UNITED STATES NAVAL POSTGRADUATE SCHOOL

FREE SPACE RADIATION IMPEDANCE OF RHOMBIC ANTENNA

---BY---

JESSE GERALD CHANEY
PROFESSOR OF ELECTRONICS

A REPORT
TO THE NAVY DEPARTMENT
BUREAU OF SHIPS
UPON AN INVESTIGATION CONDUCTED UNDER
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TECHNICAL REPORT NO. 4

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ABSTRACT

The generalized circuit[1] is partially integrated and the physical significance of certain terms is discussed. Afterwards, an unattenuated travelling wave of current is postulated as a first approximation to the current along a rhombic antenna, and the integration is completed to yield a formula for the radiation impedance of a rhombic antenna in free space. The resistive component of the impedance checks with the radiation resistance as listed by Leonard Lewin[2] in a discussion of a paper by Donald Foster[3], with Lewin apparently obtaining his formula by the solid angle Poynting vector method.

THE GENERALIZED CIRCUIT

In the generalized circuit (Fig. 1), a slice generator is assumed to exist between terminals b and c, and the current is assumed continuous through the generator. Terminals d and e may or may not be closed.

The current along the circuit is assumed to be given by $I_0 f(P)$,

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1. J. G. Chaney, " A critical study of the circuit concept", J. Appl. Phys. 22, 12, 1429 (1951).

2. Leonard Lewin, " Discussion on radiation from rhombic antenna" , Proc. I.R.E., 29, 9, 523, (1941).

3. Donald Foster, " Radiation from rhombic antenna", Proc. I.R.E., 25, 10, 1327, (1937).

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with P being any position on the circuit, and with I_0 being the current through the generator. If the length of the circuit is l , the driving point impedance is given by[1]

$$Z_{in} = lZ_i f_m^2 + j\frac{30}{k}\left(\oint_b^d + \oint_e^c \right)\left(\oint_b^d + \oint_e^c \right) \operatorname{Re}[f(P_1)^* f(P_2)]\ \bar{\bar{\nabla}}_1[e(r_{21})d\bar{r}_2]\cdot d\bar{r}_1 \tag{1}$$

in which

P_1 = any point along the axis of the wire

P_2 = any point along the inner periphery of the wire

r_{21} = distance from P_1 to P_2

$e(r_{21}) = r_{21}^{-1}\exp(-jkr_{21})$

$k = \omega(\mu_0\varepsilon_0)^{\frac{1}{2}} = \frac{2\pi}{\lambda}$

$\mu_0 = 4\pi(10^{-7})$ henries per meter

$\varepsilon_0 = (36\pi\cdot 10^{9})^{-1}$ farads per meter

Z_i = internal impedance of the circuit per meter

f_m^2 = meam square magnitude of the current distribution

$\bar{\bar{\nabla}}_1 = \nabla_1\nabla_1\cdot + k^2$ = operator deltil or delcap, with the subscript indicating the position at which the differentiations are to be performed

$*$ = the complex conjugate to be taken

Now let A indicate the integrated value of the first term of the integrand, that is

$$A = j\frac{30}{k}\left(\oint_b^d+\oint_e^c\right)\left(\oint_b^d+\oint_e^c\right)\operatorname{Re}[f(P_1)^* f(P_2)]\nabla_1[\nabla_1\cdot e(r_{21})d\bar{r}_2]\cdot d\bar{r}_1 \tag{2}$$

Upon integrating by parts along the axis, let

$$U = \mathrm{Re}[f(P_1)^* f(P_2)], \qquad dU = \nabla_1 \mathrm{Re}[f(P_1)^* f(P_2)]\cdot d\bar{r}_1$$

$$dV = \nabla_1[\nabla_1 \cdot e(r_{21}) d\bar{r}_2]\cdot d\bar{r}_1 , \quad V = \nabla_1 \cdot e(r_{21}) d\bar{r}_2 = -\nabla_2 e(r_{21})\cdot d\bar{r}_2$$

and get

$$\begin{aligned} A = j\tfrac{30}{k}\left(\oint_b^d + \oint_e^c\right)\{ & \mathrm{Re}[f(P_b)^* f(P_2)]\nabla_2 e(r_{2b}) - \mathrm{Re}[f(P_c)^* f(P_2)]\nabla_2 e(r_{2c}) \\ & + \mathrm{Re}[f(P_e)^* f(P_2)]\nabla_2 e(r_{2e}) - \mathrm{Re}[f(P_d)^* f(P_2)]\nabla_2 e(r_{2d})\}\cdot d\bar{r}_2 \qquad (3) \\ & + j\tfrac{30}{k}\left(\oint_b^d + \oint_e^c\right)\left(\oint_b^d + \oint_e^c\right)\{\nabla_1 \mathrm{Re}[f(P_1)^* f(P_2)]\cdot d\bar{r}_1\}\{\nabla_2 e(r_{21})\cdot d\bar{r}_2\} \end{aligned}$$

Let B represent the last term of equation (3), and integrate along the perimeter,

$$B = j\tfrac{30}{k}\left(\oint_b^d + \oint_e^c\right)\left(\oint_b^d + \oint_e^c\right)\{\nabla_1 \mathrm{Re}[f(P_1)^* f(P_2)]\cdot d\bar{r}_1\}\nabla_2 e(r_{21})\cdot d\bar{r}_2 \qquad (4)$$

Now let

$$U = \nabla_1 \mathrm{Re}[f(P_1)^* f(P_2)]\cdot d\bar{r}_1 , \quad dU = \nabla_2\{\nabla_1 \mathrm{Re}[f(P_1)^* f(P_2)]\cdot d\bar{r}_1\}\cdot d\bar{r}_2$$

$$dV = \nabla_2 e(r_{21})\cdot d\bar{r}_2 , \qquad V = e(r_{21})$$

and get

$$\begin{aligned} B = j\tfrac{30}{k}\left(\oint_b^d + \oint_e^c\right)\{ & e(r_{d1})\nabla_1 \mathrm{Re}[f(P_1)^* f(P_d)] \\ & - e(r_{e1})\nabla_1 \mathrm{Re}[f(P_1)^* f(P_e)] + e(r_{c1})\nabla_1 \mathrm{Re}[f(P_1)^* f(P_c)] \qquad (5) \\ & - e(r_{b1})\nabla_1 \mathrm{Re}[f(P_1)^* f(P_b)]\}\cdot d\bar{r}_1 \\ & - j\tfrac{30}{k}\left(\oint_b^d + \oint_e^c\right)\left(\oint_b^d + \oint_e^c\right) e(r_{21})\nabla_2\{\nabla_1 \mathrm{Re}[f(P_1)^* f(P_2)]\cdot d\bar{r}_1\}\cdot d\bar{r}_2 \end{aligned}$$

Recall that the current distribution is postulated to be the same along both paths of integration, and that the radius of the wire is postulated to be sufficiently small for the two paths to be interchangeable. Hence, upon interchanging the paths in the single integral terms of equation (5), and upon substituting from equation (5) into equation (3), and subsequently into equation (1), the

driving point impedance becomes

$$Z_{in} = lZ_i f_m^2 + j\frac{30}{k}\left(\oint_b^d - \oint_e^c\right)\left(\oint_b^d + \oint_e^c\right) e(r_{21})$$
$$\left\{k^2 \mathrm{Re}[f(P_1)^* f(P_2)]d\bar{r}_1 - \nabla_2\left(\nabla_1 \mathrm{Re}[f(P_1)^* f(P_2)]\cdot d\bar{r}_1\right)\right\}\cdot d\bar{r}_2$$
$$+ j\frac{30}{k}\left(\oint_b^d + \oint_e^c\right)\Big\{e(r_{2d})\nabla_2 \mathrm{Re}[f(P_d)^* f(P_2)] - \mathrm{Re}[f(P_d)^* f(P_2)]\nabla_2 e(r_{2d})$$
$$- e(r_{2e})\nabla_2 \mathrm{Re}[f(P_e)^* f(P_2)] + \mathrm{Re}[f(P_e)^* f(P_2)]\nabla_2 e(r_{2e}) \qquad (6)$$
$$+ e(r_{2c})\nabla_2 \mathrm{Re}[f(P_c)^* f(P_2)] - \mathrm{Re}[f(P_c)^* f(P_2)]\nabla_2 e(r_{2c})$$
$$- e(r_{2b})\nabla_2 \mathrm{Re}[f(P_b)^* f(P_2)] + \mathrm{Re}[f(P_b)^* f(P_2)]\nabla_2 e(r_{2b})\Big\}\cdot d\bar{r}_2$$

