# Calhoun

## Institutional Archive of the Naval Postgraduate School

**Calhoun: The NPS Institutional Archive**

---

Faculty and Researcher Publications                    Faculty and Researcher Publications

---

1991

# Trichromatic Approximation Method for Surface Illumination

## Borges, C.F.

---

# Trichromatic approximation method for surface illumination

Carlos F. Borges

*Computer Graphics Research Laboratory and Institute of Theoretical Dynamics, University of California, Davis, Davis, California 95616*

The problem of approximating the tristimulus coordinates of light reflected from a surface from those of the source and the surface is considered. A variation on a well-known and widely used approximate method for accomplishing this task is presented. This variation uses the $XYZ$ primaries that have unique properties that yield a straightforward analysis of the approximation error. We develop the notion of a colorimetric seminorm and derive an error bound by using techniques from functional analysis. This approach gives some useful insight into the factors that affect accuracy.

## INTRODUCTION

I am interested in determining the color appearance of an illuminated Lambertian surface. In an ideal physical model, light emanating from the surface has a spectral radiant power distribution that is proportional to $P_\lambda \rho(\lambda)$, the product of the spectral radiant power of the illuminant $P_\lambda$ and the spectral reflectance of the surface $\rho(\lambda)$. The tristimulus coordinates of the reflected light, and hence the color appearance of the surface, can be found by evaluating the following definite integrals:

$$R = \int_\mathcal{V} P_\lambda \rho(\lambda) \bar{r}(\lambda) d\lambda,$$

$$G = \int_\mathcal{V} P_\lambda \rho(\lambda) \bar{g}(\lambda) d\lambda,$$

$$B = \int_\mathcal{V} P_\lambda \rho(\lambda) \bar{b}(\lambda) d\lambda, \qquad (1)$$

where $\bar{r}(\lambda)$, $\bar{g}(\lambda)$, and $\bar{b}(\lambda)$ are the color-matching functions and $\mathcal{V}$ is the interval corresponding to the visible spectrum (see Ref. 1).

This is a convenient physical model but requires that the spectral properties of both the source and surface be known in full. Unfortunately, in many applications this is not the case. For example, in computer graphics it is common to have only the tristimulus coordinates of the various sources and surfaces[2]; no spectral information is known (see Ref. 3). This situation is a disadvantage but not a defeat. Since tristimulus spaces are exact models for additive mixture and scaling, many illumination phenomena can be modeled with just this information. However, the type of reflections under consideration here cannot be modeled exactly with these data. The problem stems from the fact that metameric sources (lights with distinct spectral radiant power distributions but identical tristimulus values) do not always produce the same color when they illuminate surfaces that are not spectrally white. Similarly, a pair of surfaces may have identical color appearance under spectrally white illumination, and

hence the same tristimulus values, but dissimilar appearance under some other illuminant (this phenomenon is familiar to anyone who has had their hand stamped with an invisible design that only appears under UV illumination). Since these sources and surfaces cannot be distinguished by way of their tristimulus values, it follows that the tristimulus values do not contain sufficient information to model these reflections correctly.

Given that there can be no exact method for modeling reflections with only tristimulus values, it seems worthwhile to investigate the accuracy of an approximate method. The method that will be considered is one that is common in computer graphics and simply mimics the exact method used in the spectral domain. In particular, the tristimulus values of the illuminated surface are assumed to be given by the product of the tristimulus values of the source and the surface. For example,

$$R_{\text{reflected}} \approx R_{\text{source}} R_{\text{surface}},$$

$$G_{\text{reflected}} \approx G_{\text{source}} G_{\text{surface}},$$

$$B_{\text{reflected}} \approx B_{\text{source}} B_{\text{surface}}. \qquad (2)$$

This approximation works quite well in practice. Indeed, Cowan and Ware[3] describe a similar method and note its surprising effectiveness in realistic scenes. Motivated by this, we shall consider the same approximation but will use the $XYZ$[4] primaries:

$$X_{\text{reflected}} \approx X_{\text{source}} X_{\text{surface}},$$

$$Y_{\text{reflected}} \approx Y_{\text{source}} Y_{\text{surface}},$$

$$Z_{\text{reflected}} \approx Z_{\text{source}} Z_{\text{surface}}. \qquad (3)$$

Note that the choice of primaries here is not at all arbitrary. Rather, these primaries are chosen because the $XYZ$ system has convenient properties that will simplify the derivation of bounds on the approximation error. They are as follows:

(1) All realizable stimuli (those that satisfy $P_\lambda \geq 0$) have nonnegative tristimulus coordinates.

