



Calhoun: The NPS Institutional Archive

Faculty and Researcher Publications

Faculty and Researcher Publications

1991

Cramer-von Mises Variance Estimators for Simulations

Goldsman, David

<http://hdl.handle.net/10945/38618>



Calhoun is a project of the Dudley Knox Library at NPS, furthering the precepts and goals of open government and government transparency. All information contained herein has been approved for release by the NPS Public Affairs Officer.

Dudley Knox Library / Naval Postgraduate School
411 Dyer Road / 1 University Circle
Monterey, California USA 93943

<http://www.nps.edu/library>

CRAMÉR-von MISES VARIANCE ESTIMATORS FOR SIMULATIONS

David Goldsman

Keebom Kang

School of Industrial & Systems Engineering
 Georgia Institute of Technology
 Atlanta, Georgia 30332

Department of Administrative Sciences
 Naval Postgraduate School
 Monterey, California 93943

Andrew F. Seila

Dept. of Management Sciences & Information Technology
 University of Georgia
 Athens, Georgia 30602

ABSTRACT

We study estimators for the variance parameter σ^2 of a stationary process. The estimators are based on weighted Cramér-von Mises statistics formed from the standardized time series of the process. Certain weightings yield estimators which are “first-order unbiased” for σ^2 and which have low variance. We also show how the Cramér-von Mises estimators are related to the standardized time series area estimator; we use this relationship to establish additional estimators for σ^2 .

1 INTRODUCTION

Suppose Y_1, Y_2, \dots, Y_n is a stationary process with mean μ . The estimator of choice for μ is usually the sample mean \bar{Y}_n , which is unbiased. In order to measure the precision of \bar{Y}_n , one often estimates the *variance parameter*, $\sigma^2 \equiv \lim_{n \rightarrow \infty} n \text{Var}(\bar{Y}_n)$. There is a wide body of literature devoted to the topic of estimating σ^2 (see Bratley, Fox, and Schrage 1987). This paper studies estimators for σ^2 based on weighted Cramér-von Mises (CvM) statistics formed from the *standardized time series* of the process. The standardized time series is defined as

$$T_n(t) \equiv \frac{\lfloor nt \rfloor (\bar{Y}_n - \bar{Y}_{\lfloor nt \rfloor})}{\sigma \sqrt{n}} \text{ for } 0 \leq t \leq 1,$$

where $\bar{Y}_j \equiv \sum_{k=1}^j Y_k / j$, $j = 1, \dots, n$, and $\lfloor \cdot \rfloor$ is the greatest integer function. Schruben (1983) and Glynn and Iglehart (1990) show that $T_n \xrightarrow{D} B$, where B is a standard Brownian bridge and \xrightarrow{D} denotes convergence in distribution as $n \rightarrow \infty$.

The present article is organized as follows. §2 reviews the standardized time series weighted area estimator for σ^2 . §3 presents the CvM estimators, §4 establishes some of their properties, §5 demonstrates empirical work, §6 gives extensions, and §7 concludes.

2 THE WEIGHTED AREA ESTIMATOR

If we define

$$A_0(n) \equiv \frac{\sqrt{12} \sum_{k=1}^n \sigma T_n(\frac{k}{n})}{n}$$

and

$$A_0 \equiv \sqrt{12} \int_0^1 \sigma B(t) dt,$$

then it can be shown (see Schruben 1983 and Glynn and Iglehart 1990) that

$$A_0^2(n) \xrightarrow{D} A_0^2 \sim \sigma^2 \chi_1^2.$$

We call $A_0^2(n)$ the *unweighted area estimator* for σ^2 .

