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The Diesel Submarine Flaming Datum Problem

Alan Washburn

Naval Postgraduate School
AWashburn@nps.navy.mil

Ryusuke Hohzaki

Japan National Defense Academy

ABSTRACT

The Flaming Datum problem is one of relocating an enemy target that is fleeing after momentarily revealing its position. A diesel submarine faces this problem after attacking a ship, since the ship creates a visible marker of where the submarine must once have been. The tactical problem has been studied before under the assumption that the submarine's motion is constrained only by a top speed. Here we add the constraint that the battery's capacity is also finite. The problem is bounded rather than solved. Techniques used include two-person zero-sum game theory and optimal control theory.

BACKGROUND

In World War II, it would sometimes happen that one ship in a convoy would be torpedoed by a submarine. The convoy's destroyer escorts would then attempt to locate the submarine in the vicinity of the "flaming datum" where the submarine must once have been. Since diesel-electric submarines of that era actually spent most of their time on the surface, it was also frequent for a submarine to crash-dive right after a visual detection by an aircraft, thus initiating a search in similar circumstances where the submarine's location at some time in the past is known. Regardless of the mechanism that initiates the action, we will refer to all such searches as flaming datum problems (FDPs). The main characteristics of an FDP are that a mobile platform performs some action that permits its previously unknown location to become known to an enemy, that the action occurs at a point (rather than an interval) in time, and that the platform knows that its location has been revealed. A two-person-zero-sum game ensues where a searcher or searching force attempts to relocate and attack the platform, or at least detect it by other means.

FDPs can also occur on land (Shupenus and Barr, 2000). A modern military example is when a missile is launched from a mobile vehicle. The launch can sometimes be detected and located, thus providing a location at which the vehicle must once have been. However, this article is essentially restricted to FDPs involving diesel submarines because of constraints on target motion that will be outlined below. Diesel submarines have become much quieter and generally more effective than they were in World War II, and are an important component of many modern navies (Janes [1999]). Flaming datum attacks may still be an effective countermeasure to submarine predation, as they were in WWII.

Section 2 summarizes previous work, Section 3 discusses crucial assumptions, and Section 4 provides an abstract formulation of the problem as a continuous two-person zero-sum game that is too complicated to solve exactly. Section 5 exactly solves a discrete analog. Sections 6 and 7 derive lower and upper bounds on the continuous game, and Section 8 applies the bounds to specific examples. Section 9 attempts (in vain) to reduce the gap between bounds. Section 10 is a summary.

SUMMARY OF PREVIOUS WORK

Previous work on the FDP assumes that the submarine's velocity vector must not exceed a given top speed S in absolute value, but is otherwise unconstrained. The largest distance that the submarine can move in time t is then St, the radius of the Farthest-on Circle (FOC). The earliest known analysis along these lines is Koopman (1980), who includes a WWII analysis of a problem where the submarine's direction (course) is fixed but unknown while its radial speed is known to be S. In such circumstances the searcher's best track is generally a spiral of some kind, but a spiral is in practice vulnerable to the possibility that the submarine might turn or go slower than its top speed.

More recent approaches look at the FDP as a two-person-zero-sum game with a richer set of submarine strategies. Danskin's (1968) submarine maneuvers in such a manner that its position is at all times uniform over the FOC, a tactic that will also be employed here. Other relevant papers employing game theory are Cheong (1988), Baston and Bostok (1989), and Thomas and Washburn (1991).

Let ρ be the rate at which the searcher can cover area per unit time, and assume that this search effort is distributed continuously and uniformly within the FOC at all times. Conceptually one can think of ρ as the rate at which the searcher scatters confetti in the hope that some flake will cover the point target. Since $\pi(St)^2$ is the area of the FOC at time t, a simple analysis is that $\rho/(\pi(St)^2)$ is the rate of detection (probability of detection per unit time) at time t. The integral of this rate over the appropriate time interval is the expected number of

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detections Z. If there is good reason to suppose that the actual number of detections is a Poisson random variable, then the probability of (at least one) detection is 1-exp($-Z$). Washburn (1996) shows that this model provides accurate answers in at least one experimental situation. The present work is a model of this kind. The total rate of depositing confetti over the plane will still be constrained to be ρ, consistent with the idea that the searcher is a mobile vehicle such as a helicopter or aircraft, but the confetti will not necessarily be uniformly distributed over the FOC. In other words, the present work is a generalization of the random search formula developed in World War II (Koopman, 1980).

BASIC ASSUMPTIONS ABOUT PROPULSION AND SEARCH

Submarine Propulsion

Diesel-electric submarines run on batteries when submerged, and a battery is better modeled as a fixed energy source than a fixed power source. We will therefore impose a constraint on energy consumption, as well as one on speed. This is the primary difference between the current work and previous studies.

The drag on a submarine increases with the square of its speed, so the power required to drive it goes up with the cube of its speed, approximately. In addition, part of the battery's power is wasted at high speeds because of dissipation in the battery's internal resistance, as will be explained in more detail below. High speeds are therefore unattractive from the viewpoint of conserving the battery. If the tactical situation demands high speed, as it does in the FDP, then the submarine must make the tradeoff between speed and endurance carefully.

There are reasons not related to energy conservation for avoiding high speeds. There are mechanical considerations that simply put an upper limit on speed. Submarines (of all types) also try to avoid cavitation, a noisy phenomenon associated with the collapse of air bubbles created by the propeller. Cavitation can be avoided by going deep, as well as by slowing down. We assume here that these considerations place an upper "mechanical" limit on speed, regardless of energy considerations.