(2) The tristimulus coordinates of an equal energy white ($P_\lambda \equiv 1.0$) are $X = Y = Z = 1.0$.

The first property implies that the $XYZ$ matching functions are nonnegative over the visible interval; that is, $\bar{x}(\lambda), \bar{y}(\lambda), \bar{z}(\lambda) \geq 0$ for all $\lambda \in \mathcal{V}$. The second property implies that

$$\int_{\mathcal{V}} \bar{x}(\lambda)\mathrm{d}\lambda = 1.0,$$

$$\int_{\mathcal{V}} \bar{y}(\lambda)\mathrm{d}\lambda = 1.0,$$

$$\int_{\mathcal{V}} \bar{z}(\lambda)\mathrm{d}\lambda = 1.0. \qquad (4)$$

Matching functions that satisfy Eqs. (4) will be called normalized.

## ERROR ANALYSIS

To expedite the analysis that follows, we shall consider the error associated with a single generic primary with a normalized nonnegative matching function. Determining the behavior of the error for this primary is sufficient because the results can later be applied directly to the $XYZ$ primaries since they also have normalized nonnegative matching functions. Let the matching function of this generic primary be denoted $\bar{m}(\lambda)$ and consider the approximation error given by

$$\mathrm{Err}(f, g) = \int_{\mathcal{V}} f(\lambda)g(\lambda)\bar{m}(\lambda)\mathrm{d}\lambda$$

$$- \int_{\mathcal{V}} f(\lambda)\bar{m}(\lambda)\mathrm{d}\lambda \int_{\mathcal{V}} g(\lambda)\bar{m}(\lambda)\mathrm{d}\lambda, \qquad (5)$$

where $f$ and $g$ represent a spectral power distribution and a reflectance function (there is no reason to enforce any distinction between the two objects; both are simply functions). For mathematical expedience, assume that $\bar{m}(\lambda)$, $f(\lambda)$, and $g(\lambda)$ are all elements of $C_{\mathcal{V}}$, the set of real-valued functions that are continuous on $\mathcal{V}$; this condition will ensure that all the integrals exist and are bounded.

Given the problem in this form, it is a relatively straightforward matter to bound the error with the use of techniques from functional analysis (Kreyszig[5] is an excellent reference for this material). First, notice that the error term of Eq. (5) is a Hermitian form that maps $C_{\mathcal{V}} \times C_{\mathcal{V}}$ to $\mathfrak{R}$. We can verify this term by taking $f, g, h \in C_{\mathcal{V}}$ and noting that

$$\mathrm{Err}(f + g, h) = \int_{\mathcal{V}} (f + g)h\bar{m}(\lambda)\mathrm{d}\lambda$$

$$- \int_{\mathcal{V}} (f + g)\bar{m}(\lambda)\mathrm{d}(\lambda) \int_{\mathcal{V}} h\bar{m}(\lambda)\mathrm{d}\lambda$$

$$= \int_{\mathcal{V}} fh\bar{m}(\lambda)\mathrm{d}\lambda - \int_{\mathcal{V}} f\bar{m}(\lambda)\mathrm{d}\lambda \int_{\mathcal{V}} h\bar{m}(\lambda)\mathrm{d}\lambda$$

$$+ \int_{\mathcal{V}} gh\bar{m}(\lambda)\mathrm{d}\lambda$$

$$- \int_{\mathcal{V}} g\bar{m}(\lambda)\mathrm{d}\lambda \int_{\mathcal{V}} h\bar{m}(\lambda)\mathrm{d}\lambda$$

