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Anelastic Semigeostrophic Flow over a Mountain Ridge

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ABSTRACT

Scale analysis indicates that five nondimensional parameters (R_0^2, ϵ, μ, λ and $\kappa\lambda$) characterize the disturbance generated by the steady flow of a uniform wind (U_0, V_0) incident on a mountain ridge of width a in an isothermal, uniformly rotating, uniformly stratified, vertically semi-infinite atmosphere. Here $\mu = h_0/H_R$ is the ratio of the mountain height h_0 to the deformation depth $H_R = fa/N$ where f is the Coriolis parameter and N is the static buoyancy frequency. The parameters $\lambda = H_R/H$ and $\kappa\lambda$ are the ratios of H_R to the density scale height H and the potential temperature scale height H/κ respectively. There are two Rossby numbers: One based on the incident flow that is parallel to the mountain, $\epsilon = V_0/fa$, and one normal to the mountain, $R_0 = U_0/fa$. If $R_0^2 \ll 1$, then the mountain-parallel flow is in approximate geostrophic balance and the flow is semigeostrophic.

The semigeostrophic case reduces to the quasi-geostrophic one in the limit as μ and ϵ tend to zero. If the flow is Boussinesq ($\lambda = 0$), then the semigeostrophic solutions expressed in a streamfunction coordinate can be derived from the quasi-geostrophic solutions in a geometric height coordinate.

If the flow is anelastic ($\lambda \approx 1$), no direct correspondence between the two approximations was found. However the anelastic effects are qualitatively similar for the two and lead to: (i) an increase in the strength of the mountain anticyclone, (ii) a reduction in the extent (and possible elimination) of the zone of blocked, cyclonic flow, (iii) a permanent turning of the flow proportional to the mass of air displaced by the mountain, and (iv) an increase in the ageostrophic cross-mountain flow. The last result implies an earlier breakdown of semigeostrophic theory for anelastic flow over topography.

Apart from a strengthening of the cold potential temperature anomaly over the mountain, the presence of a finite potential temperature scale height (i.e., κ nonzero) does not significantly alter the flow solution.

1. Introduction

The important problem of airflow over mountains comprises a variety of flow regimes. Gill (1982) provides a clear review of this literature. For mountains of large horizontal scale, the earth's rotation plays a major role in determining the flow structure. Meteorological studies of this regime have their origins in the work of Charney and Eliassen (1949) who presented the first quasi-geostrophic analysis of airflow over mountains. Much of the subsequent work focused on the ability of the mountain anticyclone to create a stagnation point and form a Taylor column (Bannon, 1980, reviews the barotropic literature on Taylor column formation.) Usually[1] quasi-geostrophic theory treats the lower boundary condition in linearized form.

Robinson (1960) introduced a formulation for rotating flow over topography that includes the nonlinear lower boundary condition but retains the important quasi-geostrophic feature of the filtering of internal gravity waves. Jacobs (1964), Merkine (1975), Merkine and Kalnay (1976), Pierrehumbert (1985) and Blumen and Gross (1986) have refined this Boussinesq analysis which represents an application of semigeostrophic theory (Hoskins, 1975) to mountain flows. The present study seeks to extend these analyses to include anelastic effects. It represents a generalization of the quasi-geostrophic results of Smith (1979) and Bannon (1986) to finite-amplitude mountain ridges.

A mountain immersed in an incident zonal wind on a uniformly rotating reference frame experiences a lift force, L, due to the pressure differential across the mountain which tends to push the mountain in a meridional direction:

$$L = -\mathbf{y}\cdot\int_S p\mathbf{N}dA = -\int_{MV}\frac{\partial p}{\partial y}dV, \qquad (1.1)$$

where p is the fluid pressure, $\mathbf{N}$ is the unit vector normal to the elemental surface area dA directed toward the fluid, $\mathbf{y}$ is the unit vector in the meridional (cross-stream) direction, S is the mountain surface and MV

[1] One exception is the study of Buzzi and Speranza (1979).

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is the mountain volume. The transformation of the surface integral to an integral over the volume of the mountain utilizes the scalar version of the divergence theorem and the Archimedean concept of replacing the solid body with the equivalent fluid (see Batchelor, 1967, pp. 16–17). If the uniform incident flow is in geostrophic balance and has amplitude U_0, then the contribution of the ambient pressure gradient to (1.1) is

$$L = \mathrm{MD} \times fU_0, \tag{1.2}$$

where f is the constant Coriolis parameter and MD is the mass of air displaced by the mountain

$$\mathrm{MD} = \int_{\mathrm{MV}} \rho dV.$$

As noted by Smith (1979), a mountain induced pressure field could potentially generate an $O(h_0^2)$ lift (where h_0 is the mountain height) but this cannot balance the $O(h_0)$ lift given by (1.2). Invoking airfoil theory, Smith argued that the lift (1.2) generates a far-field circulation of strength

$$\Gamma = -\frac{f \times \mathrm{MD}}{\mathrm{MC}}, \tag{1.3}$$

where MC is the mass per unit area of an air column. As (1.3) indicates, only atmospheres with finite MC will possess a nonzero circulation. For an infinitely long mountain ridge, a linear string of circulations are generated; their net effect is to produce a meridional deflection of the flow:

$$\Delta v = -\frac{f \times \mathrm{MDL}}{\mathrm{MC}}, \tag{1.4}$$

where $\Delta v = v$ (downstream) $-v$ (upstream) and MDL denotes the displaced mass per unit length of the ridge.

Bannon (1986) confirmed the validity of the far-field circulation for the anelastic quasi-geostrophic approximation. In that theory the lower boundary condition is linearized and the displaced mass is approximated by

$$\mathrm{MD} = \rho_0 MV, \tag{1.5}$$

where ρ_0 is the surface value of the density. In the present study the nonlinear lower boundary condition is incorporated and it is shown that discrepancies with the linearized condition arise from (1.5) being an overestimate.

Section 2 presents the model formulation which employs Robinson's (1960) streamfunction coordinate and follows Pierrehumbert (1985) in assuming a semi-infinite atmosphere. A scale analysis indicates that five nondimensional parameters describe the flow. It is found that the smallness of the square of the Rossby number based on the amplitude of the flow normally incident on the mountain provides justification of the semigeostrophic approximation while a mountain Richardson number measures the extent of quasi-geostrophy. Section 3 presents the closed-form solution of the nonlinear problem and the details of its numerical solution. In addition the formula (1.4) for the permanent turning is derived analytically. Section 4 describes the results for the standard anelastic case while section 5 discusses the modified case. An appendix compares the analytic quasi- and semigeostrophic solutions in the Boussinesq case.

