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2002

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http://hdl.handle.net/10945/35420

Edge Theorem for Multivariable Systems [1]

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Abstract: This paper studies robustness of multivariable systems with parametric uncertainties, and establishes a multivariable version of Edge Theorem. An illustrative example is presented.

Keywords: Complex Systems, Robustness, Uncertain Parameters, Edge Theorem, Polynomial Matrices, Interval Polynomials.

1   Introduction

Motivated by the seminal theorem of Kharitonov on robust stability of interval polynomials[1, 2], a number of papers on robustness analysis of uncertain systems have been published in the past few years[3, 4, 5, 6, 7, 8, 9, 10]. Kharitonov's theorem states that the Hurwitz stability of the real (or complex) interval polynomial family can be guaranteed by the Hurwitz stability of four (or eight) prescribed critical vertex polynomials in this family. This result is significant since it reduces checking stability of infinitely many polynomials to checking stability of finitely many polynomials, and the number of critical vertex polynomials need to be checked is independent of the order of the polynomial family. An important extension of Kharitonov's theorem is the edge theorem discovered by Bartlett, Hollot and Huang[4]. The edge theorem states that the stability of a polytope of polynomials can be guaranteed by the stability of its one-dimensional exposed edge polynomials. The significance of the edge theorem is that it allows some (affine) dependency among polynomial coefficients, and applies to more general stability regions, e.g., unit circle, left sector, shifted half plane, hyperbola region, etc. When the dependency among polynomial coefficients is nonlinear, however, Ackermann shows that checking a subset of a polynomial family generally can not guarantee the stability of the entire family[11, 12].

For Hurwitz stability of interval matrices, Bialas 'proved' that in order to guarantee robust stability, it suffices to check all vertex matrices[13]. Later, it was shown by Barmish that Bialas' result was incorrect[14]. Kokame and Mori eastblished a Kharitonov-like result on robust Hurwitz stability of interval polynomial matrices[15], and Kamal and Dahleh established some robust stability criteria for MIMO systems with fixed controllers and uncertain plants[16].

In this paper, we will study robustness of a class of MIMO systems with their transfer function matrices described by

$$
\mathcal{F}(s) = \left\{ \begin{pmatrix} a_{11}(s) & \dots & a_{1n}(s) \\ \dots & \dots & \dots \\ a_{n1}(s) & \dots & a_{nn}(s) \end{pmatrix} : \quad a_{ij}(s) \in \mathcal{A}_{ij}(s) \right\}
$$
$$
\mathcal{A}_{ij}(s) = \mathrm{conv}\left\{ b1_{ij}(s), \dots, bm_{ij}(s) \right\}
$$
(1)

where m is a given positive integer.

2   Preliminaries

Definition 1   A polynomial matrix is a matrix with all its entries being polynomials.

Definition 2   Suppose D is a simply-connected region in the complex plane. If all the roots of the determinant of a polynomial matrix lie within D, then this polynomial matrix is called D-stable. A set of polynomial matrices is called robustly D-stable, if every polynomial matrix in this set is D-stable.

Definition 3 Suppose $f_1(s), \ldots, f_m(s)$ are m given polynomials, the set

$$\left\{ \sum_{i=1}^{m} \lambda_i f_i(s) : \quad \lambda_i \geq 0, \quad \sum_{i=1}^{m} \lambda_i = 1 \right\}$$

is called the polynomial polytope generated by $f_1(s), \ldots, f_m(s)$, denoted as conv $\{f_1(s), \ldots, f_m(s)\}$.

Definition 4 The polynomial $f(s) = a_0 + a_1 s + \ldots + a_n s^n$, with $a_i \in [a_i^L, a_i^U]$ is called an interval polynomial.

Definition 5 S_n is the set of all bijections from $\{1, \ldots, n\}$ to $\{1, \ldots, n\}$.

