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Bi-orthogonality Relationships Involving Porous Media

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Abstract

Bi-orthogonality relationships are established for a vertically heterogeneous porous media in contact with a fluid, a solid, a second porous medium, and a free surface. Fraser's bi-orthogonality relation for propagation of Rayleigh-Lamb modes in a plate with traction free surfaces is shown to be a special case of the bi-orthogonality relations derived herein in which the medium can be thought of as a porous slab with zero porosity.

Introduction

Straightforward solutions to complicated problems can sometimes be found by employing bi-orthogonality relationships. These relationships can be viewed as generalizations of the orthogonality relations used in classical eigenfunction expansion techniques. Problems solved in this manner are useful for verifying numerical codes and for determining baseline characteristics of idealized problems. Bi-orthogonality methods have appeared in the literature in static plate bending, elastodynamics, and Stoke's flow problems.

Fraser[1] is often credited with establishing the bi-orthogonality relationship for elastic media, but in his work he credits Fama [2], who in turn credits Papkovich [3] for the establishment and use of bi-orthogonal "modes". Gregory [4] made the important observation that many bi-orthogonality relationships are derivable from a combination of more basic reciprocity identities and symmetries inherent in the mechanical system under consideration. The reciprocity identities are often credited to Betti [5], and are referred to as Betti's theorem (see, for example, Love's [6] treatise). However, according to Lamb's paper [7] entitled "On Reciprocal Theorems in Dynamics", credit should perhaps ultimately be given to Lagrange, who in his *Mécanique Analytique*, published in 1809, gave a very generalized form of the reciprocity relationship.

The purpose of this paper is to extend the applicability of bi-orthogonality relationships to porous, porous/elastic, and porous/fluid type layered media by applying ideas used by Gregory in [4]. In particular, the reciprocity identity for porous media given recently by Kargl and Lim [8], is used in combination with the symmetry of a layered medium to derive a bi-orthogonality relationship satisfied by waves propagating at the free surface or along the interface of two media, one of which is porous.

The special cases considered in this work are all two dimensional, and involve a porous half-space (a) with a free surface, or in contact with (b) a fluid half-space, (c) an elastic half-space, or (d) a dissimilar porous half-space. Extensions to finite media are considered in the final section of the paper.

Formulation

Biot [9] gave the following equations of motion for a porous medium comprised of a solid matrix with displacement field $\mathbf{u}$, and an interstitial liquid with displacement field $\mathbf{U}$:

$$\begin{aligned} N\nabla^2\mathbf{u} + \nabla[(D+N)\nabla\cdot\mathbf{u} + Q\nabla\cdot\mathbf{U}] &= \frac{\partial^2}{\partial t^2}[\rho_{11}\mathbf{u} + \rho_{12}\mathbf{U}] + \eta(\omega)\frac{\partial}{\partial t}(\mathbf{u}-\mathbf{U}) \\ \nabla[Q\nabla\cdot\mathbf{u} + R\nabla\cdot\mathbf{U}] &= \frac{\partial^2}{\partial t^2}[\rho_{12}\mathbf{u} + \rho_{22}\mathbf{U}] - \eta(\omega)\frac{\partial}{\partial t}(\mathbf{u}-\mathbf{U}) \end{aligned} \tag{1}$$

In (1), D, N, Q, and R are positive, constant, elastic moduli; ρ_{12}, ρ_{11}, and ρ_{22} are effective densities; and $\eta(\omega)$ is a frequency dependent dissipation function.

