perfect square. Thus, there exists a prime $Q_i \mid L_i$, such that the order at which Q_i appears in L_i (hence, in F_n) is odd. It is also clear that Q_i is odd if x is large enough (say if $u > 3$). Thus, $\prod_{i=1}^{K}(Q_i + 1)$ is a divisor of $\sigma(F_n)$, which proves that $\sigma(F_n)$ is a multiple of 2^K.

2.5 The conclusion

Let $\mathcal{A}_5(x)$ be the set of all positive integers $n \in \mathcal{A}(x)$ which are not in $\cup_{i=1}^{4}\mathcal{A}_i(x)$. Let $n_1 < n_2 < \ldots < n_\ell$ be all the elements in $\mathcal{A}_5(x)$. Then there exists $k_i \in [-2y, 2y]$ such that $m_i = n_i + k_i$ for all $i = 1, \ldots, \ell$. Furthermore, $2^K \mid \sigma(F_{n_i}) = F_{n_i} + F_{n_i+k_i}$. Let $M = \lceil 4y + 1 \rceil$. We show that if $\ell > M$, then $n_{i+M} - n_i$ is large whenever $i \le \ell - M$. Indeed, let $n_i < n_{i+1} < \ldots < n_{i+M}$. Then $k_j \in [-2y, 2y]$ for all $j = i, \ \ldots, \ i + M$, and since there are at most $2\lfloor 2y \rfloor + 1 < M + 1$ possible values of k_j and $M + 1$ possibilities for the index j, it follows that there exist $j_1 < j_2$ in $\{i, \ldots, i + M\}$ such that $k_{j_1} = k_{j_2}$. Let k denote the common value of k_{j_1} and k_{j_2}. Using the formula $F_n = (\gamma^n - \delta^n)/(\gamma - \delta)$, where $\delta = (1 - \sqrt{5})/2$ is the conjugate of γ, we note that the relation $2^K \mid F_{n_{j_1}} + F_{n_{j_1}+k}$ gives

$$\gamma^{n_{j_1}}(1 + \gamma^k) - \delta^{n_{j_1}}(1 + \delta^k) \equiv 0 \pmod{2^K}, \tag{13}$$

and similarly for n_{j_2}. Here and in what follows, we say that an algebraic integer α is a multiple of an integer m if α/m is an algebraic integer. Write $\lambda = n_{j_2} - n_{j_1}$. Then the above relation for n_{j_2} gives

$$\gamma^{n_{j_1}}\gamma^\lambda(1 + \gamma^k) - \delta^{n_{j_1}}\delta^\lambda(1 + \delta^k) \equiv 0 \pmod{2^K}. \tag{14}$$

Multiplying the congruence (13) by γ^λ and subtracting it from congruence (14), we get that

$$\delta^{n_{j_1}}(1 + \delta^k)(\gamma^\lambda - \delta^\lambda) \equiv 0 \pmod{2^K}. \tag{15}$$

Conjugating the above relation (15) and multiplying the resulting congruences, we get

$$|1 + \delta^k||1 + \gamma^k||\gamma^\lambda - \delta^\lambda|^2$$

is an integer which is a multiple of 2^{2K}. Noting that the above integer is nonzero, by taking logarithms we get

$$\lambda \log \gamma + O(M) \geq 2K,$$

therefore $\lambda \gg K$. Thus, we just proved that $n_{i+M} - n_i \gg K$, therefore

$$\#\mathcal{A}_5(x) \ll \frac{xM}{K} + M \ll \frac{x}{(\log\log x)^{2/3}}, \tag{16}$$

which together with the upper bounds (2), (4), (10) and (12) completes the proof of Theorem 1.

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