Search

Search can be either active or passive. Passive search provides no information to the target about the location of the searcher, whereas search with active sensors may provide such information if the target is within counter-detection range of the searcher's emissions. Active sensors sometimes have long detection ranges, but suffer from adverse movement on the part of the target—the adverse effect can be significant (Washburn [1996]). The analysis here is of a passive search. This does not necessarily exclude search with active sensors, provided searcher tactics make it impossible for the submarine to use the information thereby provided.

The searcher will be assumed to arrive some time after the triggering event and to have finite endurance, as is characteristic of an aircraft or helicopter. While searching, the searcher will be assumed to cover area at some constant rate ρ (area per unit time). For continuously moving sensors, ρ is the product of speed and sweepwidth. For pulsing sensors such as a dipping sonar that examine an area A every T units of time, ρ is A/T. Except for the magnitude of ρ, the nature of the search mechanism is unimportant in what follows.

The search will be assumed to consist in detail of a distribution of confetti. This notion will be made more precise in the next section, but the essential idea is that search is locally disorganized in the sense that what happens in one spatial cell is independent of what happens in neighboring cells. All hope of detecting the submarine by somehow trapping it or constructing an impermeable barrier must therefore be given up. One arrives at this assumption by considering that *both* searcher and submarine are moving, that neither can navigate perfectly, and that spatial coverage by real sensors is never perfect anyway.

DESCRIPTION OF THE ABSTRACT GAME

Consider a search game in two dimensions where a searcher attempts to detect a moving submarine. The submarine moves away from a position known to the searcher at time 0, the "flaming datum." The searcher applies confetti at rate $h(r, t)$ at time t at distance r, constrained only by the requirements that $h(r, t)$ have units of inverse time, be nonnegative, integrable and subject to the constraint

$$2\pi \int_0^\infty h(r, t) r dr \le \rho; \quad \tau \le t \le T. \qquad (1)$$

The parameters τ and T are the searcher's time of arrival and departure, respectively, with $\tau \le T$. The factor $2\pi r$ in the integrand is needed because $h(r, t)$ is the amount of confetti per unit area per unit time (not the amount of confetti per unit radius per unit time).

Note that confetti density is assumed to be radially symmetric with respect to the origin of coordinates (the flaming datum), but not necessarily uniform. To make it uniform one would require $h(r, t)$ to be a constant within some disk. Since the submarine's motion has no constraints involving angles, any departure from radial symmetry could be easily exploited by the submarine.

Our intention is to provide realistic values for parameters as they are introduced, and to employ them later in examples. We will take ρ to be 200 square nautical miles per hour. A continuously moving sensor might have a speed of 100 knots and a sweepwidth of 2 n.m., or a dipping sensor might cover 40 (n.m.^{2}) once every 12 minutes; the effect is the same.

If the submarine's distance from the origin at time t is $y(t)$, then the detection rate at time t is $h(y(t), t)$, and the game's payoff (the expected number of detections) is

$$A(h, y) = \int_\tau^T h(y(t), t) dt. \qquad (2)$$

The submarine is assumed to obtain power from a battery with open-circuit voltage V_0 (400 volts), internal resistance R (.016 ohm), and total energy E (4000 kilowatt-hours at full charge). Parenthetical values in the previous sentence are typical of modern submarines. If the electrical current through such a battery is I, then the voltage across the load is $V_0 - IR$ and the power delivered to the load is $Q = I(V_0 - IR)$. This power is maximized when $I = V_0/(2R)$, at which point the power dissipated internally in the battery is the same as that delivered to the load; namely $Q_0 = V_0^2/(4R)$. Solving the quadratic equation for I in terms of Q, we find that the total power taken from the battery is

$$P(Q) = V_0 I = 2Q_0(1 - \sqrt{1 - Q/Q_0}). \qquad (3)$$

$P(Q)$ is maximized when $Q = Q_0$, but half of the battery's power is wasted at such high power settings. Since $2Q_0 = 5000$ kilowatts, our typical submarine could travel for only 0.8 hours at this speed.

If the submarine's speed is s, assume that the power consumed is $Q(s) = ks^\gamma$, where γ is certainly larger than 2 and will be assumed to be 3 in examples. The maximum possible submarine speed is then $S \equiv (Q_0/k)^{1/\gamma}$, typically 18 knots. It follows that $k = 0.4287$ kilowatts/(knot)^3 for the typical submarine. Let S^* (15 knots) be the mechanical limit introduced in Section 3. We assume that the electrical speed limit S exceeds S^*. The submarine's endurance at the mechanical speed limit is $t^* \equiv E/P(Q(S^*))$, 2.28 hours for the typical submarine.

Let $s(t)$ be the submarine's speed at time t. The submarine is free to choose any integrable speed function that does not exceed S^* or exhaust the battery, so $s(t)$ must satisfy

$$\int_0^T P(Q(s(t)))dt \le E$$

$$\text{and } 0 \le s(t) \le S^*; 0 \le t \le T. \qquad (4)$$

The maximizing searcher chooses the confetti density function h to satisfy (1). The minimizing submarine chooses the distance function y in such a manner that the associated speed function satisfies (4). The resulting payoff is (2).

Given (3) and the above assumptions about power consumption, the power $P(Q(s))$ required for any given speed s is $Q_0 p(s/S)$, where the function p and its inverse p^{-1} are

$$\begin{aligned} p(x) &= 2(1 - \sqrt{1 - x^\gamma}); \; 0 \le x \le 1 \\ p^{-1}(y) &= (y - y^2/4)^{1/\gamma}; \; 0 \le y \le 2 \end{aligned} \qquad (5)$$

These functions will prove useful later.

A SOLVABLE DISCRETE ANALOG

The abstract FDP game is too difficult to solve exactly, so in this section we construct a discrete, one-dimensional analog where the optimal solution can be obtained by enumerating all of the submarine paths and applying linear programming. The results are to some extent a guide to the analysis in the following sections, but still this section could be skipped.