A similar set of equations for a vertically heterogeneous porous medium (i.e., one where the elastic moduli, effective densities, and porosity depend on the depth coordinate z) can be obtained by writing (1) in the form:

$$
\begin{aligned}
\nabla \cdot \boldsymbol{\sigma} &= \frac{\partial^2}{\partial t^2}[\rho_{11}\mathbf{u} + \rho_{12}\mathbf{U}] + \eta(\omega)\frac{\partial}{\partial t}(\mathbf{u} - \mathbf{U}) \\
\nabla \cdot \boldsymbol{S} &= \frac{\partial^2}{\partial t^2}[\rho_{12}\mathbf{u} + \rho_{22}\mathbf{U}] - \eta(\omega)\frac{\partial}{\partial t}(\mathbf{u} - \mathbf{U})
\end{aligned} \tag{2}
$$

where $\boldsymbol{\sigma}$ and $\boldsymbol{S}$ are the respective stress tensors in the solid and fluid components of the porous medium. If the elastic moduli and effective densities are made depth dependent, the stress tensors become:

$$
\begin{aligned}
\sigma_{ij} &= [D(z)u_{k,k} + Q(z)U_{k,k}]\delta_{ij} + N(z)(u_{i,j} + u_{j,i}) \\
S_{ij} &= s\delta_{ij} = -\beta(z)p\delta_{ij} = [Q(z)u_{k,k} + R(z)U_{k,k}]\delta_{ij}
\end{aligned} \tag{3}
$$

In (3), $\beta(z)$ is the porosity of the medium, and p is the pressure in the interstitial fluid.

Insertion of (3) into (2), yields the following equations of motion for a vertically heterogeneous porous medium:

$$
\begin{aligned}
N\nabla^2\mathbf{u} + (D+N)\nabla\nabla\cdot\mathbf{u} + Q\nabla\nabla\cdot\mathbf{U} &+ \hat{\boldsymbol{e}}_z[\frac{dD}{dz}\nabla\cdot\mathbf{u} + \frac{dQ}{dz}\nabla\cdot\mathbf{U}] \\
+\frac{dN}{dz}[2\frac{\partial\mathbf{u}}{\partial z} + \hat{\boldsymbol{e}}_z \times \nabla \times \mathbf{u}] &= \frac{\partial^2}{\partial t^2}[\rho_{11}\mathbf{u} + \rho_{12}\mathbf{U}] + \eta(\omega)\frac{\partial}{\partial t}(\mathbf{u} - \mathbf{U}) \\
Q\nabla\nabla\cdot\mathbf{u} + R\nabla\nabla\cdot\mathbf{U} + \hat{\boldsymbol{e}}_z[\frac{dQ}{dz}\nabla\cdot\mathbf{u} &+ \frac{dR}{dz}\nabla\cdot\mathbf{U}] = \\
&\quad \frac{\partial^2}{\partial t^2}[\rho_{12}\mathbf{u} + \rho_{22}\mathbf{U}] - \eta(\omega)\frac{\partial}{\partial t}(\mathbf{u} - \mathbf{U})
\end{aligned} \tag{4}
$$

Note that for $Q = 0$, $D = \lambda(z)$, and $N = \mu(z)$, the divergence of the stress tensor $\boldsymbol{\sigma}$ for the matrix (solid) in (4) is equivalent to that found in Ben-Menahem and Singh [10] for a vertically heterogeneous solid.

The generalization of the elastodynamic reciprocal identity for time harmonic waves [11] (the Betti-Rayleigh theorem) to displacements in saturated porous media is given by the identity [8]:

$$
\int_S \mathbf{n} \cdot [\boldsymbol{\sigma}^A \cdot \mathbf{u}^B - \boldsymbol{\sigma}^B \cdot \mathbf{u}^A + \boldsymbol{S}^A \cdot \mathbf{U}^B - \boldsymbol{S}^B \cdot \mathbf{U}^A] ds = 0 \quad , \tag{5}
$$

where the superscripts A, B refer to two distinct, unique solutions to (4). The terms $\boldsymbol{\sigma}^A$ and $\boldsymbol{S}^A$ are the stresses resulting from displacements $\mathbf{u}^A$ and $\mathbf{U}^A$ respectively in (3), and $\mathbf{n}$ is the outward normal to the surface S.