Definition 6 The vertex set and edge set of $\mathcal{A}_{ij}(s)$ are

$$K_{ij}(s) = \{b1_{ij}(s), \ldots, bm_{ij}(s)\}$$

$$E_{ij}(s) = \{\lambda br_{ij}(s) + (1 - \lambda)bt_{ij}(s), \quad \lambda \in [0, 1], \quad r, t \in \{1, \ldots, m\}\}$$

respectively.

Definition 7

$$\mathcal{F}_E(s) = \bigcup_{\sigma \in S_n} \left\{ (a_{ij}(s))_{n \times n} : a_{ij}(s) \left\{ \begin{array}{l} \in E_{ij}(s) \text{ if } i = \sigma(j) \\ \in K_{ij}(s) \text{ if } i \neq \sigma(j) \end{array} \right\} \right. \tag{2}$$

Lemma 1 (Edge Theorem[4]) Suppose $\Gamma \subset \mathcal{C}$ is a simply-connected region, Ω is a polynomial polytope without degree dropping. Then, Ω is Γ-stable if and only if all the edges of Ω are Γ-stable.

Lemma 2 Suppose $A(s)$ is a given $n \times (n-1)$ polynomial matrix. Then

$$\left\{ \left(\begin{array}{ccc} a_{11}(s) & & \\ \vdots & & A(s) \\ a_{n1}(s) & & \end{array} \right) : \quad \begin{array}{c} a_{i1}(s) \in \mathcal{A}_{i1}(s), \\ i = 1, \ldots, n \end{array} \right\} \text{ is robustly } D\text{-stable}$$

$$\Leftrightarrow \text{ for all } i = 1, \ldots, n, \left\{ \left(\begin{array}{ccc} a_{11}(s) & & \\ \vdots & & A(s) \\ a_{n1}(s) & & \end{array} \right) : \quad \begin{array}{cc} a_{l1}(s) \in K_{lj}(s) & l \neq i \\ a_{l1}(s) \in E_{lj}(s) & l = i \end{array} \right\} \text{ is robustly } D\text{-stable.}$$

Proof: Necessity is obvious, since the later is a subset of the former.

Sufficiency: For any $a_{i1}(s) \in \mathcal{A}_{i1}(s)$, the corresponding matrix is

$$T(s) = \left(\begin{array}{ccc} a_{11}(s) & & \\ \vdots & & A(s) \\ a_{n1}(s) & & \end{array} \right)$$

By Laplace Formula, we can expand the determinant of $T(s)$ along its first column. Then, by convexity and by Lemma 1, we know that $T(s)$ is robustly D-stable.

Lemma 3 Suppose $B(s)$ is a given $(n-1) \times n$ polynomial matrix. $*$ stands for fixed entries in a matrix. Then

$$\left\{ \left(\begin{array}{ccccc} * & a_{1i}(s) & * & a_{1j}(s) & * \\ & & B(s) & & \end{array} \right) : \quad \begin{array}{c} a_{1i}(s) \in \mathcal{A}_{1i}(s) \\ a_{1j}(s) \in \mathcal{A}_{1j}(s) \end{array} \right\} \text{ is robustly } D\text{-stable}$$

$$\Leftrightarrow \left\{ \left(\begin{array}{ccccc} * & a_{1i}(s) & * & a_{1j}(s) & * \\ & & B(s) & & \end{array} \right) : \quad \begin{array}{c} a_{1i}(s) \times a_{1j}(s) \in \\ (K_{1i}(s) \times E_{1j}(s)) \cup (E_{1i}(s) \times K_{1j}(s)) \end{array} \right\} \text{ is robustly } D\text{-stable.}$$

Proof: the proof is analogous to the proof of Lemma 2, except that the Laplace expansion is carried out along the row instead of the column.

3 Main Results

Theorem 1 $\mathcal{F}(s)$ is robustly D-stable if and only if $\mathcal{F}_E(s)$ is robustly D-stable.