$$= \mathrm{Err}(f, h) + \mathrm{Err}(g, h) \qquad (6)$$

and that

$$\mathrm{Err}(\alpha f, g) = \int_{\mathcal{V}} \alpha fg\bar{m}(\lambda)\mathrm{d}\lambda - \int_{\mathcal{V}} \alpha f\bar{m}(\lambda)\mathrm{d}\lambda \int_{\mathcal{V}} g\bar{m}(\lambda)\mathrm{d}\lambda$$

$$= \alpha\left[\int_{\mathcal{V}} fg\bar{m}(\lambda)\mathrm{d}\lambda - \int_{\mathcal{V}} f\bar{m}(\lambda)\mathrm{d}\lambda \int_{\mathcal{V}} g\bar{m}(\lambda)\mathrm{d}\lambda\right]$$

$$= \alpha\,\mathrm{Err}(f, g) \qquad (7)$$

for any $\alpha \in \mathfrak{R}$.

Second, notice that the error term of Eq. (5) is positive semidefinite. To verify this fact, consider

$$\mathrm{Err}(f, f) = \int_{\mathcal{V}} f^2\bar{m}(\lambda)\mathrm{d}\lambda - \left[\int_{\mathcal{V}} f\bar{m}(\lambda)\mathrm{d}\lambda\right]^2. \qquad (8)$$

Since $\bar{m}(\lambda)$ is nonnegative, the Schwarz inequality gives

$$\left[\int_{\mathcal{V}} f\bar{m}(\lambda)\mathrm{d}\lambda\right]^2 \leq \int_{\mathcal{V}} f^2\bar{m}(\lambda)\mathrm{d}\lambda, \qquad (9)$$

which implies that

$$0 \leq \int_{\mathcal{V}} f^2\bar{m}(\lambda)\mathrm{d}\lambda - \left[\int_{\mathcal{V}} f\bar{m}(\lambda)\mathrm{d}\lambda\right]^2. \qquad (10)$$

Hence the error functional is positive semidefinite.

The following lemma shows that the Schwarz inequality holds for any positive semidefinite Hermitian form (see Ref. 5). This will enable us to impose a bound on the magnitude of the error functional.

*Lemma 1:* Let $X$ be a vector space and $h$ a positive semidefinite Hermitian form mapping $X \times X$ to $\mathfrak{R}$. Then $h$ satisfies the Schwarz inequality

$$h^2(x, y) \leq h(x, x)h(y, y). \qquad (11)$$

*Proof:* If $y = 0$ then formula (11) clearly holds since $h^2(x, 0) = 0$. Assume that $y \neq 0$. Given any scalar $\alpha$, we have

$$0 \leq h(x - \alpha y, x - \alpha y)$$

$$= h(x, x) - \alpha h(x, y) - \alpha[h(y, x) - \alpha h(y, y)]. \qquad (12)$$

If we choose $\alpha = h(y, x)/h(y, y)$, then the term in brackets is zero and the remaining inequality is

$$0 \leq h(x, x) - \frac{h(y, x)}{h(y, y)}h(x, y). \qquad (13)$$

Using the fact that $h(x, y) = h(y, x)$ and multiplying by $h(y, y)$ yield

$$0 \leq h(x, x)h(y, y) - h^2(x, y). \qquad (14)$$

Hence

$$h^2(x, y) \leq h(x, x)h(y, y). \blacksquare \qquad (15)$$

Applying the Schwarz inequality to the error functional yields

$$[\mathrm{Err}(f, g)]^2 \leq \mathrm{Err}(f, f)\mathrm{Err}(g, g). \qquad (16)$$