Integrating the terms in the last integral containing the gradient of the exponential terms, and assuming the radius of the wire is a with

$$ka \ll 1 \text{ , } a \ll l \text{ ,}$$

one obtains

$$Z_{in} = lZ_i f_m^2 + \frac{1}{j\omega 2\pi\varepsilon_0}\Big\{\tfrac{1}{2}e(a)[|f(P_d)|^2 + |f(P_e)|^2 + 2]$$
$$- e(r_{bc})\mathrm{Re}[f(P_b)^* f(P_c)] - e(r_{de})\mathrm{Re}[f(P_d)^* f(P_e)]$$
$$- e(r_{bd})\mathrm{Re}[f(P_b)^* f(P_d)] + e(r_{cd})\mathrm{Re}[f(P_c)^* f(P_d)]$$
$$- e(r_{ce})\mathrm{Re}[f(P_c)^* f(P_e)] + e(r_{be})\mathrm{Re}[f(P_b)^* f(P_e)]\Big\} \qquad (7)$$
$$+ j\frac{60}{k}\left(\oint_b^d + \oint_e^c\right)\Big\{e(r_{2d})\nabla_2 \mathrm{Re}[f(P_d)^* f(P_2)] - e(r_{2b})\nabla_2 \mathrm{Re}[f(P_b)^* f(P_2)]$$
$$+ e(r_{2c})\nabla_2 \mathrm{Re}[f(P_c)^* f(P_2)] - e(r_{2e})\nabla_2 \mathrm{Re}[f(P_e)^* f(P_2)]\Big\}\cdot d\bar{r}_2$$
$$+ j\frac{30}{k}\left(\oint_b^d + \oint_e^c\right)\left(\oint_b^d + \oint_e^c\right) e(r_{21})$$
$$\left\{k^2 \mathrm{Re}[f(P_1)^* f(P_2)]d\bar{r}_1 - \nabla_2\left(\nabla_1 \mathrm{Re}[f(P_1)^* f(P_2)]\cdot d\bar{r}_1\right)\right\}\cdot d\bar{r}_2$$

If the antenna is symmetrically fed, that is if the antenna is physically symmetrical with respect to the generator, the sense of the path from e to c may be reversed in the single integral terms of equation (7), with the interchanging of c and e and b and d, respectively

within the interval (ec). Also, after replacing $-d\bar{r}_2$ with $+d\bar{r}_2$ along with the corresponding interchange of limits for the interval (ec), the operator $+\nabla_2$ is replaced with the operator $-\nabla_2$. Hence, for the symmetrically fed antenna, the single integral terms may be doubled and integrated only from b to d. However, if the terminal spacings bc and de are each negligibly small with respect to a wave length, the current may be assumed continuous through the terminals and hence the single integral terms of equation (7) then vanish.

The real part of the terms within the brace of equation (7) represent the reactive terms due to the end capacitances, and the imaginary part represents a correction to the component of the radiation resistance due to the distributed charges. The latter correction is necessary because the path is not closed.

In practice, either the current is supposed to vanish at the terminals de or a load impedance Z_o is assumed to be inserted at the terminals to cause the current to take on the postulated distribution. Also, the input terminals are short circuited at the generator. Thus, the real part of the terms within the brace either vanish or are usually discarded. The imaginary terms also either vanish or may be disregarded provided the terminal spacings are quite small in comparison with a wave length. Otherwise, if f_o is the current magnitude at d and at e, it may be necessary to retain the following correction terms from equation (7),

$$eo\left(\frac{\sin kr_{bc}}{kr_{bc}} - 1\right) + eo\left(\frac{\sin kr_{de}}{kr_{de}} - 1\right)f_o^2 \quad ,$$

in which it is assumed that the distances r_{bd} and r_{be} do not

appreciably differ from the distances r_{cd} and r_{ce}, respectively.

Thus, assuming the terminal spacings permit the dropping of the correction terms and that the current is continuous through the terminals so that the single integral terms in equation (7) vanish, the driving point impedance becomes[1],

$$Z_{in} = lZ_i f_m^2 + Z_0 f_0^2 \tag{8}$$
$$+ j\frac{30}{k}\oint_1\oint_2 e(r_{21})\left\{k^2 \mathrm{Re}[f(P_1)^* f(P_2)]d\bar{r}_1 - \nabla_2\left(\nabla_1 \mathrm{Re}[f(P_1)^* f(P_2)]\cdot d\bar{r}_1\right)\right\}\cdot d\bar{r}_2$$

In many cases of antenna applications , the current may be approximated sufficiently well by a distribution function which either satisfies or may be broken into the sum of functions which satisfy one of the following equations of constraint,

$$\nabla_2\left(\nabla_1 \mathrm{Re}[f(P_1)^* f(P_2)]\cdot d\bar{r}_1\right)\cdot d\bar{r}_2 \pm k^2 \mathrm{Re}[f(P_1)^* f(P_2)]dr_1 dr_2 = 0 \tag{9}$$

Substituting from equation (9) into equation (8),

$$Z_{in} = lZ_i f_m^2 + Z_0 f_0^2 + j\frac{\omega\mu_0}{4\pi}\oint_1\oint_2 e(r_{21})\mathrm{Re}[f(P_1)^* f(P_2)]d\bar{r}_1\cdot d\bar{r}_2$$
$$\pm j\frac{\omega\mu_0}{4\pi}\oint_1\oint_2 e(r_{21})\mathrm{Re}[f(P_1)^* f(P_2)]dr_1 dr_2 \tag{10}$$

The first integral is the generalized Neumann's formula, and is sometimes used alone for estimating the radiation resistance of a circuit. It appears that the second double integral term should not be neglected. However, it should be remembered that a constant current distribution does not satisfy equation (9), and that equation (10) does not contradict Neumann's formula <u>per se</u>.

If the terminals of Z_0 are sufficiently near each other for the closing of the integrals, equation (10) may be written

$$Z_{in} = lZ_i f_m^2 + f_0^2 Z_0 + j30k\oint_1\oint_2 e(r_{21})\mathrm{Re}[f(P_1)^* f(P_2)](d\bar{r}_1\cdot d\bar{r}_2 \pm dr_1 dr_2) \tag{11}$$

In terms of arc lengths, let

$$g(ks_1,ks_2) = \mathrm{Re}[f(P_1)^* f(P_2)]$$

$$h(ks_1,ks_2) = e(r_{21})$$

$$\cos\theta(s_1,s_2)\, ds_1 ds_2 = d\bar{r}_1 \cdot d\bar{r}_2$$

and equations (9) and (11) become, respectively,

$$\left(\frac{\partial^2}{\partial s_1 \partial s_2} \pm k^2\right) g(ks_1,ks_2) = 0 \qquad (12)$$

$$Z_{in} = 1Z_i f_m^2 + Z_o f_o^2 + j30k\oint_1\oint_2 g(ks_1,ks_2)h(ks_1,ks_2)[\cos\theta(s_1,s_2)\pm 1]ds_1 ds_2 \qquad (13)$$

If the algebraic sign within the parenthesis of equation (12) is positive,

$$Z_{in} = 1Z_i f_m^2 + Z_o f_o^2 + j30k\oint_1\oint_2 g(ks_1,ks_2)h(ks_1,ks_2)\cos^2\tfrac{1}{2}\theta(s_1,s_2)ds_1 ds_2 \qquad (14)$$

and if it is negative,

$$Z_{in} = 1Z_i f_m^2 + Z_o f_o^2 - j30k\oint_1\oint_2 g(ks_1,ks_2)h(ks_1,ks_2)\sin^2\tfrac{1}{2}\theta(s_1,s_2)ds_1 ds_2 \qquad (15)$$

THE RHOMBIC ANTENNA

For the purpose of obtaining a formula for the radiation impedance of a rhombic antenna in free space (Fig. 2), the generator is assumed to short circuit terminals b and c, and a terminal impedance Z_o is postulated at terminals d and e which, to a first approximation, causes the current to be an unattenuated travelling wave with no reflection. Each leg of the rhombic is assumed to be l meters in length and the vertex angle at the generator is assumed to be 2α.

Upon selecting the origin at the generator, the variable s ranges from $-2l$ at e to $+2l$ at d. For s_1 and s_2 along different but

parallel wires, $\theta = \pi$. For s_1 and s_2 along different wires one and two or three and four, $\theta = \pi - 2\alpha$. For s_1 and s_2 along different wires two and three or four and one, $\theta = 2\alpha$.

For negative s, $I = I_0 e^{jks}$, and for positive s, $I = I_0 e^{-jks}$. Thus, for s_1 and s_2 having the same sense,

$$g(ks_1, ks_2) = \cos k(s_1 - s_2) \tag{16}$$

and for s_1 and s_2 having the opposite sense,

$$g(ks_1, ks_2) = \cos k(s_1 + s_2) \tag{17}$$

The function (16) satisfies equation (12) with the negative sign, and hence when s_1 and s_2 are of the same sense, from equation (15), the contribution Z_1 to the radiation impedance

$$Z = R + j\,X \tag{18}$$

becomes

$$Z_1 = -\,j30\,k \oint_1 \oint_2 g(ks_1, ks_2)\, h(ks_1, ks_2) \sin^2 \tfrac{1}{2}\theta(s_1, s_2)\, ds_1 ds_2 \tag{19}$$

Similarly, the function (17) satisfies equation (12) with the positive sign, and hence when s_1 and s_2 are of the opposite sense, from equation (14), the contribution Z_2 to the radiation impedance becomes,

$$Z_2 = j30\,k \oint_1 \oint_2 g(ks_1, ks_2)\, h(ks_1, ks_2) \cos^2 \tfrac{1}{2}\theta(s_1, s_2)\, ds_1 ds_2 \tag{20}$$

Substituting $\theta = 0$ into equation (19) and $\theta = \pi$ into equation (20), it follows that all paths for s_1 and s_2 which are parallel, whether on the same wire or on opposite wires, are eliminated from further consideration in carrying out the integrations for the radiation impedance of a rhombic antenna.