Proof: Necessity is obvious. To prove sufficiency, we first note that interchanging any two rows (or columns) does not affect the stability of a polynomial matrix (it only changes the sign of the determinant). By Lemma 2

$$\mathcal{F}(s) \text{ is robustly } D \text{ stable}$$

$$\Leftrightarrow \text{ for all } i = 1,\ldots,n, \left\{ \begin{pmatrix} a_{11}(s) \\ \vdots & A(s) \\ a_{n1}(s) \end{pmatrix} : \begin{array}{ll} a_{l1}(s) \in K_{lj}(s) & l \neq i \\ a_{l1}(s) \in E_{lj}(s) & l = i \end{array} \right\} \text{ is robustly } D \text{ stable.}$$

$$\Leftrightarrow \text{ for all } \left\{ \begin{array}{l} i_1 = 1,\ldots,n \\ i_2 = 1,\ldots,n \end{array} \right. \left\{ \begin{pmatrix} a_{11}(s) & a_{12}(s) \\ \vdots & \vdots & A_1(s) \\ a_{n1}(s) & a_{n2}(s) \end{pmatrix} : \begin{array}{ll} a_{l1}(s) \in K_{lj}(s) & l \neq i_1 \\ a_{l1}(s) \in E_{lj}(s) & l = i_1 \\ a_{l2}(s) \in K_{lj}(s) & l \neq i_2 \\ a_{l2}(s) \in E_{lj}(s) & l = i_2 \end{array} \right\}$$

is robustly D stable.

where $A_1(s)$ is the corresponding $n \times (n-2)$ polynomial matrix. This last equivalence is based on Lemma 2 and the fact that interchanging two columns does not change the stability of a polynomial matrix. Repeating the process above, let Y_n denote the set of all mappings from $\{1,\ldots,n\}$ to $\{1,\ldots,n\}$, then

$$\mathcal{F}(s) \text{ is robustly } D \text{ stable}$$

$$\Leftrightarrow \text{ for all } \eta \in Y_n, \left\{ (a_{ij}(s)) : \begin{array}{ll} a_{ij}(s) \in K_{ij}(s) & i \neq \eta(j) \\ a_{ij}(s) \in E_{ij}(s) & i = \eta(j) \end{array} \right\} \text{ is robustly } D \text{ stable.}$$

If there exists an $\eta \in Y_n$ such that $\eta(i_1) = \eta(i_2) = k$, then the corresponding matrix $F(s) = (a_{ij}(s))$ satisfies

$$a_{i_1 k}(s) \in E_{i_1 k}(s)$$
$$a_{i_2 k}(s) \in E_{i_2 k}(s)$$

Applying Lemma 2 to column k of $F(s)$, we have

$$\mathcal{F}(s) \text{ is robustly } D \text{ stable}$$

$$\Leftrightarrow \text{ for all } \sigma \in S_n, \left\{ (a_{ij}(s)) : \begin{array}{ll} a_{ij}(s) \in K_{ij}(s) & i \neq \sigma(j) \\ a_{ij}(s) \in E_{ij}(s) & i = \sigma(j) \end{array} \right\} \text{ is robustly } D \text{ stable.}$$

$$\Leftrightarrow \mathcal{F}_E(s) \text{ is robustly } D \text{ stable.}$$

4 Interval Model

Interval model, as a simple and effective approximation of uncertain systems, has been the subject of study in robustness analysis for a long time. In a similar vein, we consider the Hurwitz stability of the following uncertain system.

$$\mathcal{G}(s) = \{(c_{ij}(s)) : c_{ij}(s) \in \mathcal{C}_{ij}(s)\}$$
$$\mathcal{C}_{ij}(s) \text{ are interval polynomials} \tag{3}$$