This is a bound on the error of the approximation. Now,

note that

$$\|f\|_{\mathbf{m}}^2 = \int_\gamma f^2 \overline{\mathbf{m}}(\lambda) \mathrm{d}\lambda - \left[ \int_\gamma f \overline{\mathbf{m}}(\lambda) \mathrm{d}\lambda \right]^2 \qquad (17)$$

is a seminorm on the space $C_\gamma$ (see Ref. 5). We shall call $\|\cdot\|_{\mathbf{m}}$ the colorimetric seminorm with respect to the matching function $\mathbf{m}(\lambda)$. The error bound is given by

$$|\mathrm{Err}(f, g)| \le \|f\|_{\mathbf{m}} \|g\|_{\mathbf{m}}, \qquad (18)$$

and it is seen that the absolute error can be no worse than the product of the lengths of the spectral reflectance function and the spectral radiant power distribution with respect to this seminorm. This is useful because it makes it possible to determine, a priori, whether this method is an appropriate approximation for a given set of sources and surfaces. If the various spectral functions are sufficiently small with respect to this seminorm then it is reasonable to assume that the method will work well.

We point out that this method (multiplying the tristimulus values) can be used with any tristimulus space. However, with more general matching functions the error analysis becomes far more involved mathematically and is omitted here (a complete analysis can be found in Ref. 6).

The reader should note that, so far, no use has been made of the fact that the matching function is normalized. To see why this condition is useful, suppose that the source (or surface) is spectrally white; that is, $f(\lambda) \equiv \alpha$ for some $\alpha \in \Re$. Equation (5) yields, after some manipulation,

$$\mathrm{Err}(\alpha, g) = \left[ 1 - \int_\gamma \overline{\mathbf{m}}(\lambda) \mathrm{d}\lambda \right] \alpha \int_\gamma g(\lambda) \overline{\mathbf{m}}(\lambda) \mathrm{d}\lambda. \qquad (19)$$

If the matching function is normalized, then the error is zero and hence the approximation is exact when applied to spectrally white sources or surfaces.

In fact, there are other conditions under which the approximation is exact. One case of particular interest is when the von Kries adapted tristimulus values for a given source and surface are the same as the tristimulus values for the surface under spectrally white illumination. In particular,

$$\frac{\displaystyle\int_\gamma P_\lambda \rho(\lambda) \overline{\mathbf{x}}(\lambda) \mathrm{d}\lambda}{\displaystyle\int_\gamma P_\lambda \overline{\mathbf{x}}(\lambda) \mathrm{d}\lambda} = \int_\gamma \rho(\lambda) \overline{\mathbf{x}}(\lambda) \mathrm{d}\lambda,$$

$$\frac{\displaystyle\int_\gamma P_\lambda \rho(\lambda) \overline{\mathbf{y}}(\lambda) \mathrm{d}\lambda}{\displaystyle\int_\gamma P_\lambda \overline{\mathbf{y}}(\lambda) \mathrm{d}\lambda} = \int_\gamma \rho(\lambda) \overline{\mathbf{y}}(\lambda) \mathrm{d}\lambda,$$

$$\frac{\displaystyle\int_\gamma P_\lambda \rho(\lambda) \overline{\mathbf{z}}(\lambda) \mathrm{d}\lambda}{\displaystyle\int_\gamma P_\lambda \overline{\mathbf{z}}(\lambda) \mathrm{d}\lambda} = \int_\gamma \rho(\lambda) \overline{\mathbf{z}}(\lambda) \mathrm{d}\lambda. \qquad (20)$$

Multiplying both sides by the tristimulus coordinates of the source shows that the trichromatic approximation is exact in this case. That is, the trichromatic approximation is exact whenever von Kries chromatic adaptation gives color constancy. The conditions under which this

will occur have been carefully analyzed in Refs. 7 and 8 and will not be considered here. See also Refs. 9 and 10 for more detailed discussions of chromatic adaptation.