2. The model

The basic state model atmosphere consists of an isothermal compressible gas in hydrostatic balance. The density/pressure scale height is $H = RT/g$ where R is the ideal gas constant, T the uniform temperature, and g the acceleration due to gravity. The constant potential temperature scale height is $H/\kappa = C_pT/g$ and implies a uniform buoyancy frequency N. Here C_p is the specific heat capacity at constant pressure and $\kappa = R/C_p$. A mountain ridge of width a and height h_0 lies at the origin of a rotating Cartesian coordinate system with constant Coriolis parameter f. A uniform wind is incident on the mountain with components U_0 normal to and V_0 parallel to the ridge. Figure 1 summarizes the model physics and geometry.

a. Basic equations

The equations of motion describing the inviscid, hydrostatic, anelastic flow are

$$\frac{Du}{Dt} - fv = -\frac{\partial \phi}{\partial x}, \tag{2.1a}$$

$$\frac{Dv}{Dt} + fu = -\frac{\partial \phi}{\partial y}, \tag{2.1b}$$

$$\frac{\partial \phi}{\partial z} - \frac{\kappa \phi}{H} = g\frac{\delta \theta}{\theta_s}, \tag{2.1c}$$

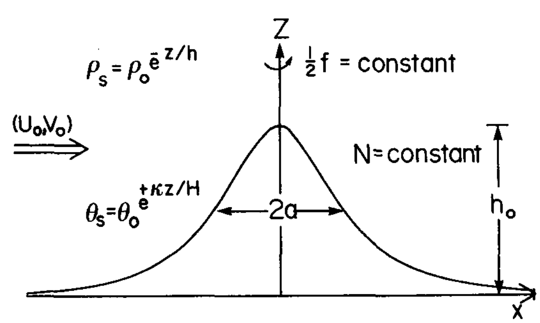

FIG. 1. A schematic illustration of the model. A uniform wind is incident on an infinite mountain ridge of height h_0 and width a on an f-plane. The wind (U_0, V_0) has components normal (U_0) and parallel (V_0) to the mountain. The isothermal basic state atmosphere has density scale height H, potential temperature scale height H/κ and uniform buoyancy frequency N.

$$\frac{\partial}{\partial x}(\rho_s u) + \frac{\partial}{\partial y}(\rho_s v) + \frac{\partial}{\partial z}(\rho_s w) = 0, \quad (2.1d)$$

$$\frac{D}{Dt}\left(N^2 z + g\frac{\delta\theta}{\theta_s}\right) = 0, \quad (2.1e)$$

where

$$\frac{D}{Dt} \equiv \frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z}, \quad (2.2a)$$

$$\rho_s = \rho_0 \exp(-z/H), \quad (2.2b)$$

$$N^2 = g\partial \ln\theta_s/\partial z = \kappa g/H. \quad (2.2c)$$

The subscript s denotes the static basic state fields and $\delta\theta$ is the dynamic contribution to the potential temperature. In (2.1) the usual meteorological convention holds for the velocity field (u, v, w) and the geopotential ϕ is related to the pressure p by $\phi = p/\rho_s(z)$. The influence of the mountain appears mathematically through the lower boundary condition

$$w = \frac{Dh}{Dt} \quad \text{at} \quad z = h, \quad (2.3)$$

where $h(x)$ describes the mountain profile.

We write the steady solution as the sum of the incident flow and that induced by the mountain:

$$u = U_0 + u_a'(x, z), \quad (2.4a)$$

$$v = V_0 + v'(x, z), \quad (2.4b)$$

$$w = w'(x, z), \quad (2.4c)$$

$$\phi = -f[U_0 y - V_0 x] + \phi'(x, z), \quad (2.4d)$$

$$g\frac{\delta\theta}{\theta_s} = \frac{\kappa f[U_0 y - V_0 x]}{H} + g\frac{\delta\theta'}{\theta_s}(x, z). \quad (2.4e)$$

The primed fields denote the disturbance generated by the mountain ridge and by symmetry are assumed to be independent of y. The subscript a on the mountain-induced zonal wind component indicates that this field is an ageostrophic one. We note that the flow in the absence of the mountain is in geostrophic balance with the uniform pressure gradient force.

Substitution of (2.4) into (2.1) and (2.3) yields

$$\frac{du_a'}{dt} - fv' = -\frac{\partial\phi'}{\partial x}, \quad (2.5a)$$

$$\frac{dv'}{dt} + fu = fU_0, \quad (2.5b)$$

$$\frac{\partial\phi'}{\partial z} - \frac{\kappa}{H}\phi' = g\frac{\delta\theta'}{\theta_s}, \quad (2.5c)$$

$$\frac{\partial}{\partial x}(\rho_s u) + \frac{\partial}{\partial z}(\rho_s w') = 0, \quad (2.5d)$$

$$\frac{d}{dt}\left(N^2 z + g\frac{\delta\theta'}{\theta_s}\right) + \frac{\kappa f}{H}(U_0 v' - V_0 u_a') = 0, \quad (2.5e)$$

where

$$\frac{d}{dt} = u\frac{\partial}{\partial x} + w'\frac{\partial}{\partial z},$$

$$w' = u\frac{\partial h}{\partial x} \quad \text{at} \quad z = h(x). \quad (2.6)$$

We note that the system (2.5) and (2.6) are fully nonlinear.

b. Scale analysis

A judicious scale analysis provides much insight into the problem posed by (2.5) with (2.6). At the outset we note that the mountain introduces zonal variations to the flow through (2.6). Thus, we choose a as the characteristic scale for x. The choice of a corresponding vertical scale is ambiguous as five such scales arise in the problem (Table 1). However the scales H and H/κ are intrinsic scales of the fluid but not of the flow. Similarly the mountain height may not represent a typical dynamical scale. Here we choose H_R rather than H_I as the appropriate vertical scale of the flow as this deformation depth is consistent with the characteristic horizontal scale a.