For any pair of wires, the axial and surface paths must be taken along both wires in succession. But due to symmetry, the integrations are the same for the axial path along one wire with the surface path along the other wire as it is if the two paths are interchanged. Hence, the integrations need to be carried out and doubled only for the four cases where the axial path lies either on wire one or wire three and the surface path lies on either wire two or wire four.

Thus, from equations (16), (17), (19), and (20),

$$\begin{aligned} Z_r = \; & j120k\int_{-l}^{0}\int_{0}^{l} \cos k(s_1+s_2)\, e(r_{12}) \sin^2\alpha \, ds_2 ds_1 \\ & -j120k\int_{-l}^{0}\int_{-2l}^{-l} \cos k(s_1-s_2)\, e(r_{14}) \sin^2\alpha \, ds_2\, ds_1 \\ & -j120k\int_{l}^{2l}\int_{0}^{l} \cos k(s_1-s_2)\, e(r_{23}) \sin^2\alpha \, ds_2 ds_1 \\ & +j120k\int_{l}^{2l}\int_{-2l}^{-l} \cos k(s_1+s_2)\, e(r_{34}) \sin^2\alpha \, ds_2 ds_1 \end{aligned} \qquad (21)$$

with

$$\begin{aligned} r_{12} &= (s_2^2 + s_1^2 + 2s_1s_2 \cos 2\alpha)^{\frac{1}{2}} \\ r_{14} &= [(l+s_1)^2 + (l+s_2)^2 + 2(l+s_1)(l+s_2) \cos 2\alpha]^{\frac{1}{2}} \\ r_{23} &= [(s_1-l)^2 + (l-s_2)^2 + 2(s_1-l)(l-s_2) \cos 2\alpha]^{\frac{1}{2}} \\ r_{34} &= [(2l-s_1)^2+(2l+s_2)^2 - 2(2l-s_1)(2l+s_2) \cos 2\alpha]^{\frac{1}{2}} \end{aligned}$$

In the first integral, let $s_1=-x_1$ and $s_2=x_2$,
in the second integral, let $s_1=x_1-l$ and $s_2=-(x_4+l)$,
in the third integral, let $s_1=x_1+l$ and $s_2=l-x_4$,
and in the fourth integral, let $s_1=2l-x_1$ and $s_2=x_2-2l$.
Then the radiation impedance becomes

$$Z_r=j240\sin^2\alpha\left[\int_0^l\int_0^l\cos k(x_2-x_1)e(r_{21})dx_2dx_1-\int_0^l\int_0^l\cos k(x_4+x_1)e(r_{41})dx_4dx_1\right] \qquad (22)$$

with

$$r_{21} =[x_1^2+x_2^2-2x_1x_2\cos 2\alpha]^{\frac{1}{2}} = [(x_1-x_2\cos 2\alpha)^2+(x_2\sin 2\alpha)^2]^{\frac{1}{2}} \qquad (23)$$

$$r_{41} =[x_1^2+x_4^2+2x_1x_4\cos 2\alpha]^{\frac{1}{2}} = [(x_1+x_4\cos 2\alpha)^2+(x_4\sin 2\alpha)^2]^{\frac{1}{2}} \qquad (24)$$

Upon writing the integrands in the exponential form,

$$Z_r = -j\ 120\ k\ \sin^2\alpha\ (I_1 + I_2 - I_3 - I_4) \tag{25}$$

with

$$I_1 = \int_0^l\int_0^l \exp[-jk(x_4+x_1+r_{41})]\ r_{41}^{-1}\ dx_4dx_1 \tag{26}$$

$$I_2 = \int_0^l\int_0^l \exp[\ jk(x_4+x_1-r_{41})]\ r_{41}^{-1}\ dx_4dx_1 \tag{27}$$

$$I_3 = \int_0^l\int_0^l \exp[\ jk(x_2-x_1-r_{21})]\ r_{21}^{-1}\ dx_2dx_1 \tag{28}$$

$$I_4 = \int_0^l\int_0^l \exp[-jk(x_2-x_1+r_{21})]\ r_{21}^{-1}\ dx_2dx_1 \tag{29}$$

Due to symmetry, x_2 and x_1 may be interchanged in I_4, yielding an equality of I_3 and I_4.

Recalling that the path for x_1 is along the axis of wire one whereas the paths for x_2 and x_4 are along the inner peripheries of wires two and four, respectively, if it is assumed that the axes of the wires are of length l, then the limits for x_2 are actually from $a\cdot\cot\alpha$ to $l - a\cdot\tan\alpha$ and those for x_4 are actually from $a\cdot\tan\alpha$ to $l - a\cdot\cot\alpha$. Hence, finally (Fig. 3)

$$Z_r = -\ j\ 120\ k\ \sin^2\alpha\ (\ I_1 + I_2 - 2I_3\ ) \tag{30}$$

with

$$I_1 = \int_0^l\int_{a\tan\alpha}^{l-a\cot\alpha} \exp[-jk(x_4+x_1+r_{41})]r_{41}^{-1}\ dx_4dx_1 \tag{31}$$

$$I_2 = \int_0^l\int_{a\tan\alpha}^{l-a\cot\alpha} \exp[jk(x_4+x_1-r_{41})]r_{41}^{-1}\ dx_4dx_1 \tag{32}$$

$$I_3 = \int_0^l\int_{a\cot\alpha}^{l-a\tan\alpha} \exp[-jk(x_2-x_1+r_{21})]r_{21}^{-1}\ dx_2dx_1 \tag{33}$$

The integrations for I_1, I_2 and I_3 may be carried out by transformations similar to those suggested by F. H. Murray[5].

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5. F. H. Murray, "Mutual impedance of two skew antenna wires", Proc. I.R.E., 21, 1, 154, (1933).

However, they should be evaluated directly and not by transforming to the exact integral given by Murray, since there is a likelihood of unsuspectingly integrating through a singularity. In this way, the following formula is obtained for the radiation impedance of a rhombic antenna in free space.†

$$
\begin{aligned}
Z_r/120 = {} & 2C + 2\ln(2kl\sin^2\alpha) + 2\mathrm{Ci}\,2kl - 2\mathrm{Ci}(2kl\sin\alpha) - \mathrm{Ci}[2kl(1+\cos\alpha)] \\
& - \mathrm{Ci}[2kl(1-\cos\alpha)] \\
& + \cos(2kl\sin^2\alpha)\{\mathrm{Ci}[2kl\cos\alpha(1+\cos\alpha)] + \mathrm{Ci}[2kl\cos\alpha(1-\cos\alpha)] \\
& \quad + \mathrm{Ci}[2kl\sin\alpha(1+\sin\alpha)] + \mathrm{Ci}[2kl\sin\alpha(1-\sin\alpha)] - 2\mathrm{Ci}[2kl\cos^2\alpha] \\
& \quad - 2\mathrm{Ci}[2kl\sin^2\alpha]\} \\
& - \sin(2kl\sin^2\alpha)\{\mathrm{Si}[2kl\cos\alpha(1+\cos\alpha)] - \mathrm{Si}[2kl\cos\alpha(1-\cos\alpha)] \\
& \quad - \mathrm{Si}[2kl\sin\alpha(1+\sin\alpha)] + \mathrm{Si}[2kl\sin\alpha(1-\sin\alpha)] - 2\mathrm{Si}[2kl\cos^2\alpha] \\
& \quad + 2\mathrm{Si}[2kl\sin^2\alpha]\} \\
& + j\Big[\mathrm{Si}[2kl(1+\cos\alpha)] - \mathrm{Si}[2kl(1-\cos\alpha)] + 2\mathrm{Si}[2kl\sin\alpha] - 2\mathrm{Si}\,2kl \\
& \quad - \cos(2kl\sin^2\alpha)\{\mathrm{Si}[2kl\cos\alpha(1+\cos\alpha)] + \mathrm{Si}[2kl\cos\alpha(1-\cos\alpha)] \\
& \quad + \mathrm{Si}[2kl\sin\alpha(1+\sin\alpha)] + \mathrm{Si}[2kl\sin\alpha(1-\sin\alpha)] - 2\mathrm{Si}[2kl\cos^2\alpha] \\
& \quad - 2\mathrm{Si}[2kl\sin^2\alpha]\} \\
& \quad - \sin(2kl\sin^2\alpha)\{\mathrm{Ci}[2kl\cos\alpha(1+\cos\alpha)] - \mathrm{Ci}[2kl\cos\alpha(1-\cos\alpha)] \\
& \quad - \mathrm{Ci}[2kl\sin\alpha(1+\sin\alpha)] + \mathrm{Ci}[2kl\sin\alpha(1-\sin\alpha)] - 2\mathrm{Ci}[2kl\cos^2\alpha] \\
& \quad + 2\mathrm{Ci}[2kl\sin^2\alpha]\}\Big]
\end{aligned}
\tag{34}
$$

The resistive component of formula (34) checks with the radiation resistance of a rhombic antenna as given by Leonard Lewin[2] in a discussion of a paper by Donald Foster[3]. From the discussion, it may be inferred that Lewin derived the formula for the radiation resistance by the solid angle Poynting vector method.

