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Abstract

A transeunt triangle of size n consists of $(n+1)\times(n+1)\times(n+1)$ 0's and 1's whose values are determined by the sum modulo 2 of two other local values. For a given n, two transeunt triangles of size n can be combined using the element-by-element modulo 2 sum to generate a third transeunt triangle. We show that, for large n, the $\frac{1}{3}2^{n+1}$ transeunt triangles of size n can be generated from a set of only $\frac{n}{3}$ generator transeunt triangles.

Index Terms. Symmetric functions, Reed-Muller expansion, transeunt triangle.

1 Introduction

A transeunt triangle of size n is completely specified by a binary $n+1$-tuple that forms the first row. The second row, a binary n-tuple, is formed by the pairwise modulo 2 sum of adjacent elements in the first row. Similarly, elements of all other rows are specified as the modulo 2 sum of adjacent elements in the row above it. The last row is a single 0 or 1. Fig. 1 shows all transeunt triangles of size 3. Here, a transeunt triangle is not repeated if it is the same as the rotation of another triangle in the figure.

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0 0 0 0 | **1** 0 0 **1**
0 0 0 | **1** 0 **1**
0 0 | **1** **1**
0 | 0

0 **1** **1** 0 | **1** **1** **1** **1**
1 0 **1** | 0 0 0
1 **1** | 0 0
0 | 0

1 **1** 0 **1** | 0 **1** 0 0
0 **1** **1** | **1** **1** 0
1 0 | 0 **1**
1 | **1**

1 0 **1** **1** | 0 0 **1** 0
1 **1** 0 | 0 **1** **1**
0 **1** | **1** 0
1 | **1**

Figure 1: All transeunt triangles of size 3.

Note that a transeunt triangle can be formed from *any* of its three sides. That is, if instead one begins with a side produced by the process described above, a rotated version of the same transeunt triangle results.

The transeunt triangle is the basis of an efficient algorithm for determining the minimal fixed polarity Reed-Muller canonical expansion of a totally symmetric switching function [1], [3], [4]. An n-variable totally symmetric function is specified by a binary $n + 1$-tuple that becomes one of the triangle's sides. Certain rectangles within the transeunt triangle each represent a circuit implementation of the symmetric function described by the edge. Transeunt triangles are related to the Sierpiński's Gasket [2]. If the first row has exactly one 1, a Sierpiński's Gasket forms below that 1. This embedded triangle is Pascal's triangle modulo 2. Transeunt triangles are also related to a specific one-dimensional cellular automata system. In such systems, the next state (0 or 1) of every cell is the modulo 2 sum of its present state and the present state of its right neighbor. This particular system is Rule 102 in Wolfram's classification scheme [5], which belongs to the class of *additive* cellular automata systems. The transeunt triangle shows the sequence of states in such a cellular automata system, where the first row is the initial state.

We are interested in determining the minimum number of generators of all transeunt triangles of size n. That is, given two transeunt triangles, a third can be generated by forming the bit-by-bit modulo 2 sum. For example, forming the bit-by-bit modulo 2 sum of any triangle with itself generates **0**, the all 0 transeunt triangle. All eight transeunt triangles

1 **1** **1** **1**   0 **1** 0 0
0 0 0   **1** **1** 0
0 0   0 **1**
0   **1**

Figure 2: Generator transeunt triangles for all transeunt triangles of size 3.

of size 3 in Fig. 1 can be formed as some combination, including rotated versions, of the two transeunt triangles shown in Fig. 2. Since one transeunt triangle is not sufficient to produce all others, the minimum number of generators needed is 2. In the general case, all transeunt triangles can be generated from transeunt triangles with a single 1 on an edge. Therefore, it is sufficient to have $n+1$ generator transeunt triangles (or even less considering transeunt triangles where two different binary $n+1$-tuples with exactly one 1 occur as sides). However, one can do better.