We can generalize this estimator by setting

$$A(n) \equiv \frac{\sum_{k=1}^n f(\frac{k}{n}) \sigma T_n(\frac{k}{n})}{n}$$

and

$$A \equiv \int_0^1 f(t) \sigma B(t) dt,$$

where (among other technical conditions) $f(t)$ is continuous and normalized so that $\text{Var}(A) = \sigma^2$. One can show (see Dzhaparidze 1986 and Goldsman, Meketon, and Schruben 1990) that

$$A^2(n) \xrightarrow{D} A^2 \sim \sigma^2 \chi_1^2.$$

We call $A^2(n)$ the *weighted area estimator* for σ^2 .

We denote the *covariance function* $R_k \equiv \text{Cov}(Y_1, Y_{1+k})$ and the quantities $\gamma \equiv -2 \sum_{k=1}^{\infty} k R_k$, $F \equiv \int_0^1 f(s) ds$, and $\bar{F} \equiv \int_0^1 \int_0^t f(s) ds dt$. Then under mild conditions (see Schmeiser and Song 1989, Foley and Goldsman 1990, and Goldsman, Meketon, and Schruben 1990),

$$E[A^2(n)] = \sigma^2 + \frac{[(F - \bar{F})^2 + \bar{F}^2]\gamma}{2n} + o\left(\frac{1}{n}\right).$$

Example 1 The expected value of the unweighted area estimator is $E[A_0^2(n)] = \sigma^2 + 3\gamma/n + o(1/n)$. The expected value of the weighted area estimator with weighting function $f(t) = \sqrt{840}(3t^2 - 3t + 1/2)$ is $E[A^2(n)] = \sigma^2 + o(1/n)$. In this case, we say that $A^2(n)$ is *first-order unbiased* for σ^2 .

Further, if $A^4(n)$ is uniformly integrable, then the asymptotic variance of the weighted area estimator is $\text{Var}(A^2) = 2\sigma^4$.

3 THE WEIGHTED CRAMÉR-von MISES ESTIMATOR

In the spirit of §2, we define the *unweighted CvM estimator* for σ^2 by

$$W_0^2(n) \equiv \frac{6 \sum_{k=1}^n (\sigma T_n(\frac{k}{n}))^2}{n}.$$

One can show that

$$W_0^2(n) \xrightarrow{D} W_0^2 \equiv 6 \int_0^1 (\sigma B(t))^2 dt.$$

Cramér (1928) and von Mises (1931) studied statistics nearly of the form of $W_0^2(n)$ for the case of independent and identically distributed Y_1, Y_2, \dots . Anderson and Darling (1952) and Smirnov (1937) derived the distribution of W_0^2 .

A generalization of $W_0^2(n)$ is the *weighted CvM estimator*,

$$W^2(n) \equiv \frac{\sum_{k=1}^n g(\frac{k}{n})(\sigma T_n(\frac{k}{n}))^2}{n}.$$

Under mild conditions,

$$W^2(n) \xrightarrow{D} W^2 \equiv \int_0^1 g(t)(\sigma B(t))^2 dt,$$

where $g(t)$ is continuous on $[0, 1]$ and normalized so that $E[W^2] = \sigma^2$. Anderson and Darling derived the distribution of W^2 with $g(t) = [t(1-t)]^{-1}$ (which is not continuous on $[0, 1]$); the distribution of W^2 with an arbitrary weighting function has not been explicitly determined (see Durbin 1973).

4 PROPERTIES OF CvM ESTIMATORS

We can express the expected value of the CvM estimator in terms of $g(t)$ and R_k . First, we need some standing assumptions.

Assumptions

1. The constants μ and σ^2 satisfy $X_n \xrightarrow{D} \sigma Z$, where Z is a standard Brownian motion and

$$X_n(t) \equiv \frac{[nt](\bar{Y}_{[nt]} - \mu)}{\sqrt{n}},$$

2. $\sum_{k=-\infty}^{\infty} R_k = \sigma^2 > 0$,
3. $\sum_{k=1}^{\infty} k^2 |R_k| < \infty$,
4. g'' exists and is bounded on $[0, 1]$, and
5. $E[W^2]/\sigma^2 = \int_0^1 g(t)t(1-t) dt = 1$.