In the next section, several cases are discussed which involve propagation of a surface or interface wave along the boundary of, or interface between, a porous medium and some other medium. We shall assume that the propagating waves are two dimensional so that fluid and solid displacements: (1) have no y dependence, (2) propagate in the $\pm x$ direction,

(3) at a prescribed frequency ω, and (4) have the proper decay as $z \to \pm\infty$. The first case considered involves a porous half-space with a free surface at $z = 0$. The remaining cases include: a porous half-space loaded by a fluid half-space with interface at $z = 0$, a porous half-space overlying an elastic half-space at $z = 0$, and two disparate porous half-spaces with interface at $z = 0$.

Bi-orthogonality relations

Case I: A porous half-space with a free surface

Refer to fig. 1a which designates the surface over which the reciprocity identity (5) is applied. Assumed forms of the displacement vectors are

$$\mathbf{u}^A = \mathbf{u}^A(z)e^{-i\omega t+ik_A x} \qquad \mathbf{U}^A = \mathbf{U}^A(z)e^{-i\omega t+ik_A x} \tag{6}$$

$$\mathbf{u}^B = \mathbf{u}^B(z)e^{-i\omega t+ik_B x} \qquad \mathbf{U}^B = \mathbf{U}^B(z)e^{-i\omega t+ik_B x} \tag{7}$$

Since all displacements tend to zero as $z \to -\infty$, (5) implies:

$$\begin{aligned} e^{-2i\omega t}\{ & e^{i(k_A+k_B)x_1}\int_{-\infty}^{0}[-\sigma_{xx}^A u_x^B - \sigma_{xz}^A u_z^B + \sigma_{xx}^B u_x^A + \sigma_{xz}^B u_z^A - s^A U_x^B + s^B U_x^A]dz \\ & + e^{i(k_A+k_B)x_2}\int_{-\infty}^{0}[\sigma_{xx}^A u_x^B + \sigma_{xz}^A u_z^B - \sigma_{xx}^B u_x^A - \sigma_{xz}^B u_z^A + s^A U_x^B - s^B U_x^A]dz \\ & + \left.\frac{e^{i(k_A+k_B)x}}{i(k_A+k_B)}\right|_{x_1}^{x_2}[\sigma_{xz}^A u_x^B + \sigma_{zz}^A u_z^B - \sigma_{xz}^B u_x^A - \sigma_{zz}^B u_z^A + s^A U_z^B - s^B U_z^A]\Big|_{z=0}\} = 0 \end{aligned} \tag{8}$$

The free surface ($z = 0$) of the porous medium is stress free, making the boundary conditions there:

$$\sigma_{zz}^B(0) = \sigma_{zz}^A(0) = \sigma_{xz}^B(0) = \sigma_{xz}^A(0) = s^A(0) = s^B(0) = 0 \tag{9}$$

Hence, the third term in (8) is zero, and because $x_1 \neq x_2$ in general, it must be true that each of the remaining integrals in (8) is identically zero. Therefore, every line integral along constant x of the following form is zero:

$$\int_{-\infty}^{0}[\sigma_{xx}^A u_x^B + \sigma_{xz}^A u_z^B - \sigma_{xx}^B u_x^A - \sigma_{xz}^B u_z^A + s^A U_x^B - s^B U_x^A]dz = 0 \tag{10}$$

To derive a bi-orthogonality relation, we shall use an argument due to Gregory [4]. Consider the eigenvalue $k_A^{(m)}$, and the eigenfunction pair

$$\mathbf{u}_{(m)}^A = \begin{pmatrix} u_x^A(z) \\ 0 \\ u_z^A(z) \end{pmatrix} e^{-i\omega t+ik_A^{(m)}x} \quad , \quad \mathbf{U}_{(m)}^A = \begin{pmatrix} U_x^A(z) \\ 0 \\ U_z^A(z) \end{pmatrix} e^{-i\omega t+ik_A^{(m)}x} \quad , \tag{11}$$

with corresponding stresses

$$s_{(m)}^A = s_{(m)}^A(z)e^{-i\omega t+ik_A^{(m)}x} \quad , \quad \boldsymbol{\sigma}_{(m)}^A = \begin{pmatrix} \sigma_{xx}(z) & 0 & \sigma_{xz}(z) \\ 0 & 0 & 0 \\ \sigma_{xz}(z) & 0 & \sigma_{zz}(z) \end{pmatrix} e^{-i\omega t+ik_A^{(m)}x} \quad , \tag{12}$$