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† $C=0.5772$, $\mathrm{Si}(x) = \int_0^x \frac{\sin u}{u}du$, $\mathrm{Ci}(x) = -\int_x^\infty \frac{\cos u}{u}du$.

THE TERMINATED VEE ANTENNA

It is interesting to note in passing that the free space radiation impedance of the terminated Vee antenna may be written from equation (7) as

$$Z_r = 60 f_0^2\left[\frac{\sin(2kl\sin a)}{2kl\sin a} - 1\right] + j\,120k\,\sin^2 a\; I_2 + j120\int_0^l [c(r_{2d}) - c(r_{2e})]\,\sin k(l-s)\,ds \qquad (25)$$

with

$$r_{2d} = [a^2 + (l-s)^2]^{\frac{1}{2}} \approx l-s$$

$$r_{2e} = [l^2+s^2-2ls\cos 2a]^{\frac{1}{2}} = [(s-l\cos 2a)^2+(l\sin 2a)^2]^{\frac{1}{2}}$$

in which each leg of the Vee is of length l and the vertex angle is $2a$.

Using the modified cosine integral function

$$Cn(x) = \int_0^x \frac{1 - \cos u}{u}\,du$$

the radiation impedance of the Vee antenna in free space becomes,

$$Z_r/60 = f_0^2\left[\frac{\sin(2kl\sin a)}{2kl\sin a} - 1\right] + 2Cn(2kl\sin a) + j\,2Si(2kl\sin a) \qquad (26)$$

In equation (26), the end capacitances are assumed to be shunted by the generator and terminal impedance, respectively, and hence have been discarded.

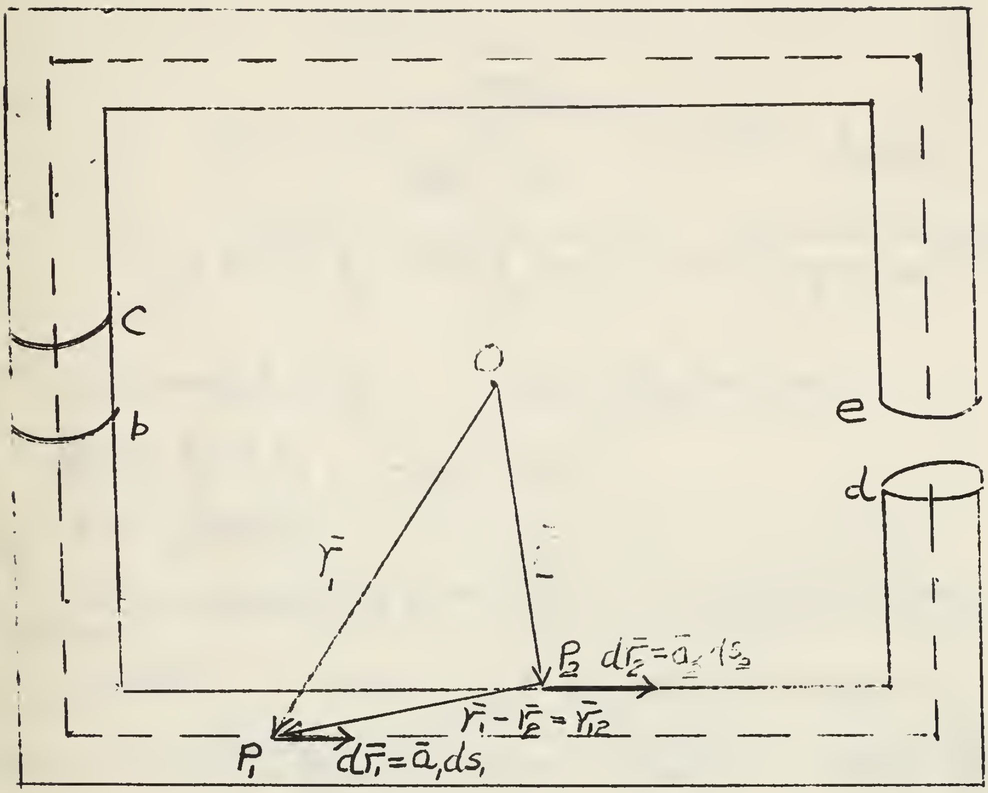


Figure 1

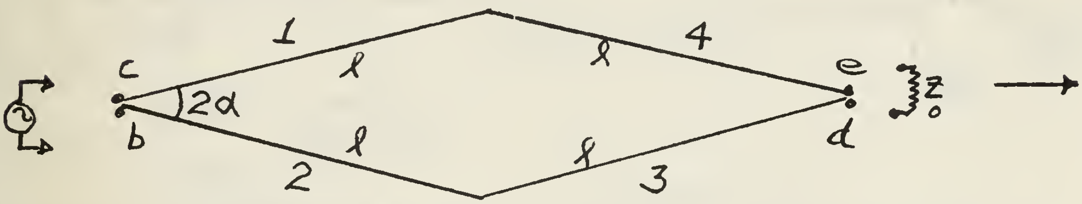


Figure 2

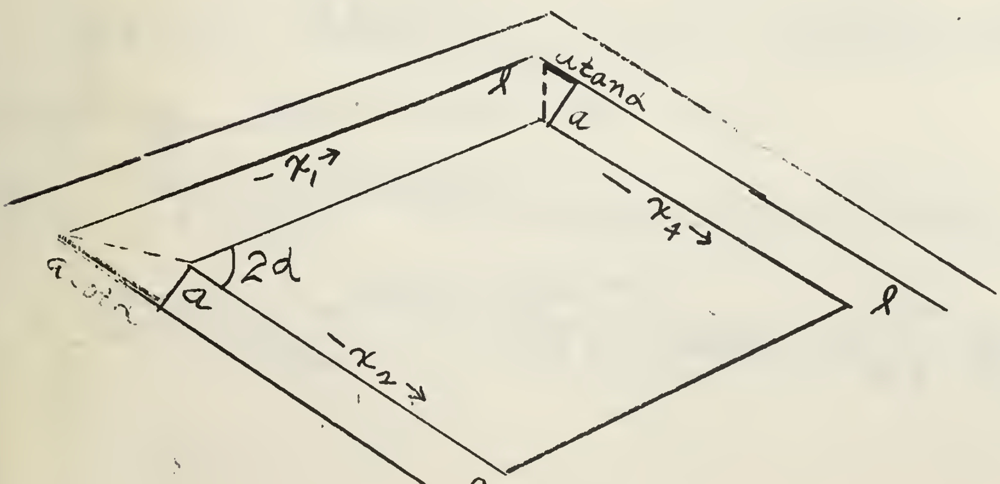


Figure 3

APPENDIX A

Given

$$I_1 = \int_0^l \int_{a\tan\alpha}^{l-a\tan\alpha} \frac{e^{-jk(x_4+x_1+r_{41})}}{r_{41}}\, dx_4\, dx_1 \qquad (1)$$

with

$$r_{41} = \sqrt{x_1^2+x_4^2+2x_1x_4\cos 2\alpha} = \sqrt{(x_1+x_4\cos 2\alpha)^2+(x_4\sin 2\alpha)^2}.$$

Let

$$m_1 t = x_1 + x_4\cos 2\alpha + r_{41} \quad , \quad m_1 = x_4 \sin 2\alpha$$

$$m_1 t_1 = x_4\cos 2\alpha + x_4 = 2x_4\cos^2\alpha$$

$$m_1 t_2 = l + x_4 \cos 2\alpha$$

$$m_1 t_1\Big]_{a\tan\alpha} = 2a\cos^2\alpha\tan\alpha = a\sin 2\alpha \;,\; m_1 t_1\Big]_l = 2l\cos^2\alpha$$

$$m_1 t_2\Big]_0 = 2l \;, \qquad m_1 t_2\Big]_l = 2l\cos\alpha(1+\cos\alpha)$$

$$m_1\, dt = \left(1+\frac{x_1+x_4\cos 2\alpha}{r_{41}}\right)dx_1 = \frac{m_1 t\, dx_1}{r_{41}} \;, \quad dx_1 = \frac{r_{41}}{t}dt$$