We consider one-dimensional discrete problems where the submarine moves among cells 1, . . ., K, starting in cell 1 at time 0. The submarine has a total energy E available, and consumes $(i - j)^2$ units of energy in moving from cell i to cell j in unit time (note that the power is 2 in this example, not 3). The searcher's first look is at time τ, his last look is at time T, and each look consists of distributing a unit of confetti over the cells. The payoff is the expected accumulated amount of confetti in the same cell as the submarine. Let Ω and Ω_i^t be the set of all feasible target paths and the set of paths occupying cell i at time t, respectively. Since only paths of length T need to be considered, both sets are finite. For $\omega \in \Omega$, $\omega(t)$ denotes the position of the submarine at time t. Let the probability of selecting path ω be $\pi(\omega)$, which is the submarine's mixed strategy and has constraints $\pi(\omega) \geq 0$. Let the distribution of search effort be $\{\varphi(i, t), t = \tau, \ldots, T\}$, which is the searcher's strategy and has constraints $\varphi(i, t) \geq 0, \Sigma_i \varphi(i, t) \leq 1, \tau \leq t \leq T$. The objective function $f(\varphi, \pi) \equiv \Sigma_\omega \pi(\omega) \Sigma_{t=\tau}^T \varphi(\omega(t), t)$ has a saddle point because it is concave-convex, and

$$\max_\varphi f(\varphi, \pi) = \max_\varphi \sum_{t=\tau}^{T} \sum_i \left(\sum_{\omega \in \Omega_i^t} \pi(\omega) \right) \varphi(i, t)$$

$$= \sum_{t=\tau}^{T} \max_i \left(\sum_{\omega \in \Omega_i^t} \pi(\omega) \right).$$

Therefore, the game can be solved by solving the following linear program:

$$\text{minimize} \quad \sum_{t=\tau}^{T} v(t)$$

$$\text{subject to} \quad \sum_{\omega \in \Omega_i^t} \pi(\omega) \leq v(t), \tau \leq t \leq T, 1 \leq i \leq K$$

$$\pi(\omega) \geq 0, \omega \in \Omega$$

$$\sum_\omega \pi(\omega) = 1.$$

For the game with parameters $T = 9$, $K = 10$, $E = 9$ and $\tau = 1$, there are 25312 feasible paths in all. The path that goes the farthest by time 9 is $\omega_1 \equiv (1,2,3,4,5,6,7,8,9,10)$, which makes 9 unit transitions at a cost of 1 unit of energy each. The path that goes the farthest by time 1 is $\omega_2 \equiv (1,4,4,4,4,4,4,4,4,4)$, which spends all 9 units of energy in moving from cell 1 at time 0 to cell 4 at time 1. Small probabilities are assigned to about one fifth of these paths in the optimal submarine mixed strategy, with the largest being $\pi(\omega_2) = 0.0983$. The resultant probability distribution of the submarine's position at time t is shown in Table 1. The searcher's optimal confetti distribution (the dual variables of the linear program used to solve the game) is shown in Table 2. The value of the game is 1.493 expected detections.

The submarine has a positive probability of visiting every feasible cell. However, the boundary path $\omega^* \equiv (1,4,5,6,6,7,8,8,9,10)$ has zero probability because it is not feasible—following it would require 15 units of energy. It is not possible for the submarine to move as far as

Table 1. Probability distribution of the submarine's position

10									.030
9								.056	.026
8						.069	.125	.118	.118
7					.094	.133	.125	.118	.118
6			.095	.167	.151	.133	.125	.118	.118
5		.200	.181	.167	.151	.133	.125	.118	.118
4	.098	.200	.181	.167	.151	.133	.125	.118	.118
3	.301	.200	.181	.167	.151	.133	.125	.118	.118
2	.301	.200	.181	.167	.151	.133	.125	.118	.118
1	.301	.200	.181	.167	.151	.133	.125	.118	.118
$t =$	1	2	3	4	5	6	7	8	9

Table 2. Distribution of search effort

10									
9									
8							.138	.111	.116
7						.046	.094	.116	.111
6				.440	.184	.159	.128	.116	.111
5		.024	.112	.328	.184	.159	.128	.116	.111
4		.244	.222	.145	.184	.159	.128	.193	.218
3	.333	.244	.222	.029	.149	.159	.128	.116	.111
2	.333	.244	.222	.029	.149	.159	.128	.116	.111
1	.333	.244	.222	.029	.149	.159	.128	.116	.111
t =	1	2	3	4	5	6	7	8	9

possible *at all times*. For example, occupying cell 4 at time 1 forces the occupation of cell 4 at all subsequent times. Note that the searcher does not bother to search cell 4 at time 1, but still guards against ω_2 by putting lots of confetti in cell 4 at times 8 and 9 when the confetti will catch several other paths in addition to ω_2.

Except for anomalies such as the one mentioned above, the searcher has a tendency to search uniformly within the boundary ω^*. A simple lower bound on the game value can be constructed from this observation, since distributing confetti uniformly within the boundary is certainly feasible for the searcher and results in $\omega^*(t)^{-1}$ units of confetti being in whatever cell the target occupies at time t. The bound is $LB \equiv \Sigma_{t=\tau}^{T} \omega^*(t)^{-1}$, the amount of confetti that accumulates on the target by time T. This bound is 1.387 in the present example.