(Glynn and Iglehart 1990 list various sets of sufficient conditions for Assumption 1 to hold; these usually involve moment and mixing conditions.)

We now state the main theorem. (All proofs are in Goldsman, Kang, and Seila 1991.)

Theorem 1 Let $G \equiv \int_0^1 g(s) ds$. Under the standing assumptions,

$$E[W^2(n)] = \sigma^2 + \frac{\gamma}{n}(G - 1) + o\left(\frac{1}{n}\right).$$

Consider the simplest case in which $g(t)$ is a constant weighting function.

Example 2 If $g(t) = 6$ for all $t \in [0, 1]$, then Theorem 1 implies that $E[W_0^2(n)] = \sigma^2 + 5\gamma/n + o(1/n)$.

If $G = 1$ (and the standing assumptions hold), Theorem 1 says that the bias of $W^2(n)$ as an estimator of σ^2 is $o(1/n)$. In this case, $W^2(n)$ is first-order unbiased for σ^2 .

Example 3 Suppose $g(t) = 51 - c/2 + ct - 150t^2$, where $t \in [0, 1]$ and c is real. Then Theorem 1 implies that $E[W^2(n)] = \sigma^2 + o(1/n)$.

We can even give exact small-sample results for certain stochastic processes.

Example 4 Consider an MA(1) process $Y_i = \theta\epsilon_{i-1} + \epsilon_i$, $i = 1, 2, \dots$, where the ϵ_i 's are independent normal $(0, 1)$; so $R_0 = 1 + \theta^2$, $R_{\pm 1} = \theta$, and $R_k = 0$, otherwise. For the weights $g(t) = 6$, we have

$$\begin{aligned} E[W_0^2(n)] &= \sigma^2 \left(1 - \frac{1}{n^2}\right) + \frac{\gamma(n-1)(5n-1)}{n^3} \\ &= \sigma^2 + \frac{5\gamma}{n} + o\left(\frac{1}{n}\right). \end{aligned}$$

For $g(t) = 51 - 150t^2$, some algebra yields

$$\begin{aligned} E[W^2(n)] &= \frac{\sigma^2(n^2 - 1)(n^2 + 5)}{n^4} + \frac{\gamma(24n^3 - 29n^2 - 5)}{n^5} \\ &= \sigma^2 + o\left(\frac{1}{n}\right). \end{aligned}$$

These results are in accord with Examples 2 and 3.

The choice of weights clearly affects the variance of $W^2(n)$. In fact, the next theorem gives a useful result on the limiting variance.

Theorem 2 *Suppose $W^4(n)$ is uniformly integrable. Then under the standing assumptions,*

$$\begin{aligned} \text{Var}(W^2(n)) &\rightarrow \text{Var}(W^2) \\ &= 4\sigma^4 \int_0^1 g(t)(1-t)^2 \int_0^t g(s)s^2 ds dt. \end{aligned}$$

Example 5 If $g(t) = 6$ (as in Example 2), then Theorem 2 implies that $\text{Var}(W_0^2) = 4\sigma^4/5$.

Example 6 Suppose $g(t) = 51 - c/2 + ct - 150t^2$, where c is real (as in Example 3). Then $\text{Var}(W^2) = (c^2 - 300c + 26856)\sigma^4/12600$. This quantity is minimized by $c = 150$, in which case $\text{Var}(W^2) = 1.729\sigma^4$.

One would like to choose a weighting function which minimizes the variance of the CvM estimator while satisfying the first-order unbiasedness and normalizing constraints; i.e., find $g(t)$ which minimizes $\text{Var}(W^2)$ subject to $G = 1 = \int_0^1 g(t)t(1-t) dt$. It is easy to show via Lagrangian multipliers that the optimal quadratic and cubic polynomial weighting function is $g(t) = -24 + 150t - 150t^2$, the choice studied in Example 6. The best quartic is

$$g(t) = \frac{-1310}{21} + \frac{19270t}{21} - \frac{25230t^2}{7} + \frac{16120t^3}{3} - \frac{8060t^4}{3},$$

for which $\text{Var}(W^2) = 1.042\sigma^4$.

