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Wavelets and Multigrid

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WAVELETS AND MULTIGRID

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1. Introducton. The last few years have seen a remarkable amount of activity and interest in the field of wavelet theory and multiresolution analysis. With this heightened level of interest, researchers in diverse fields have begun to consider wavelet-based methods. The work presented in this paper was done in an exploratory spirit, by investigating the very suggestive similarities between multiresolution analysis and multigrid methods. The results are preliminary and only point to several avenues of future work.

Like many mathematical topics that suddenly gain currency, wavelet theory has origins that are not all that recent. Both the history and the theoretical foundations of wavelets can be found in several recent and outstanding papers [2, 4, 5, 7, 8, 11]. By all accounts, the definitive treatise will be a forthcoming book by Y. Meyer [10]. In this paper we have neither the space nor the audacity to duplicate the excellent presentations that already exist in these sources. Instead, we will review the essential features of multiresolution analysis that seem to pertain to multigrid algorithms.

2. Multiresolution Analysis. A **multiresolution analysis** is a framework that consists of of a sequence of nested closed subspaces (typically of $L^2(\mathbf{R})$)

$$\cdots \subset V_2 \subset V_1 \subset V_0 \subset V_{-1} \subset V_{-2} \subset \cdots$$

whose union is dense in $L^2(\mathbf{R})$. The important features of these subsets are that:

a. V_0 is spanned by an orthonormal set consisting of integer translations of a single **scaling function** ϕ. For each integer j, V_j is spanned by an orthonormal set consisting of translations of scaled versions of ϕ:

$$V_j = \text{ span } \{\phi(2^{-j}x - k)\}_k.$$

b. For each integer j, $V_j = V_{j+1} \oplus W_{j+1}$, where each W_j is spanned by an orthonormal set consisting of translations of scaled versions of a single **wavelet function** ψ:

$$W_j = \text{ span } \{\psi(2^{-j}x - k)\}_k.$$

c. The doubly indexed set $\{\psi(2^{-j}x - k)\}_{j,k}$ spans $L^2(\mathbf{R})$.

Once a scaling function ϕ is found, the associated wavelet function ψ with all of the required orthogonality properties can be found directly. However, ϕ is clearly a rather extraordinary function and the discovery of such functions has been a major effort. Computationally, there are several ways to produce a scaling function, among them to compute its Fourier transform first. An important property of scaling functions and the associated wavelet functions is that they are highly localized in both the spatial and the frequency domain. Summarizing a wealth of fascinating work, there appear to be three general classes of scaling functions:

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a. C^∞ scaling functions due to Meyer [9] that have non-compact support and polynomial decay,

b. C^k scaling functions due to Battle [1] and Lemarie [6] that have non-compact support, exponential decay and are generated by orthogonalization of classical splines, and

c. the Daubechies scaling functions D_{2k} [2] that have compact support, but smoothness that increases slowly with k.

In practice, when dealing with discrete problems such as image processing or signal analysis, the problem is posed on (or projected onto) the space V_0 which represents the highest level of resolution that is desired. Given a function $u \in V_0$, the subspaces $V_1, \ldots, V_M$, for some M, give representations of u on increasingly coarse levels. At each level, the difference between the projection of u in the spaces V_{j+1} and V_j is given by the projection of u in the space W_{j+1}. The V_j projection retains the smooth features of u, while the W_j projection captures the detail (or oscillatory) components of u.

Within the multiresolution framework it is possible to do a very efficient decomposition of a function over all of the subspaces of interest. Given a function $u \in V_0$, we may use the orthogonality of ϕ and ψ to find coefficients c_{0k} such that

$$u(x) = \sum_k c_{0k}\phi(x-k). \tag{1}$$

This may be regarded as a fine grid representation of u. Furthermore, coefficients c_{1k} and d_{1k} may be found for a coarse grid representation of u on V_1 and W_1 of the form

$$u(x) = \sum_k c_{1k}\phi(\frac{x}{2}-k) + d_{1k}\psi(\frac{x}{2}-k). \tag{2}$$

This process may be continued by decomposing each V_j representation of u on the next coarser pair of grids V_{j+1} and W_{j+1} until the coarsest grid is reached. The efficient Pyramid Algorithm for performing this decomposition (and the inverse synthesis) has been proposed by Mallat [8, 7]. The full decomposition of an $N = 2^M$-point sample of u over M levels requires $O(N)$ operations.

3. The Multigrid Connection. With this brief survey, we turn to possible connections between multiresolution analysis and classical multigrid algorithms. We will consider a general operator equation of the form $Lu = f$ where L is a self-adjoint operator representing, for example, an elliptic boundary value problem. The notation can be simplified by letting

$$\phi_{jk} = \phi(2^{-j}x - k) \text{ and } \psi_{jk} = \psi(2^{-j}x - k).$$

In addition, $\langle u, v\rangle$ will denote the appropriate inner product for the problem.