Substituting into equation (1),

$$I_1 = \int_i^l \int_{t_1}^{t_2} e^{-jk(m_1 t + 2x_4\sin^2\alpha)}/t \; dt\, dx_4 = \int_0^l e^{-j2x_4\sin^2\alpha}\int_{km_1t_1}^{km_1t_2} e^{-jt}/t \; dt\, dx_4 \qquad (2)$$

Let

$$u = \int_{km_1t_1}^{km_1t_2} \frac{e^{-jt}}{t}dt$$

$$du = \left[\frac{e^{-jkm_1t_2}}{km_1t_2}\frac{\partial}{\partial x_4}(km_1t_2) - \frac{e^{-jkm_1t_1}}{km_1t_1}\frac{\partial}{\partial x_4}(km_1t_1)\right]dx_4$$

$$dV = e^{-j2kx_4\sin^2\alpha}dx_4 \;, \quad V = -\frac{1}{j2k\sin^2\alpha}e^{-j2kx_4\sin^2\alpha}$$

and equation (2) becomes

$$I_1 = -\frac{1}{j2k\sin^2\alpha}\left[e^{-j2kx_4\sin^2\alpha}\left\{(Ci\,km_1t_2 - Ci\,km_1t_1) - j(Si\,km_1t_2 - Si\,km_1t_1)\right\}\right]_{a\tan\alpha}^{l}$$

$$+\frac{1}{j2k\sin^2\alpha}\int_{a\tan\alpha}^{l} e^{-jk2x_4\sin\alpha}\left\{\frac{e^{-jkm_1t_2}}{m_1t_2}\frac{\partial}{\partial x_4}(m_1t_2) - \frac{e^{-jkm_1t_1}}{m_1t_1}\frac{\partial}{\partial x_4}(m_1t_1)\right\}dx_4 \qquad (3)$$

now

$$\frac{\partial(m_1 t)}{\partial x_4} = \cos 2\alpha + \frac{x_4 + x_1\cos 2\alpha}{r_{41}} = \frac{x_4 + (x_1 + r_{41})\cos 2\alpha}{r_{41}}$$

and thus if $r_{4l} = \sqrt{(x_4 + l\cos 2\alpha)^2 + (l\sin 2\alpha)^2}$

$$\frac{1}{m_1 t_1}\frac{\partial(m_1 t_1)}{\partial x_4} = \frac{x_4 + x_4\cos 2\alpha}{x_4(x_4\cos 2\alpha + x_4)} = \frac{1}{x_4}$$

$$\frac{1}{m_1 t_2}\frac{\partial(m_1 t_2)}{\partial x_4} = \frac{x_4 + (l + r_{4l})\cos 2\alpha}{r_{4l}(l + x_4\cos 2\alpha + r_{4l})}$$

Hence, if

$$A_{11} = \int_{a\tan\alpha}^{l} e^{-jk2x_4\sin^2\alpha}\left[\frac{e^{-jkm_1t_2}}{m_1t_2}\frac{\partial(m_1t_2)}{\partial x_4} - \frac{e^{-jkm_1t_1}}{m_1t_1}\frac{\partial(m_1t_1)}{\partial x_4}\right]dx_4 \qquad (4)$$

upon substituting into equation (4),

$$A_{11} = \int_{a\tan\alpha}^{l}\frac{e^{-jk(r_{4l}+l+x_4)}[x_4 + (l + r_{4l})\cos 2\alpha]}{r_{4l}(l + x_4\cos 2\alpha + r_{4l})}dx_4 - \int_{a\tan\alpha}^{l}\frac{e^{-jk2x_4}}{2x_4}2dx_4 \qquad (5)$$

Let A_{12} equal the second integral and A_{13} equal the first integral in equation (5).

$$A_{12} = -\int_{a\tan\alpha}^{l}\frac{e^{-jk2x_4}}{2x_4}2dx_4 = -\int_{2ka\tan\alpha}^{2kl}\frac{e^{-ju}}{u}du$$

$$A_{12} = \ln(\gamma 2ka\tan\alpha) - Ci\,2kl + jSi\,2kl \quad * \qquad (6)$$

now in

$$A_{13} = \int_{a\tan\alpha}^{l}\frac{e^{-jk(r_{4l}+l+x_4)}[x_4 + (l + r_{4l})\cos 2\alpha]}{r_{4l}(l + x_4\cos 2\alpha + r_{4l})}dx_4 \qquad (7)$$

let

$$m_2 y = r_{4l} + (x_4 + l\cos 2\alpha), \quad m_2 = l\sin 2\alpha$$

$$m_2 y_1 = l, \quad m_2 y_2 = \sqrt{2l^2 + 2l^2\cos 2\alpha} = 2l\cos\alpha$$

$$m_2 dy = \left[1 + \frac{x_4 + l\cos 2\alpha}{r_{4l}}\right]dx_4 = \frac{m_2 y\, dx_4}{r_{4l}}, \quad dx_4 = \frac{r_{4l}}{y}dy$$

* $\ln\gamma = C = 0.5772$, $Ci\,x = -\int_x^\infty \frac{\cos t}{t}dt$, $Si\,x = \int_0^x \frac{\sin t}{t}dt$

$$r_{4\ell}+\ell+x_4 = m_2 y - \ell\cos 2\alpha + \ell = m_2 y + 2\ell\sin^2\alpha$$

$$\frac{m_2}{y} = \frac{\ell^2\sin^2 2\alpha}{r_{4\ell}+x_4+\ell\cos 2\alpha} = \frac{\ell^2\sin^2 2\alpha\left[r_{4\ell}-(x_4+\ell\cos 2\alpha)\right]}{x_4^2+2\ell x_4\cos 2\alpha+\ell^2-x_4^2-2\ell x_4\cos 2\alpha-\ell^2\cos^2 2\alpha}$$

$$\frac{m_2}{y} = r_{4\ell}-(x_4+\ell\cos 2\alpha)$$

$$r_{4\ell} = \frac{m_2}{2}\left(y+\frac{1}{y}\right) = \frac{m_2}{2y}(y^2+1)$$

$$x_4 = m_2 y - r_{4\ell} - \ell\cos 2\alpha = m_2 y - \frac{m_2(y^2+1)}{2y} - \ell\cos 2\alpha = \frac{m_2}{2y}(y^2-1) - \ell\cos 2\alpha$$

Substituting into equation (7),

$$A_{13} = e^{-j2k\ell\sin^2\alpha}\int_{y_1}^{y_2}\frac{e^{-jkm_2 y}}{y}\,\frac{\left[\frac{m_2}{2y}(y^2-1)+\frac{m_2}{2y}(y^2+1)\cos 2\alpha\right]dy}{\ell+\frac{m_2}{2y}(y^2+1)+\frac{m_2}{2y}(y^2-1)\cos 2\alpha-\ell\cos^2 2\alpha}$$

$$= e^{-j2k\ell\sin^2\alpha}\int_{y_1}^{y_2}\frac{e^{-jkm_2 y}}{y}\,\frac{y^2\cos^2\alpha-\sin^2\alpha}{y/m_2\left[\frac{m_2}{2y}(2y^2\cos^2\alpha+2\sin^2\alpha)+\ell\sin^2 2\alpha\right]}\,dy$$

$$= e^{-j2k\ell\sin^2\alpha}\int_{y_1}^{y_2}\frac{e^{-jkm_2 y}\left[y^2-\tan^2\alpha\right]}{y\left[y^2+2xy\tan\alpha+\tan^2\alpha\right]}\,dy$$

$$A_{13} = e^{-j2k\ell\sin^2\alpha}\int_{y_1}^{y_2}e^{-jkm_2 y}\,\frac{y-\tan\alpha}{y(y+\tan\alpha)}\,dy \qquad (8)$$

Breaking the integral (8) into partial fractions,

$$A_{13} = e^{-j2k\ell\sin^2\alpha}\int_{y_1}^{y_2}e^{-jkm_2 y}\left(\frac{2}{y+\tan\alpha}-\frac{1}{y}\right)dy \qquad (9)$$

Changing the variables

$$A_{13} = 2\int_{2k\ell}^{2k\ell(1+\cos\alpha)}\frac{e^{-ju}}{u}\,du - e^{-j2k\ell\sin^2\alpha}\int_{2k\ell\cos^2\alpha}^{2k\ell\cos\alpha(1+\cos\alpha)}\frac{e^{-ju}}{u}\,du \qquad (10)$$

$$\begin{aligned}A_{13} = {} & 2\,\mathrm{Ci}[2k\ell(1+\cos\alpha)] - 2\,\mathrm{Ci}\,2k\ell - j2\{\mathrm{Si}[2k\ell(1+\cos\alpha)] - \mathrm{Si}\,2k\ell\} \\ & - \{\cos(2k\ell\sin^2\alpha) - j\sin(2k\ell\sin^2\alpha)\}\{\mathrm{Ci}[2k\ell\cos\alpha(1+\cos\alpha)] \qquad (11) \\ & - \mathrm{Ci}(2k\ell\cos^2\alpha) - j\left(\mathrm{Si}[2k\ell\cos\alpha(1+\cos\alpha)] - \mathrm{Si}[2k\ell\cos^2\alpha]\right)\}\end{aligned}$$