## PERTURBATION ANALYSIS

It is informative to examine the behavior of this method when it is subjected to small perturbations of the inputs. In particular, let us investigate the change in the error when a small amount of a contaminant light is added to the source illuminant. We shall consider the following experiment. We have a single surface with spectral reflectance $\rho(\lambda)$ and two distinct light sources with spectral power densities $I_A(\lambda)$ and $I_B(\lambda)$. We start with the surface illuminated by light source A at unit intensity and consider what happens as we add a small amount of illumination from light source B. If we let $\epsilon$ denote the amount of light source B that is added, then the total illumination is given by

$$I_A(\lambda) + \epsilon I_B(\lambda). \qquad (21)$$

The approximation error is

$$\mathrm{Err}[I_A(\lambda) + \epsilon I_B(\lambda), \rho(\lambda)]. \qquad (22)$$

The linearity of the error functional implies that

$$\mathrm{Err}[I_A(\lambda) + \epsilon I_B(\lambda), \rho(\lambda)] = \mathrm{Err}[I_A(\lambda), \rho(\lambda)] + \epsilon\, \mathrm{Err}[I_B(\lambda), \rho(\lambda)]. \qquad (23)$$

Hence the change in the error as a result of perturbing the light source is simply

$$\epsilon\, \mathrm{Err}[I_B(\lambda), \rho(\lambda)]. \qquad (24)$$

2And, since $\mathrm{Err}[I_B(\lambda), \rho(\lambda)]$ is well defined and does not change in response to the magnitude of the perturbation, it follows that the error varies linearly with the size of the perturbation. Moreover, the linear coefficient of change [i.e., the first partial derivative of Formula (22) with respect to $\epsilon$] is bounded in absolute value by the product of the colorimetric seminorms of the perturbation source and the surface.

This situation will help to illustrate one of the reasons why this approximation works so well in realistic situations. Note that most common sources of illumination are nearly white. It is reasonable to represent a nearly white source with a spectral radiant power distribution of the following form:

$$P_\lambda = \alpha + \epsilon I_P(\lambda), \qquad (25)$$

where $\alpha$ is a constant, $\epsilon$ is much smaller than $\alpha$, and $I_P(\lambda)$ is a spectral power density. If $\rho(\lambda)$ is the spectral reflectance of the surface, then applying Eqs. (6) and (19) yields

$$\mathrm{Err}[\alpha + \epsilon I_P(\lambda), \rho] = \epsilon\, \mathrm{Err}[I_P(\lambda), \rho]. \qquad (26)$$

The relative absolute error is given by

$$\frac{\epsilon |\mathrm{Err}[I_P(\lambda), \rho]|}{\displaystyle\int_\gamma [\alpha + \epsilon I_P(\lambda)] \rho(\lambda) \overline{\mathbf{m}}(\lambda) \mathrm{d}\lambda}, \qquad (27)$$

which, following formula (18) and the fact that $\epsilon I_P(\lambda)$ and
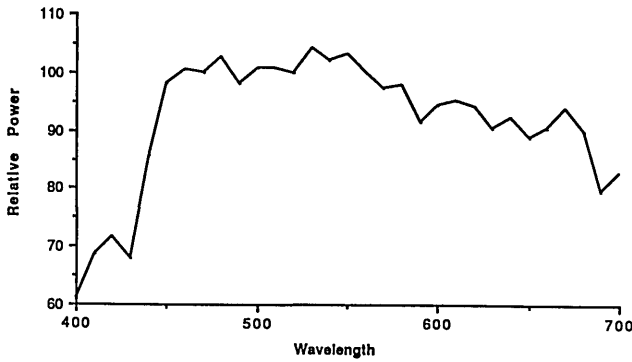
Fig. 1.    Relative spectral power of daylight (CIE standard illuminant D55).
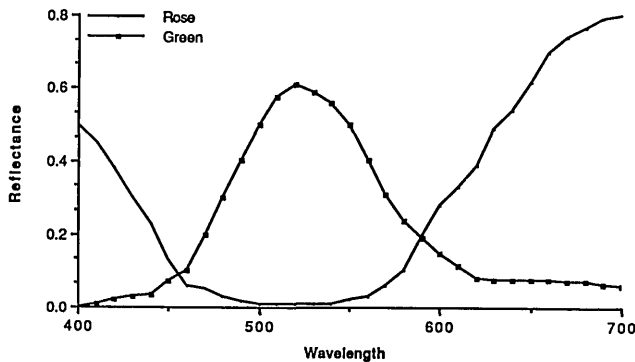


Fig. 2.    Spectral reflectance functions of the two surfaces.