Definition 8 For the interval polynomial $\mathcal{C}_{ij}(s) = \{\sum_{l=0}^n q_l(ij)s^l, \quad q_l(ij) \in [\underline{q}_l(ij), \overline{q}_l(ij)]\}$, its Kharitonov vertex set and Kharitonov edge set are defined respectively as

$$K_{ij}^I(s) = \{c_k^1(s), c_k^2(s), c_k^3(s), c_k^4(s)\}$$
$$E_{ij}^I(s) = \{\lambda c_k^r(s) + (1-\lambda)c_k^t(s), \quad (r,t) \in \{(1,2),(2,4),(4,3),(3,1)\}, \quad \lambda \in [0,1]\}$$

where

$$c_k^1(s) = \underline{q}_0(ij) + \underline{q}_1(ij)s + \overline{q}_2(ij)s^2 + \overline{q}_3(ij)s^3 + \ldots \qquad c_k^2(s) = \underline{q}_0(ij) + \overline{q}_1(ij)s + \overline{q}_2(ij)s^2 + \underline{q}_3(ij)s^3 + \ldots$$
$$c_k^3(s) = \overline{q}_0(ij) + \underline{q}_1(ij)s + \underline{q}_2(ij)s^2 + \overline{q}_3 s^3 + \ldots \qquad c_k^4(s) = \overline{q}_0(ij) + \overline{q}_1(ij)s + \underline{q}_2(ij)vs^2 + \underline{q}_3(ij)s^3 + \ldots$$

Definition 9

$$\mathcal{G}_E(s) = \bigcup_{\sigma \in S_n} \left\{ (c_{ij}(s))_{n\times n} : c_{ij}(s) \begin{array}{l} \in E_{ij}^I(s) \text{ if } i = \sigma(j) \\ \in K_{ij}^I(s) \text{ if } i \neq \sigma(j) \end{array} \right\} \tag{4}$$

Lemma 4 (Box Theorem[16]) Suppose $\Delta(s) = \{\delta(s,p) = F_1(s)P_1(s) + \ldots + F_m(s)P_m(s)\}$, $P_i(s)$ is an interval polynomial, $F_i(s)$ is a given fixed polynomial, $i = 1,\ldots,m$. And suppose $\Delta(s)$ is

degree-invariant. Then, $\Delta(s)$ is Hurwitz stable if and only if $\Delta_E(s)$ is Hurwitz stable, where $\Delta_E(s) = \cup_{l=1}^{m}\{\sum_{i=1}^{l-1} F_i(s)K_{P_i}^0(s) + F_l(s)E_{P_l}^0(s) + \sum_{i=l+1}^{m} F_i(s)K_{P_i}^0(s)\}$ (let $\sum_{i=r}^{t} f_i = 0$, if $r > t$).

By resort to the Box Theorem, and following a similar line of arguments as in the proof of Theorem 1, we can get some analogous stability verification results for the interval model.

Lemma 5 Suppose $A(s)$ is a given $n \times (n-1)$ polynomial matrix. Then

$$\left\{ \begin{pmatrix} c_{11}(s) \\ \vdots & A(s) \\ c_{n1}(s) \end{pmatrix} : \begin{array}{l} c_{i1}(s) \in \mathcal{G}_{i1}(s), \\ i = 1, \ldots, n \end{array} \right\} \text{ is robustly Hurwitz stable.}$$

$$\Leftrightarrow \text{ for all } i = 1, \ldots, n, \left\{ \begin{pmatrix} c_{11}(s) \\ \vdots & A(s) \\ c_{n1}(s) \end{pmatrix} : \begin{array}{ll} c_{l1}(s) \in K_{lj}^I(s) & l \neq i \\ c_{l1}(s) \in E_{lj}^I(s) & l = i \end{array} \right\} \text{ is robustly Hurwitz stable.}$$