5 EMPIRICAL WORK

We present the results of some Monte Carlo simulations to evaluate the performance characteristics of the CvM estimators. Consider the AR(1) process $Y_{i+1} = \phi Y_i + \epsilon_{i+1}$, $i = 1, \dots, n$, where the ϵ_i 's are independent normal $(0, 1 - \phi^2)$ random variables, $-1 < \phi < 1$, and Y_1 is initialized as a normal $(0, 1)$ random variable independent of the ϵ_i 's; so the Y_i 's are stationary with normal $(0, 1)$ marginals and covariance function $R_k = \phi^{|k|}$. Some algebra shows that the variance parameter is $\sigma^2 = (1 + \phi)/(1 - \phi)$.

For each value of $n = 2^k$, $k = 3, 4, \dots, 9$, we ran 10000 independent simulations of the process with $\phi = 0.9$ to estimate the expected values and variances of four estimators for $\sigma^2 = 19$:

- Unweighted area estimator $A_0^2(n)$.
- Weighted area estimator $A^2(n)$ with first-order unbiased weighting function $f(t) = \sqrt{840}(3t^2 - 3t + 1/2)$ (Example 1).
- Unweighted CvM estimator $W_0^2(n)$.
- Weighted CvM estimator $W^2(n)$ with first-order unbiased weighting function $g(t) = -24 + 150t - 150t^2$ (Example 6).

The results are given in Table 1. The table contains the estimated expected values of the four estimators for various n ; the numbers in parentheses are the associated standard errors of the entries above them. The standard errors allow us to estimate the variance of the estimators. We first notice that all of the estimators become less biased for σ^2 as n increases. Consider the entries for "large" n , say $n \geq 512$. As predicted by Example 1, the unweighted area estimator $A_0^2(n)$ has comparatively high bias and variance, while the first-order unbiased weighted area estimator $A^2(n)$ has much lower bias and about the same variance. The unweighted CvM estimator $W_0^2(n)$ has high bias (cf. Example 2) but very low variance (cf. Example 5). Finally, the first-order unbiased CvM estimator $W^2(n)$ has very low bias (cf. Example 3) and variance which is slightly lower than those of the area estimators (cf. Example 6).

6 EXTENSIONS

We briefly suggest some extensions to the CvM estimators, all of which are discussed in Goldsman, Kang, and Seila (1991).

6.1 Still More Estimators

Another class of estimators is based on the relationship between the unweighted area and CvM estimators. With the fact that $\text{Cov}(A_0^2, W_0^2) = 6\sigma^4/5$ in mind, we consider the estimator (cf. Durbin 1973 and Watson 1961)

$$\begin{aligned} U_0^2(n) &\equiv \frac{12}{n} \sum_{k=1}^n \left(\sigma T_n\left(\frac{k}{n}\right) - \frac{A_0(n)}{\sqrt{12}} \right)^2 \\ &= 2W_0^2(n) - A_0^2(n) \\ &\xrightarrow{D} U_0^2 \equiv 2W_0^2 - A_0^2. \end{aligned}$$

Table 1: Sample Expectations of Variance Estimators

n	$A_0^2(n)$	$A^2(n)$	$W_0^2(n)$	$W^2(n)$
8	0.97 (0.01)	0.96 (0.01)	0.69 (0.01)	0.93 (0.04)
16	2.83 (0.04)	2.69 (0.04)	2.04 (0.03)	2.75 (0.07)
32	6.50 (0.09)	6.36 (0.09)	4.90 (0.06)	6.52 (0.11)
64	11.02 (0.15)	11.58 (0.16)	8.99 (0.10)	11.64 (0.17)
128	14.95 (0.21)	16.45 (0.23)	13.15 (0.14)	16.29 (0.22)
256	16.80 (0.24)	18.01 (0.25)	15.77 (0.15)	18.07 (0.24)
512	18.12 (0.25)	18.86 (0.26)	17.41 (0.16)	18.89 (0.25)
1024	18.44 (0.26)	18.85 (0.27)	18.12 (0.17)	18.89 (0.25)