Substituting A_{13} and A_{12} into A_{11}, and afterwards substituting equation A_{11} into equation (3),

$$
\begin{aligned}
jK\sin^2\alpha\, I_1 = {} & Ci[2K\ell(1+\cos\alpha)] - Ci\,2K\ell - \ln\cos\alpha \\
& - \cos(2K\ell\sin^2\alpha)\left\{Ci[2K\ell\cos\alpha(1+\cos\alpha)] - Ci(2K\ell\cos^2\alpha)\right\} \\
& + \sin(2K\ell\sin^2\alpha)\left\{Si[2K\ell\cos\alpha(1+\cos\alpha)] - Si(2K\ell\cos^2\alpha)\right\} \\
& + j\Big[Si\,2K\ell - Si[2K\ell(1+\cos\alpha)] \\
& \quad + \cos(2K\ell\sin^2\alpha)\left\{Si[2K\ell\cos\alpha(1+\cos\alpha)] - Si(2K\ell\cos^2\alpha)\right\} \\
& \quad + \sin(2K\ell\sin^2\alpha)\left\{Ci[2K\ell\cos\alpha(1+\cos\alpha)] - Ci(2K\ell\cos^2\alpha)\right\}\Big]
\end{aligned}
\tag{12}
$$

APPENDIX B

Given $I_2 = \int_0^l \int_{a\tan\alpha}^{l-a\tan\alpha} \frac{e^{-jk(r_{41}-x_1-x_4)}}{r_{41}} dx_4 dx_1$ (1)

with $r_{41} = \sqrt{x_1^2+x_2^2+2x_1x_4\cos 2\alpha} = \sqrt{(x_1+x_4\cos 2\alpha)^2+(x_4\sin 2\alpha)^2}$

Let $m_1 t = r_{41} - x_1 - x_4\cos 2\alpha$, $m_1 = x_4 \sin 2\alpha$

$m_1 t_1 = x_4 - x_4\cos 2\alpha = 2x_4\sin^2\alpha$

$m_1 t_2 = \sqrt{x_4^2+2lx_4\cos 2\alpha + l^2} - (l + x_4\cos 2\alpha) = r_{4l} - l - x_4\cos 2\alpha$

$r_{4l} = \sqrt{(x_4+l\cos 2\alpha)^2+(l\sin 2\alpha)^2}$, $r_{40} = x_4$

$m_1 dt = \left(\frac{x_1+x_4\cos 2\alpha}{r_{41}} - 1\right)dx_1 = -\frac{m_1 t\, dx_1}{r_{41}}$, $dx_1 = -\frac{r_{41}}{t}dt$

$\frac{\partial(m_1 t)}{\partial x_4} = \frac{x_4+x_1\cos 2\alpha}{r_{41}} - \cos 2\alpha = \frac{x_4+(x_1-r_{41})\cos 2\alpha}{r_{41}}$

and substitute into equation (1),

$$\frac{I}{2} = \int_0^l \int_{t_1}^{t_2} \frac{e^{-jk(m_1 t - 2x_4\sin^2\alpha)}}{t} dt = -\int_0^l e^{jk2x_4\sin^2\alpha} \int_{km_1t_1}^{km_1t_2} \frac{e^{-jt}}{t} dt\, dx_4 \quad (2)$$

now let

$u = \int_{km_1t_1}^{km_1t_2} \frac{e^{-jt}}{t}dt$, $du = \left[\frac{e^{-jkm_1t_2}}{km_1t_2}\frac{\partial(km_1t_2)}{\partial x_4} - \frac{e^{-jkm_1t_1}}{km_1t_1}\frac{\partial(km_1t_1)}{\partial x_4}\right]dx_4$

$dV = -e^{jk2x_4\sin^2\alpha}dx_4$, $V = -\frac{1}{j2k\sin^2\alpha}e^{jk2x_4\sin^2\alpha}$

and substitute into equation (2),

$$\frac{I}{2} = -\frac{1}{j2k\sin^2\alpha}\left[e^{jk2x_4\sin^2\alpha}\left\{(Ci\,km_1t_2 - Ci\,km_1t_1) - j(Si\,km_1t_2 - Si\,km_1t_1)\right\}\right]_{a\tan\alpha}^{l}$$
$$+ \frac{1}{j2k\sin^2\alpha}\int_{a\tan\alpha}^{l} e^{jk2x_4\sin^2\alpha}\left[\frac{e^{-jkm_1t_2}}{m_1t_2}\frac{\partial(m_1t_2)}{\partial x_4} - \frac{e^{-jkm_1t_1}}{m_1t_1}\frac{\partial(m_1t_1)}{\partial x_4}\right]dx_4 \quad (3)$$

with

$m_1t_1\Big]_{a\tan\alpha} = \frac{2a\sin^3\alpha}{\cos\alpha}$, $m_1t_1\Big]_l = 2l\sin^2\alpha$

$$m_1 t_2\Big]_{a\tan\alpha} = (l + x_4\cos 2\alpha)\left[1 + \frac{(x_4 \sin 2\alpha)^2}{2(l + x_4\cos 2\alpha)^2}\right]_{a\tan\alpha} - \Big[l + x_4\cos 2\alpha\Big]_{a\tan\alpha} = \frac{2a^2\sin^4\alpha}{l}$$

$$m_1 t_2\Big]_l = \sqrt{2l^2 + 2l^2\cos 2\alpha} - l(1+\cos 2\alpha) = 2l\cos\alpha(1-\cos\alpha)$$

$$\frac{1}{m_1 t_1}\frac{\partial(m_1 t_1)}{\partial x_4} = \frac{2\sin^2\alpha}{2x_4\sin^2\alpha} = \frac{1}{x_4}$$

$$\frac{1}{m_1 t_2}\frac{\partial(m_1 t_2)}{\partial x_4} = \frac{x_4 + (l - r_{4l})\cos 2\alpha}{r_{4l}(r_{4l} - l - x_4\cos 2\alpha)}$$

Let

$$A_{21} = \int_{a\tan\alpha}^{l} e^{j2kx_4\sin^2\alpha}\left[\frac{e^{-jkm_1t_2}}{m_1 t_2}\frac{\partial(m_1 t_2)}{\partial x_4} - \frac{e^{-jkm_1t_1}}{m_1 t_1}\frac{\partial(m_1 t_1)}{\partial x_4}\right]dx_4 \qquad (4)$$

and substitute from above,

$$A_{21} = \int_{a\tan\alpha}^{l}\frac{e^{-jk(r_{4l}-l-x_4)}\left[x_4 + (l - r_{4l})\cos 2\alpha\right]}{r_{4l}(r_{4l} - l - x_4\cos 2\alpha)}dx_4 - \int_{a\tan\alpha}^{l}\frac{dx_4}{x_4} \qquad (5)$$

Now let A_{22} equal the second term and A_{23} equal the first term in equation (5),

$$A_{22} = \int_{a\tan\alpha}^{l}\frac{dx_4}{x_4} = \ln\frac{a\tan\alpha}{l} \qquad (6)$$

$$A_{23} = \int_{a\tan\alpha}^{l}\frac{e^{-jk(r_{4l}-l-x_4)}\left[x_4 + (l - r_{4l}\cos 2\alpha)\right]}{r_{4l}(r_{4l} - l - x_4\cos 2\alpha)}dx_4 \qquad (7)$$

Let

$$m_2 y = r_{4l} - (x_4 + l\cos 2\alpha), \qquad m_2 = l\sin 2\alpha$$

$$m_2 y_1 = l(1-\cos 2\alpha) = 2l\sin^2\alpha$$

$$m_2 y_2 = 2l\cos\alpha - 2l\cos^2\alpha = 2l\cos\alpha(1-\cos\alpha)$$

$$m_2\,dy = \left[\frac{x_4 + l\cos 2\alpha}{r_{4l}} - 1\right]dx_4 = -\frac{m_2 y\,dx_4}{r_{4l}}, \quad dx_4 = -\frac{r_{4l}}{y}dy$$

$$r_{4l} - l - x_4 = m_2 y + l(\cos 2\alpha - 1) = m_2 y - 2l\sin^2\alpha$$

$$\frac{m_2}{y} = \frac{l^2\sin^2 2\alpha}{r_{4l} - (x_4 + l\cos 2\alpha)} = \frac{l^2\sin^2 2\alpha\left[r_{4l} + (x_4 + l\cos 2\alpha)\right]}{l^2\sin^2 2\alpha}$$

$$\frac{m_2}{y} = r_{4l} + (x_4 + l\cos 2\alpha)$$

$$r_{4l} = \frac{m_2}{2y}(y^2+1)$$

$$x_+ = r_{4l} - l\cos 2\alpha - m_2 y = \frac{m}{2y}(y^2+1) - m_2 y - l\cos 2\alpha = -\left[\frac{m_2}{2y}(y^2-1) + l\cos 2\alpha\right]$$