The effect of the mountain is to deform the isentropes horizontally over a distance a by an amount $\Delta\theta$ where

$$\Delta\theta \sim h_0 \frac{\partial\theta_s}{\partial z} = \frac{\theta_s}{g} N^2 h_0.$$

Anticipating an approximate thermal wind balance for the mountain parallel flow,

$$\frac{\partial v'}{\partial z} \sim \frac{1}{f}\frac{\partial}{\partial x}\left(\frac{g\delta\theta'}{\theta_s}\right),$$

the scale, V_T, of the shear of the meridional wind is

$$V_T = \frac{g\Delta\theta}{fa\theta_s} = \frac{N^2 h_0}{fa}.$$

Thus we choose $v' \sim O(V_T H_R) = Nh_0$ and, by approximate geostrophy, $\phi \sim O(fNah_0)$.

The boundary condition (2.6) introduces a scale for the vertical field $w' \sim O(U_0 h_0/a)$ provides $u_a' \leqslant U_0$. By continuity, (2.5d), we choose $u_a' \sim O(U_0 h_0/H_R)$.

Based on the above considerations, we introduce the following scaling

$$x = ax'', \qquad z = H_R z'', \qquad h = h_0 h''(x), \quad (2.7a)$$

TABLE 1. Characteristic vertical scales.

h_0	mountain height
$H = RT/g$	density (pressure) scale height
H/κ	potential temperature scale height
$H_R = fa/N$	deformation depth for mountain scale a
$H_I = U_0/N$	deformation depth for inertial scale U_0/f

$$u_a' = U_0\mu u_a'', \quad v' = Nh_0 v'', \quad w' = U_0 \frac{f}{N}\mu w'', \qquad (2.7b)$$

$$\phi' = Nfah_0\phi'', \quad \frac{g\delta\theta'}{\theta_s} = N^2 h_0 \theta'', \qquad (2.7c)$$

into (2.5) and (2.6) to obtain

$$R_0^2 \frac{du_a}{dt} - v = -\frac{\partial \phi}{\partial x}, \qquad (2.8a)$$

$$\mu \frac{dv}{dt} + u = \hat{u}, \qquad (2.8b)$$

$$\frac{\partial \phi}{\partial z} - \kappa\lambda\phi = \theta, \qquad (2.8c)$$

$$\frac{\partial}{\partial x}(\hat{\rho}u) + \mu\frac{\partial}{\partial z}(\hat{\rho}w) = 0, \qquad (2.8d)$$

$$\frac{d}{dt}(z + \mu\theta) + \kappa\lambda\mu(v - \epsilon u_a) = 0, \qquad (2.8e)$$

where

$$\frac{d}{dt} = (\hat{u} + \mu u_a)\frac{\partial}{\partial x} + \mu w\frac{\partial}{\partial z}, \qquad (2.9a)$$

$$u = \hat{u} + \mu u_a, \qquad (2.9b)$$

$$w = (\hat{u} + \mu u_a)\frac{\partial h}{\partial x} \quad \text{at} \quad z = \mu h(z). \qquad (2.9c)$$

In writing the system (2.8) with (2.9) we have dropped the double primes and introduced $\hat{u}$ to denote the scaled incident wind and $\hat{\rho}$ for the scaled basic state density where

$$\hat{u} = 1, \qquad (2.10a)$$

$$\hat{\rho} = \exp(-\lambda z). \qquad (2.10b)$$

The five nondimensional parameters (R_0^2, ϵ, μ, λ and $\kappa\lambda$) describing the flow are summarized in Table 2.

c. The semigeostrophic approximation

Inspection of (2.8a) indicates that the semigeostrophic approximation of the mountain-parallel wind being in geostrophic balance is valid provided

$$R_0^2 \ll 1, \qquad (2.11a)$$

$$\mu \leqslant O(1). \qquad (2.11b)$$

TABLE 2. Nondimensional flow parameters.

$\mu = h_0/H_R = Nh_0/fa$
$\lambda = H_R/H = fa/NH$
$\kappa = R/Cp$
$\text{Ro} = H_I/H_R = U_0/fa$
$\epsilon = V_0/fa$

This differs from Merkine's (1975) criterion that $U_0^2/N^2D^2 \ll 1$ for $R_0 \leqslant 1$ where D is the depth of the fluid. Clearly, as D approaches infinity, D is no longer a characteristic scale of the flow and Merkine's criterion loses validity. Replacement of D with H_R makes his criterion equivalent to (2.11a) and overcomes some of the difficulties inherent in his nondimensionalization.

Henceforth we assume that (2.11) holds and that the flow is semigeostrophic. Then (2.8a) is replaced by

$$-v = -\frac{\partial \phi}{\partial x}. \qquad (2.8a')$$

In such a case the solutions depend only on ϵ, μ, λ and $\kappa\lambda$ and are not an explicit function of the incident normal windspeed U_0.

d. The Boussinesq case

Setting $\lambda = 0$ retrieves the Boussinesq case and the governing equations are only a function of μ. This result agrees with Pierrehumbert (1985) where $\mu = R_0Fr$ in his notation. He noted that μ is the Burger number based on the mountain slope. The present scaling indicates that μ may also be defined as

$$\mu = V_T/N = 1/\sqrt{\text{Ri}}, \qquad (2.12)$$

where Ri is the Richardson number based on the thermal shear of the mountain parallel flow. Equation (2.9a) indicates that μ is a measure of the advection by the ageostrophic flow while (2.9c) suggests that μ also measures the strength of the forcing by bottom topography.

Pierrehumbert (1985) noted a similarity between the quasi- and semigeostrophic solutions for the zonal wind. The Appendix extends this correspondence to the other flow fields.

e. The quasi-geostrophic limit

In the limit as μ and ϵ tend to zero, the set (2.8) and (2.9) with (2.8a′), reduces to

$$-v = -\frac{\partial \phi}{\partial x}, \qquad (2.13a)$$

$$\frac{dv}{dt} + u_a = 0, \qquad (2.13b)$$

$$\frac{\partial \phi}{\partial z} - \kappa\lambda\phi = \theta, \qquad (2.13c)$$

$$\frac{\partial u_a}{\partial x} + \frac{1}{\hat{\rho}}\frac{\partial}{\partial z}(\hat{\rho}w) = 0, \qquad (2.13d)$$

$$\frac{d\theta}{dt} + \kappa\lambda v + w = 0, \qquad (2.13e)$$

where

$$\frac{d}{dt} = \hat{u}\frac{\partial}{\partial x},$$

$$w = \frac{dh}{dt} \quad \text{at} \quad z = 0. \tag{2.14}$$

Solving (2.13) for the geopotential yields the quasi-geostrophic potential vorticity equation

$$\frac{d}{dt}\left[\frac{\partial^2 \phi}{\partial x^2} + \frac{1}{\hat{\rho}}\frac{\partial}{\partial z}\left(\hat{\rho}\frac{\partial \phi}{\partial z}\right)\right] = 0, \tag{2.15}$$

with the boundary condition

$$\frac{d}{dt}\left(\frac{\partial \phi}{\partial z}\right) = -\frac{dh}{dt} \quad \text{at} \quad z = 0. \tag{2.16}$$

The Green's function solution for the three-dimensional problem appears in Bannon (1986) while the Appendix treats the two-dimensional Boussinesq case.