$\rho(\lambda)$ are nonnegative, is certainly less than

$$\frac{\epsilon}{\alpha} \frac{\|I_P(\lambda)\|_m \|\rho\|_m}{\displaystyle\int_\gamma \rho(\lambda)\overline{m}(\lambda)d\lambda} . \tag{28}$$

Since we assumed that $\epsilon \ll \alpha$, then the relative error will be small as long as

$$\frac{\|I_P(\lambda)\|_m \|\rho\|_m}{\displaystyle\int_\gamma \rho(\lambda)\overline{m}(\lambda)d\lambda} \tag{29}$$

is not large.

## PROBABILISTIC APPROACH TO ERROR ANALYSIS

Note that a normalized matching function can also be thought of as a probability density function on $\mathcal{V}$. This fact permits an alternative mathematical derivation of the error term. Let $\Lambda$ be a continuous random variable with probability density

$$p(\lambda) = \begin{cases} \overline{m}(\lambda) & \lambda \in \mathcal{V} \\ 0 & \text{otherwise} \end{cases} \tag{30}$$

Consider the random variable defined by $f(\Lambda)$. Applying the fundamental theorem of expectation (see Ref. 11) yields the expected value of $f(\Lambda)$:

$$E[f(\Lambda)] = \int_\gamma f(\lambda)\overline{m}(\lambda)d\lambda . \tag{31}$$

Hence, the error term from Eq. (5) can be rewritten as

$$\begin{aligned} \mathrm{Err}(f, g) &= E[f(\Lambda)g(\Lambda)] - E[f(\Lambda)]E[g(\Lambda)] \\ &= \mathrm{Cov}[f(\Lambda), g(\Lambda)], \end{aligned} \tag{32}$$

where $E[\cdot]$ signifies expectation and $\mathrm{Cov}[f(\Lambda), g(\Lambda)]$ is the covariance. Recall that the covariance of two random variables is also given by the product of both variances and a correlation coefficient. Since the correlation coefficient of two random variables always lies between $-1$ and 1, the error bound follows directly as

$$|\mathrm{Err}(f, g)| \le \sigma_{\overline{m}}(f)\sigma_{\overline{m}}(g), \tag{33}$$

where[12]

$$\sigma_{\overline{m}}(f) = \{\mathrm{Cov}[f(\Lambda), f(\Lambda)]\}^{1/2}. \tag{34}$$

So both the error term and its analytic bound can be arrived at with a probabilistic approach. This situation does not imply that the probabilistic derivation of the error term has any particular physiological interpretation. Whether it does or does not is beyond the scope of this paper. This analysis is presented because, to many of us, probability is a far more familiar mathematical setting than is functional analysis. From a purely mathematical perspective, the two derivations are completely equivalent.

## EXAMPLE

I briefly demonstrate this method with two surfaces (one rose and one green) and a standard daylight energy source whose spectral distributions appear in Figs. 1 and 2. Only the reflection will be considered here; effects related to scene geometry will be ignored (e.g., Lambert's cosine law and the inverse square law). It is straightforward to compute the actual coordinates of the source and the two surfaces as well as the actual and approximated coordinates of the reflections (see Table 1). A brief calculation yields the error bounds and the observed approximation errors that appear in Table 2. In these examples the error bounds indicate that the approximation will be accurate to within $\pm6\%$ for the $X$ and $Y$ primaries and $\pm15\%$ for the $Z$ primary. This is an excellent result considering how little information is needed to make the approximations. By applying a standard transformation we can put the data from Table 1 in a more familiar form—RGB display coordinates for a 24-bit frame buffer. These coordi-