On the fine grid V_0 and the first coarse grid (V_1, W_1), the solution u may be represented as in (1) and (2). The data f also have a representation on V_0 and (V_1, W_1) with respective coefficients f_{0k}, f_{1k} and g_{1k}. The fine grid problem after using orthogonality in a standard Galerkin way has the form

$$\sum_k c_{0k}\langle\phi_{0j}, L\phi_{0k}\rangle = f_{0j}, \quad \forall j. \tag{3}$$

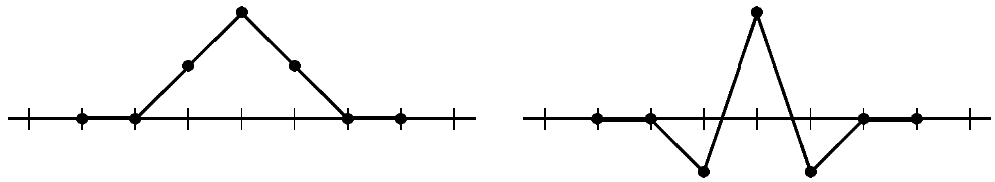

FIG. 1. *The hat functions (left) span V_1 and the range of interpolation, while the teeth functions (right) span W_1 and the null space of fullweighting.*

The known wavelets appear to have no special orthogonality properties with respect to standard elliptic operators so (3) represents a system of linear equations with generally narrow band width, but with no obvious advantages over known discretizations.

In a similar way, the problem may also be represented on the coarse grid. Substituting the (V_1, W_1) representations for u and f and using orthogonality leads to

$$\sum_k c_{1k}\langle \phi_{1j}, L\phi_{1k}\rangle + d_{1k}\langle \phi_{1j}, L\psi_{1k}\rangle = f_{1j}, \quad \forall j \tag{4}$$

and

$$\sum_k c_{1k}\langle \psi_{1j}, L\phi_{1k}\rangle + d_{1k}\langle \psi_{1j}, L\psi_{1k}\rangle = g_{1j}, \quad \forall j. \tag{5}$$

The problem given in (4) may be regarded as the coarse grid problem for the smooth components of the solution, while (5) gives the coarse grid problem for the oscillatory components.

In classical multigrid algorithms, a solution is sought on the fine grid Ω^h. The fine grid solution may be represented in terms of a basis consisting of piecewise linear hat functions ϕ_{0k}. The hat functions lack the required orthogonality to be genuine scaling functions. Nevertheless, an associated "wavelet" function ψ may be found for the hat functions (Figure 1) that allows for an orthogonal decomposition of the fine grid space. The functions ϕ_{1k} span the range of the interpolation operator I_{2h}^h and the functions ψ_{1k} span the nullspace of the full weighting operator I_h^{2h}.In multiresolution terms, we would write

$$V_0 = \text{ span } \{\phi_{1k}\} \oplus \text{ span } \{\psi_{1k}\} = V_1 \oplus W_1,$$

while in multigrid terms we would write

$$\Omega^h = \text{ span } \{\phi_{1k}\} \oplus \text{ span } \{\psi_{1k}\} = \text{ Range } \{I_{2h}^h\} \oplus \text{ Nullspace } \{I_h^{2h}\}.$$

Thus multigrid produces the same orthogonal decomposition of the fine grid Ω^h that multiresolution produces of the space V_0.

Classical multigrid deals only with the coarse grid equation for the smooth components (4). The second term representing oscillatory components is dropped and the matrix given by $\langle \phi_{1j}, L\phi_{1k}\rangle$ is precisely the multigrid coarse grid operator $I_h^{2h}L^hI_{2h}^h$.

The entire coarse grid equation for the oscillatory components is also dropped in multigrid. The rationale for neglecting the oscillatory components is that relaxation (iteration) is an extremely effective way to isolate or eliminate them. In summary, multigrid formulations use simple, near-orthogonal basis functions that still allow for an orthogonal decomposition of the fine grid space. Furthermore, multigrid does not attempt to solve for the oscillatory (W_j) components of the solution directly, but rather lets relaxation handle them indirectly. This choice of departing from orthogonality and incorporating relaxation (as well as the residual equation) accounts for the extreme efficiency of multigrid algorithms.

In closing, it should be said that preliminary work on wavelet-based multigrid algorithms has been done [3]. It appears that accuracy comparable to multigrid algorithms can be obtained using the D_{2k} compact wavelets on boundary value problems. However, a comparison of computational effort is not given. Considerable work on wavelet-based multilevel methods remains to be done.

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