3. Analysis

The method of solution follows Robinson (1960) by introducing a streamfunction coordinate transformation to form a second-order partial differential equation for the vertical displacement of the streamline. In the anelastic case treated here the equation is an elliptic one with nonconstant coefficients and we obtain solutions numerically.

a. Derivation of the governing equation

Introduction of a mass streamfunction, ψ,

$$\hat{\rho}u = \frac{\partial \psi}{\partial z}, \quad \mu\hat{\rho}w = -\frac{\partial \psi}{\partial x}, \tag{3.1}$$

satisfies the continuity equation (2.8d) identically. A convective derivative then becomes, for example,

$$\frac{dv}{dt} = \frac{1}{\hat{\rho}}J(\psi, v), \tag{3.2}$$

where $J(\psi, v) \equiv \hat{y}\cdot(\nabla\psi \times \nabla v)$ is the Jacobian.

It is convenient to introduce the meridional displacement $\eta(x, z)$ of a fluid parcel. Then by definition

$$v = \frac{d\eta}{dt}, \tag{3.3}$$

and the heat equation becomes, using (2.8b), (2.9b) and (3.2),

$$\frac{1}{\hat{\rho}}J[\psi, z + \mu\theta + \kappa\lambda\mu(\eta + \epsilon v)] = 0. \tag{3.4}$$

This equation implies that $z + \mu\theta + \kappa\lambda\mu(\eta + \epsilon v)$ is a function, z_∞ say, of ψ only. We determine $z_\infty(\psi)$ in the usual manner by examining the flow far upstream of the obstacle. There we assume θ, η, v and u_a all vanish and

$$\left.\begin{aligned} z &= z_\infty(\psi) \\ \partial\psi/\partial z &= \hat{\rho}\hat{u} \end{aligned}\right\} \quad \text{as} \quad x \to -\infty, \tag{3.5a, 3.5b}$$

The latter equation has the solution

$$\psi = -(1/\lambda)e^{-\lambda z} \quad \text{as} \quad x \to -\infty. \tag{3.5c}$$

Thus,

$$z_\infty = -(1/\lambda)\ln(-\lambda\psi) \tag{3.6}$$

is the height of a streamline far upstream of the mountain.

With these considerations the set (2.8) and (2.9) becomes

$$-v = -\partial\phi/\partial x, \tag{3.7a}$$

$$\mu J(\psi, v) = \hat{\rho}(\hat{u} - u), \tag{3.7b}$$

$$\partial\phi/\partial z - \kappa\lambda\phi = \theta, \tag{3.7c}$$

$$z + \mu\theta + \kappa\lambda\mu(\eta + \epsilon v) = z_\infty, \tag{3.7d}$$

$$\hat{\rho}v = J(\psi, \eta), \tag{3.7e}$$

with the boundary condition

$$-\frac{1}{\mu}\frac{\partial\psi}{\partial x} = \frac{\partial\psi}{\partial z}\frac{\partial h}{\partial x} \quad \text{at} \quad z = \mu h(x). \tag{3.8}$$

Elimination of ϕ and θ from (3.7a, c and d) yields

$$\frac{\partial v}{\partial z} - \kappa\lambda v = \frac{1}{\mu}\frac{\partial}{\partial x}\Theta, \tag{3.9}$$

where

$$\Theta = z_\infty(\psi) - \kappa\lambda\mu(\eta + \epsilon v), \tag{3.10a}$$

or

$$\Theta = z + \mu\theta, \tag{3.10b}$$

is the total nondimensional potential temperature field excluding the linear variations associated with the incident flow [see (2.4e)].

We next introduce a streamfunction coordinate system (χ, ψ) to replace the Cartesian system (x, z):

$$\chi = x, \quad \psi = \psi(x, z), \tag{3.11}$$

and the streamfunction height $z = z(\chi, \psi)$ is the new dependent variable. Use of the chain rule indicates that

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial \chi} + \left(\frac{\partial\psi}{\partial x}\right)\frac{\partial}{\partial\psi}, \tag{3.12a}$$

$$\frac{\partial}{\partial z} = \left(\frac{\partial\psi}{\partial z}\right)\frac{\partial}{\partial\psi}, \tag{3.12b}$$

and the Jacobian simplifies to, for example,

$$J(\psi, v) = \left(\frac{\partial\psi}{\partial z}\right)\frac{\partial v}{\partial\chi} = \hat{\rho}u\frac{\partial v}{\partial\chi}. \tag{3.12c}$$

Two other useful relations are

$$\frac{\partial z}{\partial\chi} = -(\partial\psi/\partial x)/(\partial\psi/\partial z), \tag{3.13a}$$

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial\chi} - \left(\frac{\partial z}{\partial\chi}\right)\left(\frac{\partial\psi}{\partial z}\right)\frac{\partial}{\partial\psi}. \tag{3.13b}$$

Use of the new coordinate system in (3.9), (3.7b) and (3.7e) yields

$$\frac{\partial v}{\partial \psi} - \kappa\lambda v \frac{\partial z}{\partial \psi} = \frac{1}{\mu}\left(\frac{\partial z}{\partial \psi}\frac{\partial \Theta}{\partial \chi} - \frac{\partial z}{\partial \chi}\frac{\partial \Theta}{\partial \psi}\right), \quad (3.14a)$$

$$\mu \frac{\partial v}{\partial \chi} = (\hat{u} - u)/u, \quad (3.14b)$$

$$v = u(\partial \eta/\partial \chi). \quad (3.14c)$$

The boundary condition (3.8) with (3.13a) becomes

$$z = \mu h \quad \text{at} \quad \psi = -1/\lambda. \quad (3.15)$$

A second-order partial differential equation for $z(\chi, \psi)$ emerges from (3.14a and b) upon cross-differentiation. The result is

$$\frac{\partial}{\partial \chi}\left(\frac{\partial \Theta}{\partial \psi}\frac{\partial z}{\partial \chi}\right) - \frac{\partial}{\partial \chi}\left[\left(\frac{\partial \Theta}{\partial \chi} + \kappa\lambda\mu v\right)\frac{\partial z}{\partial \psi}\right] + \frac{\partial}{\partial \psi}\left(\hat{\rho}\hat{u}\frac{\partial z}{\partial \psi}\right) = 0. \quad (3.16)$$