The submarine's position is also approximately uniform within ω^*, except near the boundary where the occupation probabilities are necessarily somewhat depressed because ω^* itself is not feasible. If the submarine could arrange to make its distribution uniform within some other feasible path ω^{**}, then, since the total amount of confetti at each time is 1, the expected amount of confetti on the target at time t cannot exceed $\omega^{**}(t)^{-1}$. The corresponding upper bound on the game value would be $UB \equiv \Sigma_{t=\tau}^{T} \omega^{**}(t)^{-1}$. For the present example, the feasible path that minimizes UB is ω^{**} = (1,3,4,5,6,7,8,8,8,8), and the associated upper bound is 1.593.

Both of these bounding ideas are applied to the continuous FDP in subsequent sections, but without having an actual game solution to compare them to. It is perhaps noteworthy that the game value is very close to being midway between the bounds in this discrete example.

There are discrete cases where the bounds are equal. One is where $T = 4$, $K = 5$, $\tau = 1$, and $E = 6$. In this case ω^* and ω^{**} are both (1,3,4,5), and the submarine can make the distribution of his position be uniform beneath ω^{**} at all times. It is not obvious in discrete problems that the submarine can always do this, but the situation will prove simpler in the continuous case. Both bounds are equal to the value of the game, namely 0.78 expected detections.

We now return to the continuous, two-dimensional FDP.

A LOWER BOUND

Since $|dy(t)/dt|$ is constrained to not exceed the submarine's speed $s(t)$, the maximum possible value of $y(t)$ will occur when $dy(t)/dt = s(t)$ for $0 \le t \le T$; that is, when the submarine's motion is entirely radial. Since $P(Q(s))$ is a convex function of s, the speed function that maximizes $y(t)$ for any specific time t will be $s(u) = s^*(t)$ for $0 \le u \le t$; that is, the speed to obtain maximum distance should be constant at some level $s^*(t)$ before the referenced time t, and zero thereafter. If t is smaller than t^*, the endurance at the mechanical limit S^*, then the constant speed should be S^*. Otherwise, $s^*(t)$ should satisfy $tP(Q(s^*(t))) = E$. Using (5), we find that the solution of this equation is $s^*(t) = Sp^{-1}(E/(tQ_0))$.

The maximum possible $y(t)$ is just $y^*(t) = ts^*(t)$. Like Soto (2000), we will refer to the expanding circle with this radius as the MPD ("maximum possible distance") circle to distinguish it from the smaller FOC to be defined in the next section. The analog of $y^*(t)$ in the discrete problem considered earlier is the boundary ω^*. It is not in general possible for the submarine to make $y(t) = y^*(t)$ *for all* t, just as ω^* is not a feasible path in the discrete game. The derivative of $y^*(t)$ is a "virtual speed" that may not satisfy the energy constraint in (4). Nonetheless, if the searcher distributes confetti uniformly over the MPD circle at all times, then the rate of detection will be $h(y(t), t) = \rho/(\pi y^*(t)^2)$ regardless of $y(t)$, since $y(t) \le y^*(t)$ for $0 \le t \le T$ regardless of how the submarine moves.

If $\tau \le t^* \le T$, then a lower bound on the payoff is

$$\underline{v} \equiv (\rho/\pi)\int_{\tau}^{T}(y^*(t))^{-2}dt = (\rho/\pi)\int_{\tau}^{t^*}(tS^*)^{-2}dt + (\rho/\pi)\int_{t^*}^{T}\left(tSp^{-1}\left(\frac{E}{tQ_0}\right)\right)^{-2}dt. \tag{6}$$

Let ε be the speed ratio S^*/S (5/6 for the typical submarine), and define the functions

$$g(x;\varepsilon) \equiv \begin{cases} x;\ x \le 1 \\ \dfrac{x}{\varepsilon}p^{-1}\left(\dfrac{p(\varepsilon)}{x}\right);\ x \ge 1 \end{cases}$$
$$\text{and } G(y;\varepsilon) \equiv \int_{y}^{\infty} g(x;\varepsilon)^{-2}dx \tag{7}$$

Then (6) can be expressed as

$$\underline{v} = \frac{\rho}{\pi t^*(S^*)^2}\int_{\tau/t^*}^{T/t^*} g(x;\varepsilon)^{-2}dx. \tag{8}$$

Equation (8) can be simplified to

$$\underline{v} = \alpha\{G(\tau/t^*;\varepsilon) - G(T/t^*;\varepsilon)\},\ \text{where } \alpha \equiv \frac{\rho}{\pi t^*(S^*)^2}. \tag{9}$$

The constant α can be thought of as a dimensionless search capacity. It is the ratio of the area that can be searched in time t^* to the area of the MPD circle at that time. The function $G(y, \varepsilon)$ is shown in Figure 1 for $1 \le y \le 4$. Formula (9) is actually correct even if $t^* < \tau$ or $t^* > T$.

A lower bound on $G(y, \varepsilon)$ can be obtained by omitting the $y^2/4$ term in (5) and then extending the linear segment in (6) to maintain continuity. The result is

$$\underline{G}(y;\varepsilon) = \begin{cases} 1/y - 1/Y + \underline{G}(Y;\varepsilon) & \text{for } y \le Y \\ \dfrac{\gamma/y}{\gamma-2}(y/Y)^{2/\gamma} & \text{for } y \ge Y \\ \text{where } Y \equiv p(\varepsilon)/\varepsilon^{\gamma} > 1 & \end{cases} \tag{10}$$

Note that use of the first formula in (10) requires use of the second. This lower bound is tight for large y or small ε; the worst comparison in Figure 1 is to approximate $G(1, 5/6) = 2.72$ by $\underline{G}(1, 5/6) = 2.65$. When $\varepsilon = 0$, $G(y, 0)$ is