Substitute into equation (7),

$$A_{23} = e^{j2Kl\sin^2\alpha} \int_{y_1}^{y_2} \frac{e^{-jKm_2 y}}{y} \frac{\left[\frac{m_2}{2y}(y^2-1) + \frac{m_2}{2y}(y^2+1)\cos 2\alpha\right] dy}{\frac{m_2}{2y}(y^2+1) - l + l\cos^2 2\alpha + \frac{m_2}{2y}(y^2-1)\cos 2\alpha}$$

$$= e^{j2Kl\sin^2\alpha} \int_{y_1}^{y_2} \frac{e^{-jKm_2 y}}{y} \frac{(y^2\cos^2\alpha - \sin^2\alpha)}{\left[\frac{1}{2}(y^2+1) - \frac{l y \sin^2 2\alpha}{l \sin 2\alpha} + \frac{1}{2}(y^2-1)\cos 2\alpha\right]}$$

$$= e^{j2Kl\sin^2\alpha} \int_{y_1}^{y_2} \frac{e^{-jKm_2 y}}{y} \frac{y^2 - \tan^2\alpha}{y^2 - 2y\tan\alpha + \tan^2\alpha}\, dy$$

$$= e^{j2Kl\sin^2\alpha} \int_{y_1}^{y_2} e^{-jKm_2 y} \frac{y + \tan\alpha}{y(y-\tan\alpha)}\, dy$$

$$= e^{j2Kl\sin^2\alpha} \int_{y_1}^{y_2} e^{-jKm_2 y} \left(\frac{2}{y-\tan\alpha} - \frac{1}{y}\right) dy$$

$$A_{23} = 2\int_{\frac{2Ka\sin^3\alpha}{\cos\alpha}}^{2Kl(1-\cos\alpha)} \frac{e^{ju}}{u}\, du - e^{j2Kl\sin^2\alpha} \int_{2Kl\sin^2\alpha}^{2Kl\cos\alpha(1-\cos\alpha)} \frac{e^{-ju}}{u}\, du \qquad (8)$$

where

$$-u = Km_2(y - \tan\alpha) = Km_2 y - 2Kl\sin^2\alpha, \quad du = -Km_2\, dy$$

$$-jKm_2 y = ju - j2Kl\sin^2\alpha$$

$$-u_2 = Km_2 y_2 - 2Kl\sin^2\alpha = Kr_{ll} - K(l + l\cos 2\alpha) - 2Kl\sin^2\alpha = 2Kl(\cos\alpha - 1)$$

$$-u_1 = Km_2 y_1 - 2Kl\sin^2\alpha = Kr_{0l} - Kl\cos 2\alpha - 2Kl\sin^2\alpha$$

$$= Kr_{0l} - Kl = -\frac{2Ka\sin^3\alpha}{\cos\alpha}$$

Substituting A_{23} and A_{22} into A_{21} and subsequently into equation (3),

$$
\begin{aligned}
jK\sin^2\alpha\, I_2 = {} & \ln\cos\alpha - \ln(\gamma 2Kl\sin^2\alpha) + Ci[2Kl(1-\cos\alpha)] \\
& - \cos(2Kl\sin^2\alpha)\{Ci[2Kl\cos\alpha(1-\cos\alpha)] - Ci(2Kl\sin^2\alpha)\} \\
& - \sin(2Kl\sin^2\alpha)\{Si[2Kl\cos\alpha(1-\cos\alpha)] - Si(2Kl\sin^2\alpha)\} \\
& + j\Big[Si[2Kl(1-\cos\alpha)] \\
& + \cos(2Kl\sin^2\alpha)\{Si[2Kl\cos\alpha(1-\cos\alpha)] - Si(2Kl\sin^2\alpha)\} \\
& - \sin(2Kl\sin^2\alpha\{Ci[2Kl\cos\alpha(1-\cos\alpha)] - Ci(2Kl\sin^2\alpha)\}
\end{aligned}
\qquad (9)
$$

APPENDIX C

Given

$$I_3 = \int_0^l \int_{a\cot\alpha}^{l-a\cot\alpha} \frac{e^{-jk(r_{21}+x_1-x_2)}}{r_{21}} dx_2 dx_1 \qquad (1)$$

$$r_{21} = \sqrt{x_1^2 + x_2^2 - 2x_1x_2\cos 2\alpha} \quad \sqrt{(x_1-x_2\cos2\alpha)^2+(x_2\sin2\alpha)^2}$$

Let

$$m_1 t = r_{21} - (x_1 - x_2\cos 2\alpha), \quad m_1 = x_2 \sin 2\alpha, \quad r_{ll} = 2l\sin\alpha$$

$$m_1 t_1 = x_2 + x_2\cos 2\alpha = 2x_2\cos^2\alpha$$

$$m_1 t_2 = \sqrt{x_2^2 - 2lx_2\cos2\alpha + l^2} - (l - x_2\cos 2\alpha) = r_{2l} - l + x_2\cos2\alpha$$

$$m_1 t_1\Big]_{a\cot\alpha} = 2a\frac{\cos\alpha}{\sin\alpha}\cos^2\alpha = \frac{2a\cos^3\alpha}{\sin\alpha}$$

$$m_1 t_2\Big]_{a\cot\alpha} = \frac{(x_2\sin2\alpha)^2}{2l}\Big]_{a\cot\alpha} = \frac{4a^2(\cos^2\alpha)^2}{2l} = \frac{2a^2\cos^4\alpha}{l}$$

$$m_1 t_1\Big]_l = 2l\cos^2\alpha$$

$$m_1 t_2\Big]_l = r_{ll} - l(1-\cos2\alpha) = 2l\sin\alpha(1-\sin\alpha)$$

$$\frac{1}{m_1t_1}\frac{\partial(m_1t_1)}{\partial x_2} = \frac{1}{x_2}, \quad \frac{\partial(m_1 t)}{\partial x_2} = \frac{x_2 - x_1\cos2\alpha}{r_{21}} + \cos 2\alpha = \frac{x_2-(x_1-r_{21})\cos2\alpha}{r_{21}}$$

$$\frac{1}{m_1t_2}\frac{\partial(m_1t_2)}{\partial x_2} = \frac{x_2 + (r_{2l}-l)\cos2\alpha}{r_{2l}(r_{2l}-l+x_2\cos2\alpha)}$$

$$m_1 dt = \left(\frac{x_1 - x_2\cos2\alpha}{r_{21}} - 1\right)dx_1 = -\frac{m_1 t}{r_{21}}dx_1, \quad dx_1 = -\frac{r_{21}}{t}dt$$

and substitute into equation (1),

$$I_3 = -\int_0^l\int_{t_1}^{t_2} \frac{e^{-jk(m_1t+2x_2\sin^2\alpha)}}{t} dt = -\int_0^l e^{-jk2x_2\sin^2\alpha}\int_{km_1t_1}^{km_1t_2}\frac{e^{-jt}}{t}dt\,dx_2 \qquad (2)$$

Let $u = \int_{km_1t_1}^{km_1t_2}\frac{e^{-jt}}{t}dt$, $du = \left[\frac{e^{-jkm_1t_2}}{m_1t_2}\frac{\partial(m_1t_2)}{\partial x_2} - \frac{e^{-jkm_1t_1}}{m_1t_1}\frac{\partial(m_1t_1)}{\partial x_2}\right]dx_2$

$$dv = -e^{-j2kx_2\sin^2\alpha}dx_2, \quad v = \frac{1}{j2k\sin^2\alpha}e^{-j2kx_2\sin^2\alpha}$$

Hence, upon substituting into equation (2),

$$I_3 = \frac{1}{j2K\sin^2\alpha}\left[e^{-j2Kx_2\sin^2\alpha}\left\{ Ci\,Km_1t_2 - Ci\,Km_1t_1 - j(Si\,Km_1t_2 - Si\,Km_1t_1)\right\}\right]_{a\cot\alpha}^{l} \tag{3}$$

$$-\frac{1}{j2K\sin^2\alpha}\int_0^l e^{-jK2x_2\sin^2\alpha}\left[\frac{e^{-jKm_1t_2}}{m_1t_2}\frac{\partial(m_1t_2)}{\partial x_2} - \frac{e^{-jKm_1t_1}}{m_1t_1}\frac{\partial(m_1t_1)}{\partial x_2}\right]dx_2$$

Let

$$A_{31} = \int_0^l e^{-j2Kx_2\sin^2\alpha}\left[\frac{e^{-jKm_1t_2}}{m_1t_2}\frac{\partial(m_1t_2)}{\partial x_2} - \frac{e^{-jKm_1t_1}}{m_1t_1}\frac{\partial(m_1t_1)}{\partial x_2}\right]dx_2 \tag{4}$$

$$= \int_0^l \frac{e^{-jK(r_{2l}-l+x_2)}\left[x_2+(r_{2l}-l)\cos 2\alpha\right]}{r_{2l}\left[r_{2l}-l+x_2\cos 2\alpha\right]}dx_2 - \int_{a\cot\alpha}^l \frac{e^{-jKx_2}}{x_2}dx_2 \tag{5}$$