It can readily be shown that this equation reduces to the Boussinesq problem if $\lambda = 0$. The equation is elliptic provided

$$\frac{\hat{\rho}\hat{u}(\partial \Theta/\partial \psi)}{(\partial \Theta/\partial \chi + \kappa\lambda\mu v)^2} > \frac{1}{4}. \quad (3.17)$$

We note that the numerator represents the static stability of the flow while the denominator is the square of a measure of the thermal wind shear [cf. (3.9)]. Thus the ellipticity condition (3.17) requires that the effective Richardson number of the flow in (χ, ψ) space be greater than ¼. Using (3.10a) and (3.14c) we rewrite (3.17) as

$$\frac{\hat{\rho}\hat{u}(\partial \Theta/\partial \psi)}{(\kappa\lambda\mu)^2\left(\dfrac{v}{u}\right)^2(u - \hat{u})^2} > \frac{1}{4},$$

and we conclude that the standard anelastic case ($\kappa = 0$) is always elliptic. The cases investigated here all satisfied (3.17).

b. Permanent turning

Here we derive the expression (1.4) for the net meridional displacement of the flow. First we note that this displacement is barotropic. Use of (3.1) and (3.10a) into (3.9) yields

$$\frac{\partial v}{\partial z} = \kappa\lambda\left(v - \frac{\partial \eta}{\partial x} - \epsilon\frac{\partial v}{\partial x}\right) - \hat{\rho}w\left(\frac{\partial z_\infty}{\partial \psi}\right). \quad (3.18)$$

Infinitely far upstream or downstream of the ridge, the ageostrophic motions vanish. Then, from (3.3), $v = \partial\eta/\partial x$, and, from (2.8b), $\partial v/\partial x = 0$, and the meridional wind is independent of height:

$$\frac{\partial v}{\partial z} = 0 \quad \text{at} \quad |x| = \infty. \quad (3.19)$$

The variation of the meridional wind along a streamline is, from (3.14b),

$$\frac{\partial v}{\partial \chi} = \frac{1}{\mu}\left(\frac{1}{u} - 1\right). \quad (3.20)$$

Integration along a streamline from $\chi = -\infty$ to $+\infty$ and then across ψ from the surface ($\psi = -1/\lambda$) to infinity ($\psi = 0$) yields

$$\Delta v = \frac{\lambda}{\mu}\int_{-1/\lambda}^{0} d\psi \int_{-\infty}^{+\infty} d\chi\left(\frac{1}{u} - 1\right) \quad (3.21)$$

where we have used the fact that the permanent turning Δv is independent of ψ. We use (3.1) to rewrite the integrand in the form

$$\left(\frac{1}{u} - 1\right) = \left(\hat{\rho}\frac{\partial z}{\partial \psi} - 1\right) = -\frac{1}{\lambda}\frac{\partial}{\partial \psi}[\lambda\psi + e^{-\lambda z}] \quad (3.22)$$

using (2.10b). Interchanging the order of integration in (3.21) and substituting (3.22) enables the integration over ψ to be performed. Since $z(\psi = 0) = \infty$ and $z(\psi = -1/\lambda) = \mu h$, the expression for Δv is

$$\Delta v = -\frac{1}{\mu}\int_{-\infty}^{+\infty} d\chi[1 - \exp(-\lambda\mu h)]. \quad (3.23)$$

This result is the nondimensional version of the permanent turning (1.4) for an isothermal atmosphere. Inspection of (3.23) indicates that Δv vanishes for the Boussinesq case ($\lambda = 0$) and that $\Delta v = -\lambda$ MV for the anelastic quasi-geostrophic ($\mu = 0$) case. Finally we note that the permanent turning is not a function of κ.

c. Method of solution

The coefficients of the elliptic equation (3.16) for z are nonconstant and are, from (3.10a), functions of v and η. We find expressions for these variables using (3.14b) and (3.14c):

$$v = \int_{-\infty}^{\chi} \frac{(\hat{u} - u)}{\mu u} d\chi, \quad (3.24a)$$

$$\eta = \int_{-\infty}^{\chi} \left(\frac{v}{u}\right) d\chi, \quad (3.24b)$$

where

$$u = 1/(\hat{\rho}\partial z/\partial \psi). \quad (3.24c)$$

In writing the definite integrals we have applied the upstream condition that v and η vanish at $\chi = -\infty$. Thus (3.16) and (3.24) form a coupled system of equations for z.

We obtain numerical solutions by solving (3.16) and (3.24) iteratively. At each iteration we solve (3.16) using standard techniques for fixed coefficients. We use the new approximate solution for z to calculate revised coefficients from (3.24). The technique is convergent after about 10 iterations. We take a numerical solution to the Boussinesq case ($\lambda = 0$) which has constant coefficients as the initial guess.

We map the infinite strip domain ($-\infty \leq \chi \leq +\infty$, $-\lambda^{-1} \leq \psi \leq 0$) into a rectangle ($-1 \leq \bar{\chi} \leq +1$, $1 \geq \bar{\psi} \geq 0$) by the transformations

$$\bar{\chi} = \tanh(0.2\chi), \qquad (3.25a)$$

$$\bar{\psi} = (-\lambda\psi)^{1/\lambda}. \qquad (3.25b)$$

A uniform grid of 101 × 51 points covers the 2 × 1 domain in ($\bar{\chi}$, $\bar{\psi}$) space. In physical space the horizontal resolution is $0.1a$ near the mountain and the first interior points are $\pm 11.5a$ from the mountain peak. The vertical resolution near the lower surface is $0.02H_R$ with the highest interior point at $z = 3.9H_R$ from the surface. We use second-order differencing and the trapezoidal rule.