Definition 1.1. Let $g_{\text{approx}}(n)$ and $g_{\text{exact}}(n)$ be two functions of n. $g_{\text{exact}}(n) \sim g_{\text{approx}}(n)$ if and only if

$$\lim_{n\to\infty} \frac{g_{\text{approx}}(n)}{g_{\text{exact}}(n)} = 1. \tag{1.1}$$

In what follows, $g_{\text{approx}}(n)$ is a *simple*, approximate expression for $g_{\text{exact}}(n)$.

Theorem 1.1. *The number $N_{\text{tt}}(n)$ of unoriented transeunt triangles of size n is*

$$N_{\text{tt}}(n) \sim \frac{1}{3} 2^{n+1}, \tag{1.2}$$

and the minimum number $\tau_{\text{tt}}(n)$ of (generator) transeunt triangles needed to generate these triangles is

$$\tau_{\text{tt}}(n) \sim \frac{n}{3}\ . \tag{1.3}$$

2 Number of transeunt triangles and self-similar transeunt triangles

In this section, we prove the first part of Theorem 1.1, namely (1.2). Note that four of the transeunt triangles in Fig. 1 (those in the left column) are unchanged by a rotation of 120° and 240°.

Definition 2.1. A transeunt triangle T is *self-similar* if and only if a rotation of 120° and 240° leaves T unchanged.

Theorem 2.1. *The number $N_{\text{ss}}(n)$ of self-similar transeunt triangles of size n is*

$$N_{\text{ss}}(n) = 2^{\left\lfloor \frac{n-1}{3} \right\rfloor + (n-1) \bmod 3}\ . \tag{2.1}$$

Proof For $n = 1$, 2, and 3, the number of self-similar transeunt triangles is 1 ([00]), 2 ([000] & [010]), and 4 ([0000], [0110], [1101], & [1011]), respectively. For $n = 1$, 2, and 3, (2.1) yields 1, 2, and 4, respectively. Consider a self-similar transeunt triangle of size $n > 3$. It embeds another triangle T' of size $(n-2) \times (n-2) \times (n-2)$, where $n - 2 > 1$. Since T is invariant under rotation, so also is T'.

Given any $(n-2) \times (n-2) \times (n-2)$ self-similar triangle T', there are two ways to form an $(n+1) \times (n+1) \times (n+1)$ self-similar triangle T (i.e. two circumferences that are complements). Thus,

$$N_{\text{ss}}(n) = 2N_{\text{ss}}(n-3). \tag{2.2}$$

But, this implies (2.1), since substituting the inductive hypothesis,

$$N_{\text{ss}}(n-3) = 2^{\left\lfloor \frac{n-4}{3} \right\rfloor + (n-4) \bmod 3}, \tag{2.3}$$

into (2.2), yields (2.1).

∎

Theorem 2.2. *The number of unoriented transeunt triangles of size n is*

$$N_{\text{tt}}(n) = \frac{2^{n+1} + 2^{\left\lfloor \frac{n+2}{3} \right\rfloor + (n-1) \bmod 3}}{3}. \tag{2.4}$$

Proof Among the three sides in a transeunt triangle of size n are one, two, or three distinct binary $n+1$-tuples. As shown in Fig. 1, one and three are possible. For example, a transeunt triangle with one tuple is self-similar. Transeunt triangles with exactly two distinct tuples are impossible because of symmetry. It follows that the number of unoriented transeunt triangles is

$$N_{\text{tt}}(n) = \frac{2^{n+1} - N_{\text{ss}}(n)}{3} + N_{\text{ss}}(n) = \frac{2^{n+1} + 2N_{\text{ss}}(n)}{3}. \tag{2.5}$$

Substituting (2.3) into (2.5) yields (2.4)

∎

Note that $N_{tt}(n) \sim \frac{2^{n+1}}{3}$. As the number n of variables increases without bound, the number of unoriented transeunt triangles approaches one-third the number of oriented transeunt triangles. This proves the first part of Theorem 1.1. This shows that most large transeunt triangles are *not* self-similar.