In equation (5), let

$$A_{32} = \int_{a\cot\alpha}^l \frac{e^{-jKx_2}}{x_2}dx_2 = \ln(\gamma 2Ka\cot\alpha) - \left[Ci\,2Kl - j\,Si\,2Kl\right] \tag{6}$$

$$A_{33} = \int_0^l \frac{e^{-jK(r_{2l}-l+x_2)}\left[x_2+(r_{2l}-l)\cos 2\alpha\right]}{r_{2l}\left[r_{2l}-l+x_2\cos 2\alpha\right]}dx_2 \tag{7}$$

Let

$$m_2y = r_{2l} + (x_2 - l\cos 2\alpha),\quad m_2 = l\sin 2\alpha,\quad r_{2l} = \sqrt{(x_2-l\cos 2\alpha)^2+(l\sin 2\alpha)^2}$$

$$m_2y_1\Big] = (l - x_2\cos\alpha)\Big]_{a\cot\alpha} + \frac{(2x_2\sin 2\alpha)^2}{2l}\Big]_{a\cot\alpha} + (x_2 - l\cos 2\alpha)\Big]_{a\cot\alpha} = Ka\sin 2\alpha$$

$$m_2y_2\Big] = r_{ll} + l(1-\cos 2\alpha) = 2l\sin\alpha + 2l\sin^2\alpha = 2l\sin\alpha(1+\sin\alpha)$$

$$m_2dy = \left[\frac{x_2 - l\cos 2\alpha}{r_{2l}} + 1\right]dx_2 = \frac{m_2y}{r_{2l}}dx_2,\qquad dx_2 = \frac{r_{2l}}{y}dy$$

$$r_{2l} = \frac{m_2}{2}\left(y+\frac{1}{y}\right) = \frac{m_2}{2y}(y^2+1)$$

$$x_2 = m_2y - r_{2l} + l\cos 2\alpha = m_2y - \frac{m_2}{2y}(y^2+1) + l\cos 2\alpha$$

$$= \frac{m_2}{2y}(y^2-1) + l\cos 2\alpha$$

Substituting into equation (7),

$$A_{33} = e^{j2Kl\sin^2\alpha}\int_{y_1}^{y_2}\frac{e^{-jKm_2y}}{y}\,\frac{\frac{m_2}{2y}(y^2-1) + \frac{m_2}{2y}(y^2+1)\cos 2\alpha}{\left[\frac{m_2}{2y}(y^2+1) - l + l\cos^2 2\alpha + \frac{m_2}{2y}(y^2-1)\cos 2\alpha\right]}dy$$

$$= e^{j2Kl\sin^2\alpha}\int_{y_1}^{y_2}\frac{e^{-jKm_2y}}{y}\,\frac{y^2\cos^2\alpha - \sin^2\alpha}{\frac{1}{2}(y^2+1) - \frac{(l\sin^2 2\alpha)}{l\sin 2\alpha}y + \frac{1}{2}(y^2-1)\cos 2\alpha}dy$$

$$A_{33} = e^{j2kl\sin^2\alpha} \int_{y_1}^{y_2} \frac{e^{-jkm_2 y}(y^2 - \tan^2\alpha)}{y^2 - 2y\tan\alpha + \tan^2\alpha}\, dy$$

$$= e^{-j2kl\sin^2\alpha} \int_{y_1}^{y_2} e^{-jkm_2 y}\left(\frac{2}{y-\tan\alpha} - \frac{1}{y}\right) dy$$

$$= 2\int_{ka\sin 2\alpha}^{2kl\sin\alpha} \frac{e^{-ju}}{u}\, du - e^{j2kl\sin^2\alpha} \int_{2kl\sin^2\alpha}^{2kl\sin\alpha(1+\sin\alpha)} \frac{e^{-ju}}{u}\, du \quad (8)$$

upon substituting A_{33} and A_{32} into A_{31} and subsequently into equation (3),

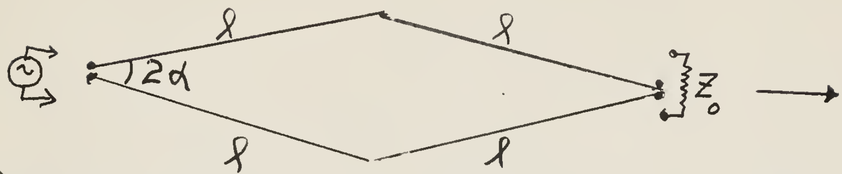
$$\begin{aligned}
j2k\sin^2\alpha\, I_3 &= \ln(\gamma 2kl\sin^2\alpha) + Ci\,2kl - 2Ci(2kl\sin\alpha) \\
&+ \cos(2kl\sin^2\alpha)\{Ci[2kl\sin\alpha(1+\sin\alpha)] + Ci[2kl\sin\alpha(1-\sin\alpha)] - Ci(2kl\cos^2\alpha) - Ci(2kl\sin^2\alpha)\} \\
&+ \sin(2kl\sin^2\alpha)\{Si[2kl\sin\alpha(1+\sin\alpha)] - Si[2kl\sin\alpha(1-\sin\alpha)] + Si(2kl\cos^2\alpha) - Si(2kl\sin^2\alpha)\} \\
&+ j\Big[2Si(2kl\sin\alpha) - Si\,2kl \\
&- \cos(2kl\sin^2\alpha)\{Si[2kl\sin\alpha(1+\sin\alpha)] + Si[2kl\sin\alpha(1-\sin\alpha)] - Si(2kl\cos^2\alpha) \\
&\qquad - Si(2kl\sin^2\alpha)\} \\
&+ \sin(2kl\sin^2\alpha)\{Ci[2kl\sin\alpha(1+\sin\alpha)] - Ci[2kl\sin\alpha(1-\sin\alpha)] + Ci(2kl\cos^2\alpha) \\
&\qquad - Ci(2kl\sin^2\alpha)\}\Big]
\end{aligned} \quad (9)$$

Substituting into

$$Z_r = -j120k\sin^2\alpha\,(I_1 + I_2 - 2I_3)$$

one gets the radiation impedance of a rhombic antenna in free space.

FREE SPACE RADIATION IMPEDANCE OF RHOMBIC ANTENNA



$$\ln \gamma = C = 0.5772, \quad Si\,x = \int_0^x \frac{\sin u}{u}\,du, \quad Ci\,x = -\int_x^\infty \frac{\cos u}{u}\,du$$

$$\begin{aligned}
Z_\pi/120 = {} & 2C + 2\ln(2k\ell\sin^2\alpha) + 2\,Ci(2k\ell) - 2\,Ci(2k\ell\sin\alpha) - Ci[2k\ell(1+\cos\alpha)] - Ci[2k\ell(1-\cos\alpha)] \\
& + \cos(2k\ell\sin^2\alpha)\{Ci[2k\ell\cos\alpha(1+\cos\alpha)] + Ci[2k\ell\cos\alpha(1-\cos\alpha)] + Ci[2k\ell\sin\alpha(1+\sin\alpha)] \\
& \quad + Ci[2k\ell\sin\alpha(1-\sin\alpha)] - 2\,Ci(2k\ell\cos^2\alpha) - 2\,Ci(2k\ell\sin^2\alpha)\} \\
& - \sin(2k\ell\sin^2\alpha)\{Si[2k\ell\cos\alpha(1+\cos\alpha)] - Si[2k\ell\cos\alpha(1-\cos\alpha)] - Si[2k\ell\sin\alpha(1+\sin\alpha)] \\
& \quad + Si[2k\ell\sin\alpha(1-\sin\alpha)] - 2\,Si(2k\ell\cos^2\alpha) + 2\,Si(2k\ell\sin^2\alpha)\} \\
& + j\Big[Si[2k\ell(1+\cos\alpha)] - Si[2k\ell(1-\cos\alpha)] + 2\,Si(2k\ell\sin\alpha) - 2\,Si(2k\ell) \\
& - \cos(2k\ell\sin^2\alpha)\{Si[2k\ell\cos\alpha(1+\cos\alpha)] + Si[2k\ell\cos\alpha(1-\cos\alpha)] + Si[2k\ell\sin\alpha(1+\sin\alpha)] \\
& \quad + Si[2k\ell\sin\alpha(1-\sin\alpha)] - 2\,Si(2k\ell\cos^2\alpha) - 2\,Si(2k\ell\sin^2\alpha)\} \\
& - \sin(2k\ell\sin^2\alpha)\{Ci[2k\ell\cos\alpha(1+\cos\alpha)] - Ci[2k\ell\cos\alpha(1-\cos\alpha)] - Ci[2k\ell\sin\alpha(1+\sin\alpha)] \\
& \quad + Ci[2k\ell\sin\alpha(1-\sin\alpha)] - 2\,Ci(2k\ell\cos^2\alpha) + 2\,Ci(2k\ell\sin^2\alpha)\}\Big]
\end{aligned}$$

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