The boundary conditions for (3.16) are

$$\delta = 0 \quad \text{at} \quad \bar{\chi} = \pm 1, \qquad (3.26a)$$

$$\delta = h(\bar{\chi}) \quad \text{at} \quad \bar{\psi} = 1, \qquad (3.26b)$$

$$\delta = \delta_T \quad \text{at} \quad \bar{\psi} = 0, \qquad (3.26c)$$

where

$$\delta = (z - z_\infty)/\mu \qquad (3.27)$$

is the deviation of the streamfunction height from its value far upstream. These conditions express the physical constraints that δ vanish far upstream and downstream of the mountain and that the lower boundary correspond to a streamfunction.

The nonhomogeneous boundary condition, (3.26c), at the top of the domain requires some justification. The constant δ_T is defined by

$$\delta_T \equiv \frac{1}{2\mu l}\int_{-l}^{+l} dx \int_0^{\mu h} d z \hat{\rho} \qquad (3.28)$$

and is the constant displacement over the internal ($-l \leq x \leq l$) having the same normalized mass per unit length as that displaced by the mountain. For our isothermal atmosphere, we can perform the integration over z analytically to yield

$$\delta_T = \frac{1}{2l}\int_{-l}^{+l} dx[1 - \exp(-\lambda\mu h(x))]/\lambda\mu, \qquad (3.29)$$

which has the limiting value

$$\delta_T = \frac{1}{2l}\int_{-l}^{+l} dx h(x) \quad \text{as} \quad (\lambda \text{ or } \mu) \to 0, \qquad (3.30)$$

and $2l\delta_T$ is the cross-sectional area of the mountain. Condition (3.28) is therefore consistent with the anelastic quasi-geostrophic (Bannon, 1986) and Boussinesq semigeostrophic (Pierrehumbert, 1985) results for which the displaced volume equals the mountain volume. For a horizontally infinite domain, $l \to \infty$ and $\delta_T \to 0$. Use of this homogeneous condition in the numerical scheme yielded solutions similar to those with a rigid lid. This suggested that despite the transformation (3.25a), the effective horizontal domain is finite. Solutions with $l = 11.5$ (corresponding to the first interior grid point) in (3.30) showed excellent agreement with the anelastic solutions for the two limiting cases. All solutions presented here used (3.26) with $l = 11.5$.

4. Results for the standard ($\kappa = 0$) theory

We refer to the case with $\kappa = 0$ as the standard theory which ignores the finiteness of the potential temperature scale height. In this case the governing equation (3.16) with (3.10a) and (3.24c) reduces to

$$\frac{\partial z_\infty}{\partial \psi}\frac{\partial^2 z}{\partial x^2} + \frac{\partial}{\partial \psi}\left[\left(\frac{\partial z_\infty}{\partial \psi}\right)^{-1}\frac{\partial z}{\partial \psi}\right] = 0. \qquad (4.1)$$

This result and the boundary conditions indicate that the solution is symmetric about $x = 0$ for symmetric mountains. In addition the solution is independent of ϵ.

Figures 2–6 display the flow fields for the case of a Gaussian mountain

$$h = \exp(-x^2), \qquad (4.2)$$

with nondimensional parameter settings $\lambda = 1$ and $\mu = 0.4$. Experiments in which these parameters are changed indicate that the results vary monotonically with both λ and μ. The choice displayed here is for flow over strong topography with significant anelastic effects. Dimensionally these values roughly correspond to a mountain of height $h_0 \approx 3$ km and width $a \approx 800$ km in an atmosphere with scale height $H = 8$ km, $f = 10^{-4}$ s^{-1}, and $N = 10^{-2}$ s^{-1}. The semigeostrophic assumption will be a good approximation (e.g., $Ro^2 \leq 0.10$) for typical atmospheric wind speeds (e.g. U_0

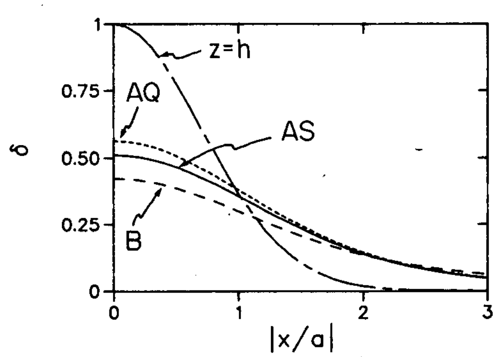

FIG. 2. Horizontal variation of the vertical displacement, δ (in units of h_0), of the streamline from its far-field value of one deformation depth ($z_\infty^* = fa/N$). Also shown is the displacement of the surface streamline which corresponds to the nondimensional mountain height $z = h(x)$. Here and in Figs. 3 through 6, $\mu = 0.4$, $\lambda = 1$, and $\kappa = 0$, and the labels AS, AQ and B denote the anelastic semigeostrophic, and anelastic quasi-geostrophic, and the Boussinesq cases respectively.

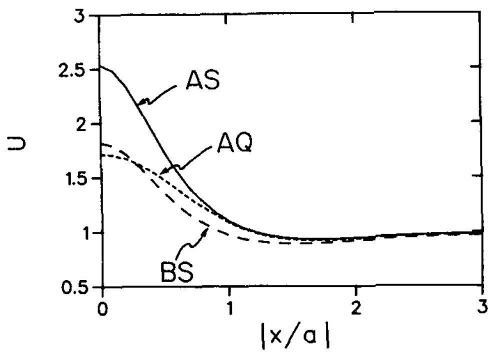

FIG. 3. The cross-mountain wind at the surface (in units of U_0) as a function of absolute horizontal distance. BS denotes the Boussinesq semigeostrophic case.

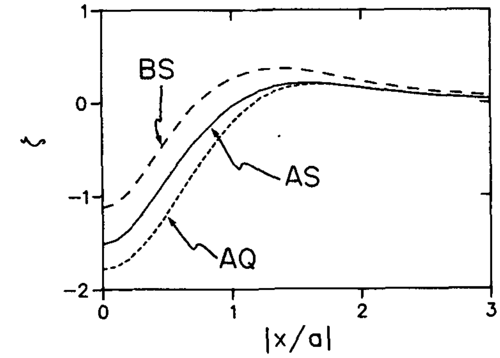

FIG. 5. The surface relative vorticity (in units of Nh_0/a) as a function of absolute horizontal distance.

$\leqslant 25$ m s^{-1}). For purposes of comparison the figures also display the anelastic quasi-geostrophic (denoted AQ) and Boussinesq semigeostrophic (denoted BS) solution. In cases where the BS results correspond with the quasi-geostrophic ones (see the Appendix) they are denoted B. The new anelastic results are denoted AS.