3 Generators of transeunt triangles

In this section, we prove the second part of Theorem 1.1, namely (1.3). Given a transeunt triangle T, T_{120} and T_{240} are T rotated by 120° and 240°, respectively. Let $\oplus$ be the bit-by-bit modulo 2 sum of two transeunt triangles.

Lemma 3.1. *Given any transeunt triangle T, $T \oplus T_{120} \oplus T_{240}$ is a self-similar transeunt triangle.*

Definition 3.1. Transeunt triangle T is a T_{ss}*-self-similar-group transeunt triangle*, where $T_{ss} = T \oplus T_{120} \oplus T_{240}$.

Since $T_{ss} \oplus T_{ss120} \oplus T_{ss240} = T_{ss}$, T_{ss} is a T_{ss}self-similar-group transeunt triangle. Further, there is exactly one self-similar T_{ss}-self-similar-group transeunt triangle, namely T_{ss}.

Definition 3.2. T is a *basic transeunt triangle* if it is a T_{ss}-self-similar-group transeunt triangle, where $T_{ss} = \mathbf{0}$, the transeunt triangle with all 0 entries.

Given transeunt triangle T, $T' = T \oplus T_{ss}$ is a basic transeunt triangle, since $T' \oplus T'_{120} \oplus T'_{240} = (T \oplus T_{ss}) \oplus (T_{120} \oplus T_{120ss}) \oplus (T_{240} \oplus T_{240ss}) = T_{ss} \oplus T_{ss} = \mathbf{0}$. Further, $T' = T \oplus T \oplus T_{120} \oplus T_{240} = T_{120} \oplus T_{240} = T \oplus T_{120}$.

Definition 3.3. Transeunt triangle T is a T_b-basic-group transeunt triangle, where $T_b = T \oplus T_{120}$.

Theorem 3.2. *The number of unoriented basic transeunt triangles of size n is*

$$N_{\mathrm{b}}(n) = \frac{4^{\lceil \frac{n}{3} \rceil} + 2}{3}. \tag{3.1}$$

Proof For every unoriented basic transeunt triangle T', there is a unique triangle in self-similar-group T_{ss}, that is obtained as $T' \oplus T_{ss}$. Similarly, for every unoriented transeunt triangle T'' in self-similar-group T''_{ss}, there is a unique unoriented basic transeunt triangle, that is obtained as $T'' \oplus T''_{ss}$. Thus, each self-similar-group represents an equal sized block in a partition of all unoriented transeunt triangles. Therefore, the number of unoriented basic transeunt triangles is

$$N_{\mathrm{b}}(n) = \frac{N_{tt}(n)}{N_{ss}(n)}. \tag{3.2}$$

Substituting (2.4) and (2.1) into (3.2) yields (3.1). ∎

From Theorem 3.2, $N_b(n) \sim \frac{1}{3}4^{\lceil \frac{n}{3} \rceil}$. From Definitions 3.1 and 3.3, it follows that each transeunt triangle T is characterized by the self-similar transeunt triangle T_{ss} and by the basic transeunt triangle T_b, and that

$$T = T_{ss} \oplus T_b \ . \tag{3.3}$$

Therefore, if we can generate all self-similar and all basic transeunt triangles, we can generate all transeunt triangles. Both the set of basic transeunt triangles and the set of self-similar transeunt triangles are closed under the $\oplus$ operation. Thus, $T'' = T \oplus T'$ is a basic transeunt triangle if T and T' are basic, and T'' is self-similar if T and T' are self-similar. Thus, each set can be generated from a subset. Indeed,

Theorem 3.3. *The set of self-similar transeunt triangles of size $n > 1$ can be generated from a set of $\alpha(n)$ nonredundant self-similar generator transeunt triangles, where*

$$\alpha(n) = \left\lfloor \frac{n-1}{3} \right\rfloor + (n-1) \bmod 3. \tag{3.4}$$