**Table 1.    CIE $XYZ$ Coordinates of the Source and Surfaces from the Example and the Actual and Approximated Coordinates for the Diffuse Reflections**

| Coordinate | Daylight | Rose | Green | Daylight–Rose | | Daylight–Green | |
|---|---|---|---|---|---|---|---|
| | | | | Actual | Approximate | Actual | Approximate |
| $X$ | 94.008545 | 0.246095 | 0.190821 | 22.473387 | 23.135039 | 18.611805 | 17.938802 |
| $Y$ | 98.369507 | 0.119954 | 0.360694 | 11.241696 | 11.799859 | 36.161922 | 35.481281 |
| $Z$ | 90.521194 | 0.151974 | 0.132420 | 12.321855 | 13.756877 | 13.032540 | 11.986803 |

**Table 2.  Observed Errors and Analytic Error Bounds of the Trichromatic Approximations from the Example**

| Coordinate | Daylight–Rose | | Daylight–Green | |
|---|---|---|---|---|
| | Error | Bound | Error | Bound |
| $X$ | 0.661652 | 1.470889 | 0.673003 | 1.142348 |
| $Y$ | 0.558163 | 0.694311 | 0.680641 | 0.804540 |
| $Z$ | 1.435022 | 1.683958 | 1.045737 | 1.932217 |

**Table 3.  Actual and Approximated Tristimulus Values for the Example in Standard RGB Coordinates for a 24-Bit Frame Buffer**

| Coordinate | Daylight–Rose | | Daylight–Green | |
|---|---|---|---|---|
| | Actual | Approximate | Actual | Approximate |
| R | 83 | 85 | 31 | 30 |
| G | 0 | 1 | 134 | 132 |
| B | 28 | 31 | 21 | 19 |

nates appear in Table 3 and show that the approximation is quite accurate (the actual and approximated colors are virtually indistinguishable on a standard display device).

## CONCLUSIONS

I have presented a variation on a widely used approximation for surface illumination. This method is appealing because of its low computational costs and its minimal storage requirements. I derived simple analytic bounds on the error associated with this approximation, using classical results from functional analysis (and probability theory), and gave some indication of why it might be expected to work well in realistic scenes. Furthermore, I have indicated that there is a relationship between this approximation and the von Kries coefficient law. In particular, this approximation is exact when used with a source–surface pair that yields color constancy under von Kries chromatic adaptation. I have also introduced the notion of a colorimetric seminorm that can be used to bound the approximation error and yields valuable in-

sights into the change in approximation error as a result of perturbations.

## REFERENCES AND NOTES

1. G. I. Wyszecki and W. S. Stiles, *Color Science: Concepts and Methods, Quantitative Data and Formulae* (Wiley, New York, 1982).
2. Of course, surfaces do not have tristimulus values. However, for pedagogical expedience it is common to call the tristimulus values of a given surface under spectrally white illumination of unit intensity the tristimulus values of the surface.
3. W. B. Cowan and C. Ware, "Tutorial on color perception," course notes prepared for the 10th Conference on Computer Graphics and Interactive Techniques, sponsored by the Association for Computing Machinery's Special Interest Group on Computer Graphics, Detroit, July 1983.
4. The experimental results presented herein use the $XYZ$ matching functions for the CIE 1931 standard observer.
5. E. Kreyszig, *Introductory Functional Analysis with Applications* (Wiley, New York, 1978).
6. C. F. Borges, "Numerical methods for illumination models in realistic image synthesis," Ph.D. dissertation (University of California, Davis, Davis, Calif., 1990).
7. G. West and M. H. Brill, "Necessary and sufficient conditions for von Kries chromatic adaptation to give color constancy," J. Math. Biol. **15**, 249–258 (1982).
8. J. A. Worthey, "Limitations of color constancy," J. Opt. Soc. Am. A **2**, 1014–1026 (1985).
9. H. Helson, D. B. Judd, and M. H. Warren, "Object–color changes from daylight to incandescent filament illumination," Illum. Eng. (NY) **47**, 221–233 (1952).
10. W. D. Wright, "The measurement and analysis of colour adaptation phenomena," Proc. R. Soc. London Ser. B **115**, 49–87 (1934).
11. S. Ross, *A First Course in Probability* (Macmillan, New York, 1984).
12. Note that $\sigma_{\bar{m}}(\cdot)$ and $\|\cdot\|_m$ are mathematically equivalent.