Figure 2 displays the displacement of the $z_\infty = 1$ streamline which far upstream lies at the elevation of one Rossby height above the surface. Comparison with the displacement of the surface streamline (which corresponds to the mountain height) indicates that the amplitude of the displacement decreases with height but broadens laterally. Both the B and AQ displacements conserve the mountain volume per unit length given by (3.30) while the AS results conserves the normalized displaced mass (3.29). While both anelastic cases predict greater displacements aloft than the Boussinesq case, the (linearized) AQ result is an overprediction.

The well-known enhancement of the cross-mountain flow imposed by mass conservation is illustrated in Fig. 3. Both the BS and AQ cases underestimate this enhancement and hence the associated breakdown of semigeostrophic theory (Pierrehumbert, 1985). [We note in passing that the zonal windspeed need not be infinite for the theory to fail. Large but finite speeds would generate sufficiently large advections that their neglect in (2.8a) would no longer be justifiable.] Farther down the mountain slope the zonal wind is decelerated below its upstream value. Anelasticity reduces this upstream blocking. The zonal wind is symmetric about $x = 0$. Thus, apart from the regions of weak blocking, the flow is divergent upstream and convergent downstream of the mountain top.

We plot the mountain-parallel wind in Fig. 4 with the value of V_0 set to produce an antisymmetric field with $v(x = 0) = 0$. The permanent turnings calculated numerically at $x = -11.5$ are $\Delta v = 2V_0/Nh_0 = 0.12$, 1.76 and 1.53 for the cases B, AQ and AS, respectively. Theoretical values for cases B and AQ are $\Delta v = 0$ and $\sqrt{\pi}$ (=1.77), respectively. This good agreement with theory verifies the numerical technique and gives an estimate of the numerical error. As noted in the Introduction, the overprediction for the AQ case with its linearized boundary condition is given by (see 1.4) the ratio

$$\rho_0 \mathrm{MV}/\mathrm{MD} = 1.15, \tag{4.3}$$

and agrees with the difference in the anelastic predictions for the meridional deflection Δv. This agreement

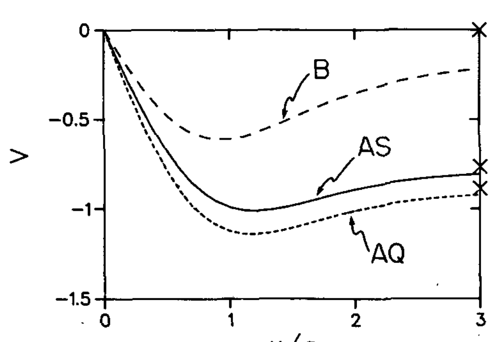

FIG. 4. The total mountain-parallel wind at the surface (in units of Nh_0) as a function of distance downstream of the mountain top. The fields are antisymmetric about the origin. The crosses on the right ordinate denote the asymptotic far-field value for each case.

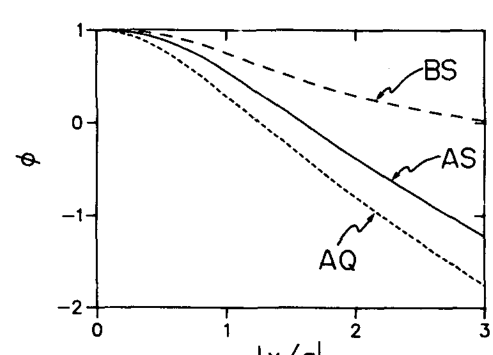

FIG. 6. The mountain-induced geopotential at the surface (in units of Nh_0fa) as a function of absolute horizontal distance.

provides a posteriori justification for the boundary condition (3.26).

The overprediction by quasi-geostrophic theory is further reflected in Figs. 5 and 6 where one sees that the strength of the AS mountain anticyclone is reduced compared to the AQ case.

We next note that some of the features of the u, v and ζ curves are interrelated. Operating (3.13b) on v and using (3.14a) with $\lambda = 0$ and (3.14b) yields an expression for the relative vorticity

$$\zeta = \frac{\partial v}{\partial x} = \frac{\hat{u} - u}{\mu u} + \frac{1}{\mu}\left(\frac{\partial z}{\partial x}\right)^2 \frac{u}{\hat{u}}, \qquad (4.4)$$

Since $\partial z/\partial x = \mu \partial h/\partial x$ at $z = \mu h(x)$, we have

$$\zeta = \frac{\partial v}{\partial x} = \frac{\hat{u} - u}{\mu u} + \mu\left(\frac{\partial h}{\partial x}\right)^2 \frac{u}{\hat{u}} \quad \text{at} \quad z = \mu h(x). \qquad (4.5)$$

We obtain insight into this expression by taking the quasi-geostrophic ($\mu \to 0$) limit, to obtain

$$\zeta = \frac{\partial v}{\partial x} = -u_a \quad \text{at} \quad z = 0. \qquad (4.6)$$

We note from (2.13b) that the quasi-geostrophic result holds at all levels. Since the second rhs term in (4.5) is $O(\mu)$, the first term generally dominates. Then anticyclonic (cyclonic) flow results for $u > \hat{u}$ ($u < \hat{u}$). Inspection of Figs. 3 and 5 indicate that the region of blocked flow is typically cyclonic. Further the overshooting of the v field seen in Fig. 4 requires cyclonic flow. As noted by Pierrehumbert (1985), the blocking depends on the mountain profile. Sensitivity experiments (not shown) for the anelastic case qualitatively confirm his findings. For example, solutions for a bell-shaped mountain profile (which has gentler slopes than the Gaussian displayed here) exhibit no region of blocking, no cyclonic far-field vorticity and the v-field approaches its asymptotic value monotonically.

5. Results for the modified ($\kappa = 2/7$) theory

Here we retain a finite potential temperature scale height and set $\kappa = 2/7$. In such a case the governing equation (3.16) must be solved with (3.10a), and the solution is a function of ϵ. In order to obtain a symmetric solution for z we choose $\epsilon = \mu\Delta v/2$. As in section 4, this choice makes the total meridional wind field antisymmetric about $x = 0$.

Inspection of (3.10a) indicates that the effect of a nonzero κ is measured by the parameter grouping

$$\kappa\lambda\mu = \kappa h_0/H,$$

which is the ratio of the mountain height to the potential temperature scale height. For the parameter settings $\lambda = 1$ and $\mu = 0.4$ used in the preceding section, $\kappa\lambda\mu = 0.114$ and the effects are small. In the quasi-geostrophic case, $\kappa\lambda\mu \to 0$, and as noted by Bannon (1986), the modified results are identical to the standard ones except for the strengthening of the potential temperature anomaly over the mountain due to cold advection of the ambient meridional gradient (2.4e).