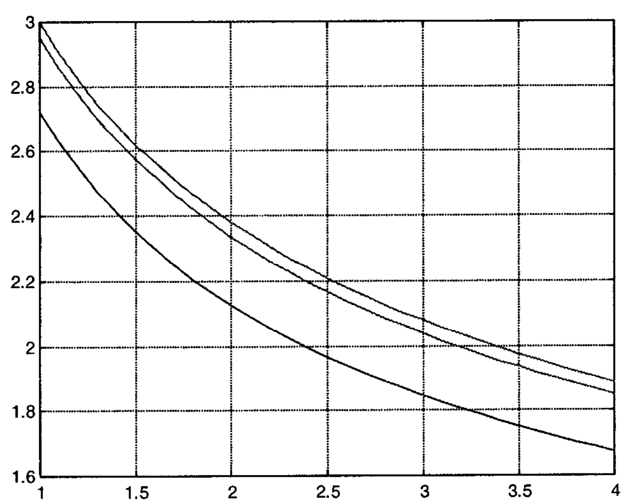

Figure 1. $G(y;\varepsilon)$ versus y for $\varepsilon = 1/6$ (top), 1/2 (middle) and 5/6 (bottom).

given exactly by (10) with $Y = 1$. This will be the case for an ideal battery with no internal resistance, since S is infinite in that case.

For $\varepsilon = 0$ and $\tau \geq t^*$, the first argument of G is never smaller than 1 in (9), so

$$\underline{v} = \alpha \frac{\gamma}{\gamma - 2}((\tau/t^*)^{-(\gamma-2)/\gamma} - (T/t^*)^{-(\gamma-2)/\gamma}); \quad \varepsilon = 0, \tau \geq t^*. \qquad (11)$$

Since $(S^*)^2 = (E/kt^*)^{2/\gamma}$ when $\varepsilon = 0$, (11) can also be written:

$$\underline{v} = \frac{\rho}{\pi\tau}\left(\frac{k\tau}{E}\right)^{2/\gamma}\{1 - (T/\tau)^{-(\gamma-2)/\gamma}\}; \quad \varepsilon = 0, \tau \geq t^*. \qquad (12)$$

This formula nicely exposes how the lower bound on the expected number of detections depends on fundamental quantities such as E, τ, and T. Note that the lower bound does not approach infinity with T.

AN UPPER BOUND

If $s(t)$ is any speed function that is feasible according to (4), then the speed function $Xs(t)$ is also feasible as long as $0 \leq X \leq 1$, even if X is random. Furthermore the distance traveled at time t is $Xy(t)$, where $y(t)$ is as before; that is, $dy(t)/dt = s(t)$. Suppose that X has the triangular density $2x$ over the unit interval. Multiplication by X converts $s(t)$ into the radial distance of a point selected uniformly at random in two dimensions within the circle with radius $s(t)$ (Shupenus and Barr (2000)). The expected value of the detection rate $h(Xy(t), t)$ is then

$$E(h(Xy(t), t)) = \int_0^1 h(xy(t), t)(2x)dx$$

$$= (2/y(t)^2)\int_0^{y(t)} h(r, t)rdr$$

$$\leq \rho/(\pi y(t)^2),$$

where the last inequality is due to (1). It follows from (2) that the expected number of detections cannot exceed

$$Z \equiv (\rho/\pi)\int_\tau^T y(t)^{-2}dt. \qquad (13)$$

In this section the submarine will be assumed to choose $y(t)$ to minimize Z, with the minimized value of Z being an upper bound on the expected number of detections. Specifically, the goal is to minimize (13) subject to (4). With $y(t)$ thus determined, the expanding circle with radius $y(t)$ will be called the FOC. The analog of the FOC in the discrete problem considered earlier is the feasible path ω^{**}.

The upper bound obtained in this manner will not be a tight one. The reason for this is that the submarine is assumed to consume power according to the largest speed that *might* be used, rather than the speed *actually* used. A submarine that chooses a small value for X will therefore end up with excess energy at time T. Tactical guidance might be that $Xs(t)$ should be the submarine's *radial* speed (speed in the direction away from the flaming datum), as distinct from the actual speed $s(t)$. Permitting the radial speed to be smaller than the actual speed lets the submarine's path be curved or zig-zag, which has its own tactical advantages. Small values of X would thus be converted to nonlinear motion, rather than excess energy. Of course there are also tactical advantages to having some "excess" energy at T.

We establish an upper bound by minimizing expected detections (13) subject to constraints on speed and energy (4), with the distance function $y(t)$ being the decision variable. First substitute $x(t) = y(t\tau)/(S\tau)$ into (13) and (4) to produce a dimensionless minimization problem in which τ is the unit of time (so t is from now on dimensionless):

$$\text{minimize} \quad z = \int_0^{T/\tau} I(t)x(t)^{-2}dt$$
$$\text{subject to} \quad \int_0^{T/\tau} p(v(t))dt \leq \frac{E}{Q_0\tau}, \qquad (14)$$
$$\text{and } 0 \leq v(t) \leq \varepsilon$$

where $v(t)$ is the derivative of $x(t)$ and $I(t) = 0$ for $t < 1$ or 1 for $t \geq 1$. The step function $I(t)$ is needed in (14) because the lower limit of the first integral is 0, rather than 1. The upper bound is then $Z = \beta z$, where $\beta \equiv \rho/(\pi\tau S^2)$. The constant β plays no role in (14), so the submarine's behavior will not depend on the search rate ρ.