and suppose the displacements and stresses in (11) and (12) satisfy the equations of motion (4), free surface boundary conditions, and decay to zero as $z \to -\infty$. We claim that a second "set" of displacements and stresses, similar to (11) and (12) except for a change in sign of the $k_A^{(m)}$ can also be found. Specifically, since any line "x = constant" is an axis of symmetry, (due to the medium's vertical heterogeneity), a second eigenfunction pair with wavenumber $-k_A^{(m)}$ exists, with corresponding displacements and stresses given by:

$$\mathbf{u}^A_{(m)} = \begin{pmatrix} -u_x^A(z) \\ 0 \\ u_z^A(z) \end{pmatrix} e^{-i\omega t - ik_A^{(m)}x} \quad ; \quad \mathbf{U}^A_{(m)} = \begin{pmatrix} -U_x^A(z) \\ 0 \\ U_z^A(z) \end{pmatrix} e^{-i\omega t - ik_A^{(m)}x} \quad , \tag{13}$$

and

$$s^A_{(m)} = s^A_{(m)}(z) e^{-i\omega t - ik_A^{(m)}x} \quad , \quad \boldsymbol{\sigma}^A_{(m)} = \begin{pmatrix} \sigma_{xx}(z) & 0 & -\sigma_{xz}(z) \\ 0 & 0 & 0 \\ -\sigma_{xz}(z) & 0 & \sigma_{zz}(z) \end{pmatrix} e^{-i\omega t - ik_A^{(m)}x} \quad . \tag{14}$$

Substitution of this second eigenfunction set into the reciprocity relationship (10) yields

$$\int_{-\infty}^{0} [\sigma^A_{xx} u^B_x - \sigma^A_{xz} u^B_z + \sigma^B_{xx} u^A_x - \sigma^B_{xz} u^A_z + s^A U^B_x + s^B U^A_x] dz = 0 \tag{15}$$

Adding (10) and (15) then gives the desired bi-orthogonality relationship

$$\int_{-\infty}^{0} [\sigma^A_{xx} u^B_x - \sigma^B_{xz} u^A_z + s^A U^B_x] dz = 0 \quad . \tag{16}$$

For the special case of zero porosity ($\beta(z) = 0$), we have $s^A = -\beta(z) p^A = 0$, and the bi-orthogonality relationship in (16) reduces to Fraser's bi-orthogonality relation for elastic media.

Case II: A fluid-loaded porous half-space

See fig. 1b for a schematic of the problem. In addition to the reciprocity relation for porous media (eq. (5)), the reciprocity relation for a fluid is [12]:

$$\int_S \mathbf{n} \cdot [p^A \mathbf{U}^B - p^B \mathbf{U}^A] ds = 0 \tag{17}$$

In standard form the $\mathbf{U}$'s are velocities, but since this paper only addresses the time harmonic problem, it will be assumed that the $\mathbf{U}$'s are displacements, and the $-i\omega$ factor has been dropped. As before, the assumed forms of the pressure and fluid displacements are given by:

$$p = p(z) e^{-i\omega t + ikx} \qquad \mathbf{U} = \begin{pmatrix} U_x(z) \\ 0 \\ U_z(z) \end{pmatrix} e^{-i\omega t + ikx} \tag{18}$$

Taking the surface integral around the strip between $x_1 \leq x \leq x_2$ for $z \geq 0$, and neglecting the integral as $z \to \infty$ yields:

$$e^{-2i\omega t}\{e^{i(k_A+k_B)x_1}\int_0^\infty[-p^A U_x^B + p^B U_x^A]dz$$

$$+e^{i(k_A+k_B)x_2}\int_0^\infty[p^A U_x^B - p^B U_x^A]dz + \left.\frac{e^{i(k_A+k_B)x}}{i(k_A+k_B)}\right|_{x_1}^{x_2}\left.[-p^A U_z^B + p^B U_z^A]\right|_{z=0}\} = 0 \quad (19)$$