Numerical solutions for $\kappa = 2/7$, $\lambda = 1$, and $\mu = 0.4$ are very similar to those displayed in Figs. 2–6 for $\kappa = 0$ and are not shown. For example, the maximum cross-mountain windspeed is 2.65 (2.53), the minimum zonal windspeed is 0.92 (0.93), the minimum relative vorticity is -1.55 (-1.51), the maximum meridional windspeed is 1.02 (1.01), and the permanent turning is $\Delta v = -1.49$ (-1.53) for $\kappa = 2/7$ (0). These comparisons suggest that, to within numerical error, the solutions are the same.

6. Conclusion

This study has presented an investigation of anelastic flow over a finite-amplitude mountain ridge in a rotating, isothermal atmosphere. The results indicate that use of the linearized boundary condition, consistent with quasi-geostrophic theory, underestimates the strength of the cross-mountain flow but overestimates the mountain anticyclone and the effect of the lift force in producing a permanent turning of the flow.

The lift force (1.1) may be generalized for the beta effect and nonuniform incident flow to

$$L = \int_{MV} \rho(z) f(y) U(y, z) dV, \qquad (6.1)$$

provided the incident flow is in geostrophic balance. The permanent turning/far-field circulation associated with (6.1) should be modified accordingly. However, inclusion of vertical shear in an anelastic flow and the beta effect introduces a gradient to the ambient potential vorticity field. Thus Rossby waves will comprise a part of the solution.

It is important to note that the lift (6.1) depends crucially on the presence of a nonzero incident wind U_0. Thus even though Γ in (1.3) and Δv in (1.4) are not explicit functions of U_0, both the far-field circulation and permanent turning would *not* be present in an atmosphere at rest.

A final remark concerns the drag force D acting to push the mountain downstream,

$$D = -\hat{x} \cdot \int_S p\mathbf{N} dA = -\int_{MV} \frac{\partial p}{\partial x} dV. \qquad (6.2)$$

If the mountain-parallel wind v is in geostrophic balance, then this expression may be written as

$$D = -\int_{MV} \rho(z) f v dV. \qquad (6.3)$$

In general the drag (6.3) does not vanish unless v is antisymmetric [assuming $h(x)$ is symmetric]. The antisymmetry displayed in Fig. 4 is achieved by choosing the appropriate value of the upstream mountain-parallel flow V_0 ($=Nh_0\Delta v/2$) such that half of the permanent turning occurs on the upstream side of the mountain and half on the downstream side. Then the drag

TABLE 3. A comparison of the quasi-geostrophic and semigeostrophic solutions for the Boussinesq approximation.

Quantity	Semigeostrophic	Quasi-geostrophic
Streamline displacement	$\delta(\chi, \psi) = \int_{-\infty}^{+\infty} dkh(k) \exp(ik\chi - \|k\|\psi)$	$\delta(x, z) = \int_{-\infty}^{+\infty} dkh(k) \exp(ikx - \|k\|z)$
Cross-mountain wind	$u(\chi, \psi) = \hat{u}/(1 + \mu\partial\delta/\partial\psi)$	$u(x, z) = \hat{u}(1 - \mu\partial\delta/\partial z)$
Mountain-parallel wind	$v(\chi, \psi) = \int (\partial\delta/\partial\psi)d\chi$	$v(x, z) = +\int (\partial\delta/\partial z)dx$
Vertical motion	$w(\chi, \psi) = u(\partial\delta/\partial\chi)$	$w(x, z) = \hat{u}(\partial\delta/\partial x)$
Cross-mountain geopotential gradient	$\frac{\partial\phi}{\partial\chi}(\chi, \psi) = v - \left(\frac{\mu}{2}\right)(\partial\delta^2/\partial\chi)$	$\frac{\partial\phi}{\partial x}(x, z) = v$
Relative vorticity	$\zeta(\chi, \psi) = \partial v/\partial\chi + \left(\frac{\mu}{2}\right)\left(\frac{u}{\hat{u}}\right)\left(\frac{\partial\delta}{\partial\chi}\right)^2$	$\zeta(x, z) = \partial v/\partial x$

vanishes but the mountain has an upstream influence. This upstream influence, however, is consistent with the far-field circulation (1.3). We further note that for this choice of V_0, $\epsilon = \mu\Delta v/2$ and ϵ is not a free parameter. For the choice $V_0 = 0$, $\epsilon = 0$, and the v-field is asymmetric (e.g., Blumen and Gross, 1986). As the flow is inviscid, adiabatic, and without wave motion, the implied nonzero drag must (Bannon, 1985) be the result of flow transience. Here the transience is associated not with the incident wind but rather with the starting vortex shed during the initial setup of the flow and forever advecting downstream.

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APPENDIX

The Boussinesq Case

If the flow is Boussinesq, $\lambda = 0$ and the governing equation (3.16) reduces to Laplace's equation for the streamline displacement δ. Pierrehumbert (1985) presented the solution for a semi-infinite atmosphere in terms of the Fourier transform $h(k)$ of the mountain profile $h(x)$. Table 3 summarizes the solutions in (χ, ψ) space for δ and related flow variables. The table also displays the corresponding solutions in (x, z) space for the quasi-geostrophic case.

Inspection of Table 3 indicates that the semigeostrophic solutions reduce to the quasi-geostrophic ones in the limit as $\mu \to 0$. Note also that since $\psi = z + \mu\delta$, the coordinates (χ, ψ) reduce to (x, z) in that limit. Even for finite μ the two solutions possess a close correspondence. In particular, knowledge of the quasi-geostrophic solution is sufficient to determine the semigeostrophic one completely. For example, the solution for δ, v and θ ($= -\delta$) have the identical form.

Comparison of the two cases indicates that the mountain anticyclone has weaker pressure gradients in the semigeostrophic cases. The vorticity fields differ in that the anticyclonic region is narrower and the far-field cyclonic flow is stronger for the semigeostrophic case. Despite the addition of vortex tilting and relative vorticity stretching to the semigeostrophic vorticity dynamics, the anticyclonic vorticity at the mountain top (where $\partial\delta/\partial\chi$ vanishes) is the same for the two cases.

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