Lemma 6 Suppose $B(s)$ is a given $(n-1) \times n$ polynomial matrix. $*$ stands for fixed entries in a matrix. Then

$$\left\{ \begin{pmatrix} * & c_{1i}(s) & * & c_{1j}(s) & * \\ & & B(s) \end{pmatrix} : \begin{array}{l} c_{1i}(s) \in \mathcal{C}_{1i}(s) \\ c_{1j}(s) \in \mathcal{C}_{1j}(s) \end{array} \right\} \text{ is robustly Hurwitz stable}$$

$$\Leftrightarrow \left\{ \begin{pmatrix} * & c_{1i}(s) & * & c_{1j}(s) & * \\ & & B(s) \end{pmatrix} : \begin{array}{c} c_{1i}(s) \times c_{1j}(s) \in \\ \left(K_{1i}^I(s) \times E_{1j}^I(s)\right) \cup \left(E_{1i}^I(s) \times K_{1j}^I(s)\right) \end{array} \right\} \text{ is robustly Hurwitz stable.}$$

Theorem 2 $\mathcal{G}(s)$ is robustly Hurwitz stable if and only if $\mathcal{G}_E(s)$ is robustly Hurwitz stable.

Remark: Theorem 2 is consistent with the result in [15]. In [15], the authors obtained their result using some theorem in signal processing. Our proof is based on the properties of matrix determinant, hence is more straightforward and self-contained.

5 Example

Consider the uncertain polynomial matrix

$$\mathcal{A}(s) = \{(a_{ij}(s))_{3 \times 3}\}$$
$$a_{11}(s) = \lambda_{11} b1_{11}(s) + (1 - \lambda_{11}) b2_{11}(s)$$
$$a_{12}(s) = \lambda_{12} b1_{12}(s) + (1 - \lambda_{12}) b2_{12}(s)$$
$$a_{13}(s) = \lambda_{13} b1_{13}(s) + (1 - \lambda_{13}) b2_{13}(s)$$
$$a_{21}(s) = \lambda_{21} b1_{21}(s) + (1 - \lambda_{21}) b2_{21}(s)$$
$$a_{22}(s) = \lambda_{22} b1_{22}(s) + (1 - \lambda_{22}) b2_{22}(s)$$
$$a_{23}(s) = \lambda_{23} b1_{23}(s) + (1 - \lambda_{23}) b2_{23}(s)$$
$$a_{31}(s) = \lambda_{31} b1_{31}(s) + (1 - \lambda_{31}) b2_{31}(s)$$
$$a_{32}(s) = \lambda_{32} b1_{32}(s) + (1 - \lambda_{32}) b2_{32}(s)$$
$$a_{33}(s) = \lambda_{33} b1_{33}(s) + (1 - \lambda_{33}) b2_{33}(s)$$

Then,

$$E_{ij}(s) = \{a_{ij}(s)\}$$
$$K_{ij}(s) = \{b1_{ij}(s), b2_{ij}(s)\}$$
$$S_3 = \{\sigma_1, \ldots, \sigma_6\}$$
$$\sigma_1 : 1 \to 1; 2 \to 2; 3 \to 3$$
$$\sigma_2 : 1 \to 2; 2 \to 3; 3 \to 1$$
$$\sigma_3 : 1 \to 3; 2 \to 1; 3 \to 2$$
$$\sigma_4 : 1 \to 1; 2 \to 3; 3 \to 2$$
$$\sigma_5 : 1 \to 2; 2 \to 1; 3 \to 3$$
$$\sigma_6 : 1 \to 3; 2 \to 2; 3 \to 1$$

Let

$$\mathcal{A}_E(s) = \bigcup_{\sigma \in S_3} \left\{ (a_{ij}(s))_{3 \times 3} : a_{ij}(s) \begin{array}{l} \in E_{ij}(s) \text{ if } i = \sigma(j) \\ \in K_{ij}(s) \text{ if } i \neq \sigma(j) \end{array} \right\}$$

By Theorem 1, $\mathcal{A}(s)$ is robustly D stable if and only if $\mathcal{A}_E(s)$ is robustly D stable.

6 Conclusions

This paper discussed the robust D-stability problems for MIMO uncertain systems. The Edge Theorem and Kharitonov Theorem have been generalized to multivariable case.

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