As before, we can generalize $U_0^2(n)$ to obtain additional estimators for σ^2 . Define

$$U^2(n) \equiv \frac{1}{n} \sum_{k=1}^n h\left(\frac{k}{n}\right) \left(\sigma T_n\left(\frac{k}{n}\right) - \frac{A_0(n)}{\sqrt{12}} \right)^2$$

$$\xrightarrow{D} U^2 \equiv \sigma^2 \int_0^1 h(t)(B(t) - \bar{B})^2 dt,$$

where $\bar{B} \equiv \int_0^1 B(t) dt$ and $h(t)$ is a continuous, bounded weighting function on $[0, 1]$, normalized so that $\int_0^1 h(t) dt = 12$. Under mild conditions,

$$E[U^2(n)] \rightarrow E[U^2] = \sigma^2$$

and

$$\text{Var}(U^2(n)) \rightarrow \text{Var}(U^2)$$

$$= 4\sigma^4 \int_0^1 \int_0^t h(s)h(t)c^2(s, t) ds dt,$$

where $c(s, t) \equiv s(1 - t) - \frac{s-s^2}{2} - \frac{t-t^2}{2} + \frac{1}{12}$.

We mention in passing that it is possible to devise estimators for σ^2 based on other functionals of Brownian bridges – for instance, the Anderson-Darling statistic or $\int_0^1 |B(t)| dt$.

6.2 Estimators Using Batching

All of our work so far has assumed that we have one long batch of n observations. Alternatively, we can break the n observations into b contiguous,

nonoverlapping batches, each of size m (assume $n = bm$). Then, e.g., let $W_i^2(m)$, $i = 1, \dots, b$, denote the CvM estimator formed exclusively from the i th batch of observations, $Y_{(i-1)m+1}, Y_{(i-1)m+2}, \dots, Y_{im}$. The CvM batch estimator for σ^2 is $\bar{W}^2(m) \equiv \sum_{i=1}^b W_i^2(m)/b$. Of course, $E[\bar{W}^2(m)] = E[W^2(m)]$ and, if the $W_i^2(m)$'s are approximately independent, $\text{Var}(\bar{W}^2(m)) \approx \text{Var}(W^2(m))/b$.

6.3 Overlapping Estimators

We can also apply the methodology of Meketon and Schmeiser (1984) in which the n observations are broken into $n - m + 1$ overlapping batches, each of size m . Then, e.g., let $W^2(i, m)$, $i = 1, \dots, n - m + 1$, denote the CvM estimator formed exclusively from the observations $Y_i, Y_{i+1}, \dots, Y_{i+m-1}$. The CvM overlapping estimator for σ^2 is $\bar{W}^2(m) \equiv \sum_{i=1}^{n-m+1} W^2(i, m)/(n - m + 1)$. Clearly, $E[\bar{W}^2(m)] = E[W^2(m)]$; further, in the special case that each $W^2(i, m)$ uses the weighting function $g(t) = 6$, Goldsman and Meketon (1990) show that $\text{Var}(\bar{W}^2(m)) \approx \frac{11}{21} \text{Var}(W^2(m))$.

7 CONCLUSIONS

In this article, we introduced a class of CvM estimators for σ^2 , derived expectation and variance properties, discussed some empirical results, and proposed extensions to the initial work. Although the estimators are all asymptotically unbiased for σ^2 , they can be quite biased for finite samples. Luckily, we were able to find first-order unbiased estimators having comparatively low variance.

ACKNOWLEDGMENTS

David Goldsman's work was supported by National Science Foundation Grant No. DDM-90-12020. Keebom Kang's work was supported by the Naval Weapons Support Center, Crane, Indiana. We thank George Fishman and Bruce Schmeiser for many interesting discussions.