Proof There exists a set $S_{ss} = \{T_1, T_2, \ldots T_m\}$ of self-similar transeunt triangles from which all self-similar transeunt triangles can be generated as the modulo 2 sum of elements from S_{ss} (e.g. S_{ss} can be all self-similar transeunt triangles). If S_{ss} is nonredundant, then no $T_i \in S$ is the modulo 2 sum of any other transeunt triangles in S. Since $T_i \oplus T_i = \mathbf{0}$, no element in S is chosen more than once. It follows that $m \geq \lfloor \frac{n-1}{3} \rfloor + (n-1) \bmod 3$; otherwise, there are not enough combinations (2^m) of transeunt triangles in S to form all $2^{\lfloor \frac{n-1}{3} \rfloor + (n-1) \bmod 3}$ self-similar transeunt triangles (for convenience, we substitute $\mathbf{0}$ for the combination consisting of choosing *no* element of S). We show that equality holds by showing that the modulo 2 sum of a combination of elements from S is distinct from any other combination. On the contrary, suppose

$$T_{i_1} \oplus T_{i_2} \oplus \ldots \oplus T_{i_\alpha} \oplus T_{j_1} \oplus T_{j_2} \oplus \ldots \oplus T_{j_\beta} = T_{k_1} \oplus T_{k_2} \oplus \ldots \oplus T_{k_\gamma} \oplus T_{j_1} \oplus T_{j_2} \oplus \ldots \oplus T_{j_\beta}, \tag{3.5}$$

where the only T's common to both sides of (3.5) are T_{j_1}, T_{j_2}, ... , and T_{j_β}. It follows that

$$T_{i_1} \oplus T_{i_2} \oplus \ldots \oplus T_{i_\alpha} = T_{k_1} \oplus T_{k_2} \oplus \ldots \oplus T_{k_\gamma}, \tag{3.6}$$

and that

$$T_{i_1} = T_{k_1} \oplus T_{k_2} \oplus \ldots \oplus T_{k_\gamma} \oplus T_{i_2} \oplus \ldots \oplus T_{i_\alpha}. \tag{3.7}$$

But, this is impossible, as no element of S is the modulo 2 sum of other elements of S. The theorem statement follows.

∎

Theorem 3.4. *The set of unoriented basic transeunt triangles of size n can be generated from a set of $\beta(n)$ nonredundant unoriented basic generator transeunt triangles, where*

$$\beta(n) = \left\lceil \frac{n}{3} \right\rceil. \tag{3.8}$$

Proof There exists a set $S_b = \{T_1, T_2, \ldots T_p\}$ of basic generator transeunt triangles from which all basic transeunt triangles can be generated as the modulo 2 sum of elements from S_b. The all 0 triangle can be generated as the element-by-element modulo 2 sum of any non-zero generator triangle with itself. The remaining $(4^{\lceil \frac{n}{3} \rceil} - 1)/3$ non-zero basic transeunt triangles can be generated as follows. Each generator transeunt triangle T can be used in four ways 1. omitted, 2. as T, 3. as T_{120}, and 4. as T_{240}. For each combination of at least one generator, rotating *all* generator triangles by 120° or by 240° creates a resulting triangle that is rotated by 120° or 240°, respectively; however this produces the *same* unoriented triangle. This accounts for the divisor of 3 in $(4^{\lceil \frac{n}{3} \rceil} - 1)/3$. The one exception, where all generator triangles are omitted, accounts for the -1 in $(4^{\lceil \frac{n}{3} \rceil} - 1)/3$. The argument regarding uniqueness of combinations is similar to that of Theorem 3.3.

∎

For S_b and S_{ss}, the set of basic and self-similar generator transeunt triangles, respectively, we can form a set $S_{b/ss} = \{T_{b/ss} = T_b \oplus T_{ss} | T_b \in S_b,\ \ T_{ss} \in S_{ss}\}$. We can then form every basic generator transeunt triangle and every self-similar transeunt triangle, as $T_{b/ss120} \oplus T_{b/ss240}$ and $T_{b/ss} \oplus T_{b/ss120} \oplus T_{b/ss240}$, respectively. From the generator triangles, we can form all basic and all self-similar transeunt triangles. This proves the following.