The difference of (8) and (19) produces a term resulting from the integration along the interface ($z = 0$) given by:

$$e^{-2i\omega t}\left.\frac{e^{i(k_A+k_B)x}}{i(k_A+k_B)}\right|_{x_1}^{x_2}[(\sigma_{xz}^A u_x^B + \sigma_{zz}^A u_z^B - \sigma_{xz}^B u_x^A - \sigma_{zz}^B u_z^A$$

$$\left.+s^A U_z^B - s^B U_z^A)_{\text{porous media}} - (-p^A U_z^B + p^B U_z^A)_{\text{fluid media}}\,]\right|_{z=0} \quad (20)$$

The appropriate interface boundary conditions are [13]:

$$\begin{aligned} \sigma_{xz}(0) = 0 \qquad & [(1-\beta(0))u_z(0) + \beta(0)U_z(0)]_{\text{porous}} = U_z(0)_{\text{fluid}} \\ [\sigma_{zz}(0) + s(0)]_{\text{porous}} = -p(0)_{\text{fluid}} \qquad & p(0)_{\text{fluid}} = p(0)_{\text{porous}} \end{aligned} \quad (21)$$

The second equation in (21) actually involves fluid velocities, but with time harmonic variation assumed, the common factor of $-i\omega$ drops out. The elimination of $\sigma_{zz}(0)$ in favor of $-(1-\beta(0))p(0)$, and equality of fluid pressures at the interface, implies that the expression in (20) is identically zero for boundary conditions given by (21).

Applying the same argument used for the porous half-space medium, the remaining integrals from the difference of (8) and (19) yield an identity which corresponds to that given by (10):

$$\int_{-\infty}^0[\sigma_{xx}^A u_x^B + \sigma_{xz}^A u_z^B - \sigma_{xx}^B u_x^A - \sigma_{xz}^B u_z^A + s^A U_x^B - s^B U_x^A]_{\text{porous}}\,dz$$

$$-\int_0^\infty[p^A U_x^B - p^B U_x^A]_{\text{fluid}}\,dz = 0 \quad (22)$$

which is true along any line in the medium on which x is constant. A straightforward symmetry argument (as used in the half-space problem) supposes an eigenvalue/eigenfunction pairing, with a second eigenvalue (the negative of the first) and corresponding eigenfunction related to the first by appropriate sign changes in the shear stresses and displacements in the $\pm x$ directions.

Adding this *second* reciprocity equation to the first (eq. (22)) yields a bi-orthogonality relationship for a fluid-loaded porous half-space:

$$\int_{-\infty}^0[\sigma_{xx}^A u_x^B - \sigma_{xz}^B u_z^A + s^A U_x^B]_{\text{porous}}\,dz + \int_0^\infty[-p^A U_x^B]_{\text{fluid}}\,dz = 0 \quad (23)$$

If we define the porosity of the fluid medium to be equal to 1 for values of $z > 0$, then the fluid pressure for $z > 0$ is simply $-s$ and (23) can be written

$$\int_{-\infty}^0[\sigma_{xx}^A u_x^B - \sigma_{xz}^B u_z^A]\,dz + \int_{-\infty}^\infty[s^A U_x^B]\,dz = 0 \quad (24)$$

Cases III & IV: A porous-loaded elastic half-space & joined porous half-spaces

The development of these two cases follow lines similar to the previous two. Figs. 1c and 1d display the geometry of each problem. In each case, the reciprocity integrals ((10) and (22)) are first found, and then symmetry arguments are used to reduce the number of terms in them. The intermediate step in each case applies the appropriate boundary conditions between the two media to eliminate the interface term resulting from the surface integrations along $z = 0$. The appropriate boundary conditions [13] for an elastic/porous interface are:

$$\begin{aligned}
\sigma_{xz}(0)_{\text{porous}} &= \sigma_{xz}(0)_{\text{solid}} \quad ; \\
[\sigma_{zz}(0) + s(0)]_{\text{porous}} &= \sigma_{zz}(0)_{\text{solid}} \quad , \\
u_z(0)_{\text{porous}} &= u_z(0)_{\text{solid}} \quad , \\
u_x(0)_{\text{porous}} &= u_x(0)_{\text{solid}} \quad ; \\
[U_z(0) - u_z(0)]_{\text{porous}} &= 0 \quad .
\end{aligned} \tag{25}$$