Problem (14) is an optimal control problem (Intriligator (1971)). We solve it by introducing a Lagrange multiplier λ for the integral constraint and considering instead the problem

$$\text{minimize} \int_0^{T/\tau} \{I(t)x(t)^{-2} + \lambda p(v(t))\}dt, \text{ subject to } 0 \le v(t) \le \varepsilon. \quad (15)$$

The Hamiltonian function for this minimization problem is

$$H(t) \equiv I(t)x(t)^{-2} + \lambda p(v(t)) - u(t)v(t) \quad (16)$$

where $u(t)$ is the costate variable whose time derivative is the partial derivative of H with respect to x; that is

$$du(t)/dt = \begin{cases} 0 \text{ if } t \le 1 \\ -2x(t)^{-3} \text{ if } t > 1. \end{cases} \quad (17)$$

The optimal speed $v(t)$ must at all times minimize H. Letting $U \equiv \lambda p'(\varepsilon)$, it is therefore true that

$$\begin{cases} v(t) = 0 & \text{if } u(t) \le 0 \\ p'(v(t)) = u(t)/\lambda & \text{if } 0 \le u(t) \le U \\ v(t) = \varepsilon & \text{if } u(t) \ge U \end{cases} \quad (18)$$

Since $u(t)$ is nonincreasing and $p(v)$ is strictly convex, it follows from (18) that $v(t)$ is also nonincreasing. Negative values of $u(0)$ correspond to uninteresting tracks where $v(t) \equiv 0$, so assume $u(0) > 0$. There will be an initial interval $[0, b]$ over which $v(t)$ is constant at some positive speed V. The time b is at least 1 because of (17), but may be larger than 1 because of (18) if $u(0) > U$. Since the center formula of (18) must hold for $t > b$, $u(b) = U$ and therefore

$$H(b) = (Vb)^{-2} + \lambda(p(V) - Vp'(V)). \quad (19)$$

The optimized Hamiltonian must be constant for $t \ge b$ because $I(t)$ is constant over that interval. Since the Hamiltonian must be constantly $H(b)$, it is possible to solve for the optimal $v(t)$ as a function of $x(t)$ for $t > b$, thus setting up an ordinary differential equation for $x(t)$ over that interval. Omitting the time arguments of x and v for clarity, we obtain

$$x^{-2} - H(b) = \lambda(vp'(v) - p(v)); b \le t \le T/\tau$$
$$= \lambda\left(\frac{\gamma v^\gamma}{\sqrt{1 - v^\gamma}} - 2 + 2\sqrt{1 - v^\gamma}\right). \quad (20)$$

For convenience, define

$$\delta = 1 + \{x^{-2} - H(b)\}/(2\lambda). \quad (21)$$

Then the solution of (20) for v^γ can be obtained by solving a quadratic equation:

$$v^\gamma = \frac{2(\delta^2 - 1)}{\gamma - 2 + \delta^2 + \delta\sqrt{2(\gamma - 2) + (\gamma - 2)^2 + \delta^2}}. \quad (22)$$

Since v is the rate of change of x, (21) and (22) establish the desired first order differential equation. The parameters b, V and λ must be jointly manipulated to solve (14), subject to the additional constraints $b \ge 1$ and $0 \le V \le \varepsilon$. It is also necessary for optimality that $V = \varepsilon$ if $b > 1$ (these are the trajectories where $u(0) \ge U$), and $b = 1$ if $V < \varepsilon$ (these are the trajectories where $u(0) \le U$). The pair (V, b) is operationally equivalent to $u(0)$, but more directly meaningful.

A COMPARISON

Consider the typical submarine when the searcher's arrival time is $\tau = 1$ hour and the search completion time is $T = 3$ hours, in which case $E/(Q_0\tau) = 1.6$. The upper bound has the submarine proceeding at the mechanical limit $S^* = 15$ knots (dimensionless speed $v = 5/6$) until time 1.82 hours. Figure 2 shows how dimensionless speed decreases after that time. Suppose $\rho = 200$ square nautical miles per hour. Then $\beta = \rho/(\pi\tau S^2) = 0.196$ and the optimized z is 0.978, giving an upper bound of $\beta z = 0.192$ on the expected number of detections. The parameter α introduced in equation (9) is $\beta(\varepsilon^{-2}p(\varepsilon)/(E/(Q_0\tau)) = (0.196)(0.632) = 0.124$. Using (9) with a G-difference of 1.54, we have a lower bound of $\alpha(1.54) = 0.191$. The bounds are close, which is not surprising because the sub-

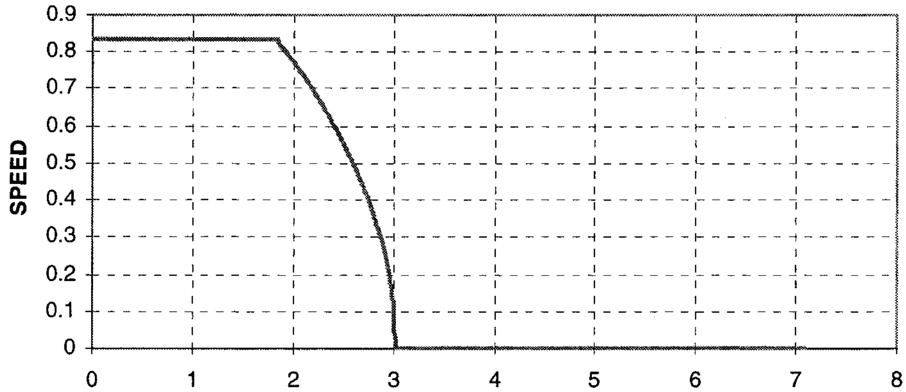

Figure 2. Dimensionless speed function for the case $(E/(Q_0\tau), T/\tau) = (1.6, 3)$.

marine spends much of its time at the mechanical limit. The submarine's chances of escape are good, even if the searcher knows the speed function.

Suppose now that $\tau = 0.5$ hour and $T = 3$ hours, so that the searcher arrives half an hour earlier and spends an additional half hour on station. The submarine now holds top speed until dimensionless time 3.7, as can be seen in Figure 3. The upper bound is now $(0.392)(1.21) = 0.474$. The G-difference is now 3.81, and, since α is unchanged, the lower bound is $(0.124)(3.81) = 0.473$. The bounds are again close. The submarine will still probably escape, but shortening the searcher arrival time by ½ hour is significant.