Theorem 3.5. *All basic and all self-similar transeunt triangles can be generated from $\gamma(n)$ generator transeunt triangles, where*

$$\gamma(n) = \max\left\{ \left\lceil \frac{\mathrm{n}}{3} \right\rceil, \left\lfloor \frac{\mathrm{n}-1}{3} \right\rfloor + (\mathrm{n}-1) \bmod 3 \right\}. \tag{3.9}$$

From all basic and all self-similar transeunt triangles, we can generate all transeunt triangles. Note that $\gamma(n)$ is also the *minimum* number of transeunt triangles from which all transeunt triangles can be generated. On the contrary, with fewer than $\gamma(n)$ generators, it is impossible to generate either all basic and/or all self-similar transeunt triangles, as observed in the proofs of Theorems 3.3 and 3.4. Observing that $\gamma(n)$ approaches $\frac{n}{3}$ as n increases without bound proves the second part of Theorem 1.1. Table 1 shows the number of transeunt triangle, the number of self-similar transeunt triangles, and the number of basic transeunt triangles, for $1 \leq n \leq 20$.

4 Concluding remarks

In this paper, we have shown that the minimum number of generators of transeunt triangles is small, approaching $\frac{n}{3}$ as the size n increases without bound. An appreciation for the reduction achievable can be obtained by an examination of Table 1, which shows, for example, that the number of transeunt triangles of size $n = 20$ is 699,136, while the minimum number of generators for this set is only 7.

References

[1] J. T. Butler, G. W. Dueck, V. P. Shmerko, and S. N. Yanushkevich. An efficient algorithm for the synthesis of fixed polarity Reed-Muller expansions of symmetric functions. *preprint*, 1999.

	Number of Transeunt Triangles			Number of Generators	
n	Total $N_{\mathrm{tt}}(n)$	Self-Similar $N_{\mathrm{ss}}(n)$	Unoriented Basic $N_{\mathrm{b}}(n)$	Self-Similar $\alpha(n)$	Unoriented Basic $\beta(n)$
1	2	1	2	0	1
2	4	2	2	1	1
3	8	4	2	2	1
4	12	2	6	1	2
5	24	4	6	2	2
6	48	8	6	3	2
7	88	4	22	2	3
8	176	8	22	3	3
9	352	16	22	4	3
10	688	8	86	3	4
11	1,376	16	86	4	4
12	2,752	32	86	5	4
13	5,472	16	342	4	5
14	10,944	32	342	5	5
15	21,888	64	342	6	5
16	43,712	32	1,366	5	6
17	87,424	64	1,366	6	6
18	174,848	128	1,366	7	6
19	349,568	64	5,462	6	7
20	699,136	128	5,462	7	7
∞	$\frac{1}{3}2^{n+1}$	$2^{\lfloor\frac{n-1}{3}\rfloor}$	$\frac{1}{3}4^{\lceil\frac{n}{3}\rceil}$	$\frac{n}{3}$	$\frac{n}{3}$

Table 1: Number of n-variable transeunt triangles, self-similar, basic unoriented, generator self-similar, and generator basic transeunt triangles, for $1 \leq n \leq 20$.

[2] I. Stewart. Four encounters with Sierpiński's Gasket. *J. Mathematical Intellingencer*, 17(1):52–64, 1999.

[3] V. P. Suprun. Polynomial expression of symmetric Boolean functions. *Izvestija AN USSR. Techn. Kibernetika (in Russian)*, (4):123–127, 1985.

[4] V. P. Suprun. Fixed polarity Reed-Muller expressions of symmetric Boolean functions. In *Proc. IFIP WG 10.5 Workshop on Application of the Reed-Muller Expansions in Circuit Design*, pages 246–249, 1995.

[5] S. Wolfram. Theory and applications of cellular automata. *World Scientific Publishing Co. Ltd.*, pages 485–557, 1986.