REFERENCES

Anderson, T. W., and D. A. Darling. 1952. Asymptotic theory of certain 'goodness of fit' criteria based on stochastic processes. *Annals of Mathematical Statistics* **23**, 193–212.

Bratley, P., B. L. Fox, and L. E. Schrage. 1987. *A Guide to Simulation*, 2nd Ed. New York: Springer-Verlag.

- Cramér, H. 1928. On the composition of elementary errors. Second paper: statistical applications. *Skand. Aktuariidskr.* **11**, 141–180.
- Durbin, J. 1973. *Distribution Theory for Tests Based on the Sample Distribution Function*. Philadelphia: Society for Industrial and Applied Mathematics.
- Dzhaparidze, K. 1986. *Parameter Estimation and Hypothesis Testing in Spectral Analysis of Stationary Time Series*, New York: Springer-Verlag.
- Foley, R. D., and D. Goldsmán. 1990. Confidence intervals using orthonormally weighted standardized time series. Technical Report, School of Industrial and Systems Engineering, Georgia Institute of Technology, Atlanta, Georgia.
- Glynn, P., and D. L. Iglehart. 1990. Simulation output analysis using standardized time series. *Mathematics of Operations Research* **15**, 1–16.
- Goldsmán, D., K. Kang, and A. F. Seila. 1991. Cramér-von Mises variance estimators for simulations. Technical Report, School of Industrial and Systems Engineering, Georgia Institute of Technology, Atlanta, Georgia.
- Goldsmán, D., and M. S. Meketon. 1990. A comparison of several variance estimators. Technical Report, School of Industrial and Systems Engineering, Georgia Institute of Technology, Atlanta, Georgia.
- Goldsmán, D., M. S. Meketon, and L. Schruben. 1990. Properties of standardized time series weighted area variance estimators. *Management Science* **36**, 602–612.
- Meketon, M. S., and B. W. Schmeiser. 1984. Overlapping batch means: Something for nothing? *Proceedings of the 1984 Winter Simulation Conference*, 227–230.
- Schmeiser, B. W., and W.-M. Song. 1989. Optimal mean-squared-error batch sizes. Technical Report, School of Industrial Engineering, Purdue University, West Lafayette, Indiana.
- Schruben, L. 1983. Confidence interval estimation using standardized time series. *Operations Research* **31**, 1090–1108.
- Smirnov, N. V. 1937. On the distribution of the von Mises ω^2 -criterion (in Russian). *Matem Sbornik.* **5**, 973–993.
- von Mises, R. 1931. *Wahrscheinlichkeitsrechnung*. Leipzig: Wein.
- Watson, G. S. 1961. Goodness-of-fit tests on a circle. *Biometrika* **48**, 109–114.

AUTHOR BIOGRAPHIES

DAVID GOLDSMAN is an Associate Professor in the School of Industrial and Systems Engineering at the Georgia Institute of Technology. His research in-

terests include simulation output analysis and ranking and selection. He is Secretary-Treasurer of the TIMS College on Simulation.

KEEBOM KANG is on the faculty of the logistics group in the Administrative Sciences Department at the Naval Postgraduate School. He received his Ph.D. in Industrial Engineering from Purdue University. His research interests are stochastic modeling, and probabilistic and statistical aspects of computer simulation. He is currently conducting weapon reliability research sponsored by the Naval Weapons Support Center, Crane, Indiana, and military logistics research supported by the U. S. Navy.

ANDREW F. SEILA is an Associate Professor of Management Sciences at The University of Georgia. He received the Ph.D. degree in Operations Research from the University of North Carolina at Chapel Hill. His research interests include simulation model development and validation, simulation program development, and output analysis. Dr. Seila is a member of TIMS, ORSA, ASA, and the TIMS College on Simulation, and he is Program Chairman for the 1994 Winter Simulation Conference.