Finally, suppose that $\tau = 0.5$ hour and $T =$ 3 hours, but that the battery is only 25% charged, so that $E/(Q_0\tau) = 0.8$. Figure 4 shows that the mechanical speed limit is never an active constraint, and that the optimal dimensionless speed starts decreasing from its initial value of 0.733 (13.2 knots) immediately when the searcher arrives at time 1. In this case the upper bound is $(0.392)(2.094) = 0.821$, while the lower bound on the expected number of detections is $(0.496)(1.34) = 0.665$. The submarine's best strategy is to preserve its limited battery by going slowly, and the chances of at least one detection are $1 - e^{-0.665} = 0.486$. This assumes that the searcher is aware of the battery status and therefore searches within an appropriately small area. If the searcher behaved as if the submarine had a fully charged battery, the lower bound would be the same 0.312 as in the paragraph above. The difference between the bounds is comparatively large in this case. If both sides carried out their bounding strategies with the searcher believing that the battery is fully charged, part of the searcher's confetti would be placed outside of the disc of possible submarine positions.

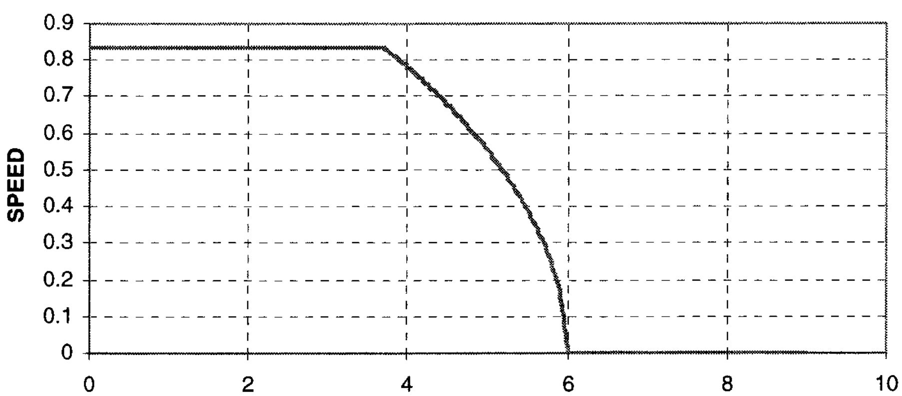

Figure 3. Dimensionless speed function for the case $(E/(Q_0\tau), T/\tau) = (3.2, 6)$.

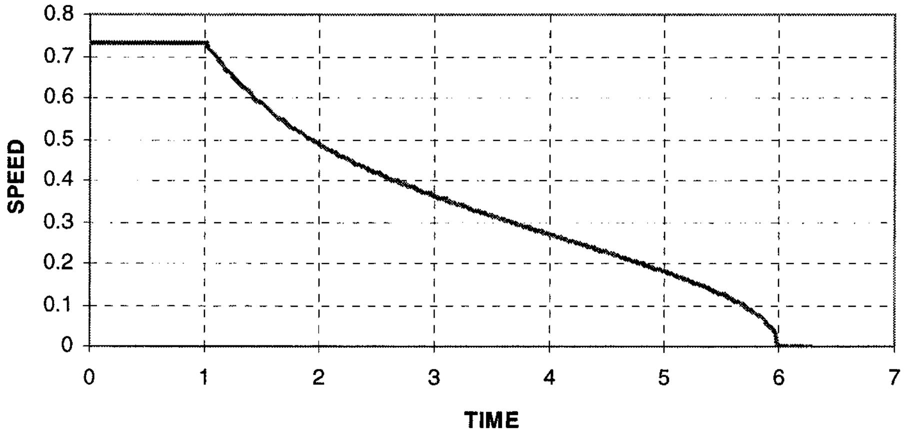

Figure 4. Speed function for the case $(E/(Q_0\tau), T/\tau) = (.8, 6)$.

ON REDUCING THE GAP

We speculate that the gap between upper and lower bound will be hard to reduce. The speculation is partly based on the disappointing results of pursuing the following idea for increasing the lower bound. Since the submarine is unable to make $y(t)$ as large as $y^*(t)$ at all times, one might argue that the searcher is being overly conservative in searching uniformly over the MPD; there should be more confetti in the center than on the edge. The idea, then, is to linearly "shade" the density near the edge of the MPD. Linear shading turns out to be a bad idea, so the reader who is not surprised by this may wish to skip the rest of this section. To simplify calculations, we consider only the case of the ideal battery (no internal resistance) and $\gamma = 3$.

The Δ-shaded confetti density is given by

$$h(r, t) = \frac{D}{y^*(t)^2}\left\{1 - \Delta\frac{r}{y^*(t)}\right\}^+;$$

$0 \le r \le y^*(t)$; $\tau \le t \le T$, where

$$y^*(t) = (E/k)^{1/3}t^{2/3} \text{ and } D = \frac{\rho}{\pi(1 - 2\Delta/3)}. \tag{23}$$

The formula for $y^*(t)$ in (23) is the limit of $ts^*(t)$ as Q_0 becomes large and $\gamma = 3$. The nonnegative parameter $\Delta \in [0, 1]$ causes the confetti distribution to be higher in the middle than on the edge, and D is whatever it has to be to make the total rate of applying confetti be ρ.

From here on we will take $k = 1$, since the only effect of k is to normalize E.