The resulting bi-orthogonality condition for a porous half space over an elastic half-space is

$$\int_{-\infty}^{0} [\sigma_{xx}^A u_x^B - \sigma_{xz}^B u_z^A]_{\text{solid}}\, dz + \int_0^{\infty} [\sigma_{xx}^A u_x^B - \sigma_{xz}^B u_z^A + s^A U_x^B]_{\text{porous}}\, dz = 0 \tag{26}$$

Again, noting that for $z \leq 0$ we have $\beta(z) = 0$, (26) can be expressed as a single integral for a "composite porous" medium:

$$\int_{-\infty}^{\infty} [\sigma_{xx}^A u_x^B - \sigma_{xz}^B u_z^A + s^A U_x^B]\, dz = 0 \quad . \tag{27}$$

When two disparate porous medium are in contact along the $z = 0$ axis, the appropriate boundary conditions [13] are:

$$\begin{aligned}
\sigma_{xz}(0)_{\text{medium 1}} &= \sigma_{xz}(0)_{\text{medium 2}} \quad ; \\
[\sigma_{zz}(0) + s(0)]_{\text{medium 1}} &= [\sigma_{zz}(0) + s(0)]_{\text{medium 2}} \quad , \\
u_z(0)_{\text{medium 1}} &= u_z(0)_{\text{medium 2}} \quad ; \\
u_x(0)_{\text{medium 1}} &= u_x(0)_{\text{medium 2}} \quad ; \\
p(0)_{\text{medium 1}} - p(0)_{\text{medium 2}} &= \frac{-i\omega}{\kappa_s}\{\beta(0)[U_z(0) - u_z(0)]\}_{\text{medium 1}} \quad , \\
\{\beta(0)[U_z(0) - u_z(0)]\}_{\text{medium 1}} &= \{\beta(0)[U_z(0) - u_z(0)]\}_{\text{medium 2}} .
\end{aligned} \tag{28}$$

where κ_s has the dimensions of hydraulic permeability per unit length (inverse of the coefficient of resistance used in [13]) and is a measure of the permeability (or alignment of the pores) between the two porous media.

The resulting bi-orthogonality relationship can be written:

$$\int_{-\infty}^{0} [\sigma_{xx}^A u_x^B - \sigma_{xz}^B u_z^A + s^A U_x^B] dz + \int_0^{\infty} [\sigma_{xx}^A u_x^B - \sigma_{xz}^B u_z^A + s^A U_x^B] dz = 0 \quad , \tag{29}$$

or, as the single integral

$$\int_{-\infty}^{\infty} [\sigma_{xx}^A u_x^B - \sigma_{xz}^B u_z^A + s^A U_x^B] dz = 0 \qquad , \tag{30}$$

where it is understood that the material properties of the two porous media are discontinuous across the interface at $z = 0$.

Layers of finite thickness

The aforementioned bi-orthogonality relationships have been written for infinite or semi-infinite media, but the extension to finite media is straightforward. For example, in the case of a slab of porous media of depth H, the bi-orthogonality relation (eq 16) is merely amended by the substitution of $-H$ for $-\infty$, so that

$$\int_{-H}^{0} [\sigma_{xx}^A u_x^B - \sigma_{xz}^B u_z^A + s^A U_x^B] dz = 0 \qquad . \tag{31}$$

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FIGURE CAPTIONS

Figure 1a. Geometry of the problem in which a porous half-space is in contact with a free surface.

Figure 1b. Geometry of the problem in which a porous half-space is in contact with a fluid half-space.

Figure 1c. Geometry of the problem in which a porous half-space is in contact with an underlying solid half-space.

Figure 1d. Geometry of the problem in which two porous half-spaces are in contact.