The objective function is

$$Z = D\int_\tau^T \left\{\frac{1}{y^*(t)^2} - \Delta\frac{y(t)}{y^*(t)^3}\right\}dt \tag{24}$$

which the submarine desires to minimize subject to an energy constraint. The positive-part symbol (+) has been omitted from (24), since $h(y(t), t) > 0$ as long as $t < T$ for any feasible $y(t)$. Since $y^*(t)^3$ is proportional to t^2, Z will be minimized regardless of Δ when $\int_\tau^T y(t)t^{-2}\,dt$ is maximized, subject to the constraint that $\int_\tau^T (dy(t)/dt)^3\,dt \le E - e$, where e is the energy used before time τ. This leads to a Control Theory problem similar to the one in Section 7, except that the integrand is $F(y(t), dy(t)/dt, t) \equiv y(t)t^{-2} + \lambda(dy(t)/dt)^3$ for some multiplier λ. The Euler conditions require that $dy(t)/dt$ have the form $K(1/t - 1/T)^{1/2}$ for $\tau \le t \le T$, and the energy constraint requires that $K^3 = T^{1/2}(E - e)/M(\tau/T)$, where

$$M(x) \equiv \int_x^1 (u^{-1} - 1)^{3/2}du = 2(x - x^2)^{3/2}/x^2$$

$$+ 3(x - x^2)^{1/2} - 1.5\text{a}\cos(2x - 1); 0 < x \le 1.$$

The submarine's distance is then $y(t) = y(\tau) + K\int_\tau^t (1/u - 1/T)^{1/2}\,du$ for $\tau \le t \le T$. Since $y(\tau)$ should be the maximum value permitted by an energy expenditure of e, $y(\tau) = \tau^{2/3}e^{1/3}$ and

$$\int_{\tau}^{T} y(t)t^{-2}dt = \tau^{2/3}e^{1/3}(1/\tau - 1/T) + T^{-1/3}(E - e)^{1/3}M(\tau/T)^{2/3}. \quad (25)$$

This integral should be maximized by choosing e in the interval $[0, E]$. This can be accomplished by differentiation. Letting $x \equiv \tau/T$ and $z = (1 - x)^{3/2}/x^{1/2}$, the maximizing e is $zE/(M(x) + z)$. Substituting this into (25) and then (25) into (24), we obtain

$$Z = D\tau^{-1/3}E^{-2/3}[3(1 - x^{1/3}) - \varepsilon x^{1/3}(z + M(x))^{2/3}]. \quad (26)$$

Upon substituting D into (26) and letting the coefficient of ε inside the [ ] be $R(x)$, where $x = \tau/T$ as before, (26) becomes

$$Z = \frac{\rho(\tau/E)^{2/3}}{\pi\tau}[3(1 - x^{1/3}) - \varepsilon R(x)]/(1 - 2\varepsilon/3). \quad (27)$$

The crucial function $R(x)$ can be somewhat simplified:

$$R(x) = [3(1 - x)^{3/2} + 3(1 - x)^{1/2} - 1.5x^{1/2}\text{a}\cos(2x - 1)]^{2/3}. \quad (28)$$

We finally come to the question of whether shading is a good idea; that is, should Δ be 1 or 0? The answer should be 1 if $R(x) < 2(1 - x^{1/3})$, or otherwise 0. It is a simple computational matter to show that $R(x)$ is never small enough to invite shading. In other words, even though the submarine cannot make $y(t) = y^*(t)$ all the time, it can come sufficiently close when motivated to make shading a bad idea. A uniform confetti distribution is always better than a linearly shaded one if the distribution must be announced to the submarine. It is possible, of course, that some other style of shading would be a good idea, but intuition argues that not much is to be gained here.

CONCLUSIONS ABOUT STRATEGIES

The abstract FDP is a complicated game that we have only succeeded in bounding. The lower bound strategy for the searcher (distribute confetti uniformly within the MPD) and the upper bound strategy for the submarine (move radially and follow the appropriate speed function) are each simple enough to emulate in practice after making adjustments for theoretical artificialities, and the two bounds are often close. Furthermore each has the virtue of being "optimal" in a certain sense, so it is reasonable to consider using each of them as a tactical guide.

For example, Soto (2000) includes a Monte Carlo simulation of a FDP where the searcher is a helicopter with a dipping sonar searching for a submarine with an ideal battery. The "confetti" with such a sonar consists of only a few large circular coverage areas. The location of the first dip is especially important and edge effects must be considered. Still, the idea of placing the center of the next dip uniformly within the MPD, as Soto does, is viable and reasonable. Soto's measured detection probabilities for the engagement are close to the theoretical lower bound regardless of what the submarine does, so the confetti approximation is not a bad one.

While it makes no difference what the submarine does when the searcher uses his lower bound strategy, a good submarine strategy will avoid being exploitable by other searcher tactics. Submarine speed is a strongly decreasing function of time (see Figures 2–4) in the upper bound strategy derived above. After considerable analytic work we have not been successful at finding equally good submarine strategies of the form "pick a speed at random and stick to it until the battery is exhausted"; whatever is the optimal strategy for the submarine in the FDP, it appears to be a continuous function of time that starts out fast and gradually slows down. Soto's (2000) simulated submarine follows such a theoretical speed function except for making shallow turns to avoid strict radial movement.

The time-late parameter τ will be known to the searcher, but probably not to the submarine. Perversely, the searcher's lower bound strategy does not require knowing τ, while the submarine's upper bound strategy does. In theory one might generalize the FDP so that τ is selected from some probability distribution, with τ being known to the searcher but only the distribution known to the submarine. But the FDP is a difficult game even without this generalization, and anyway it is not obvious what the distribution should be. The searcher is in a similar position with respect to the battery charge, but here there is an obvious and reasonable worst-case assumption. These considerations hint that the lower bound strategy for the searcher may be of more practical use than the upper bound strategy for the submarine.

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