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FEEDBACK CONTROL SYSTEMS: DESIGN WITH REGARD TO SENSITIVITY.

UNITED STATES
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FEEDBACK CONTROL SYSTEMS:

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ABSTRACT

The purpose of using feedback in a control system is not merely to improve its static and dynamic performance and eliminate or minimize the effect of noise, but also to eliminate or minimize the effect of unpredictable changes within the plant itself. Such changes are expressed in terms of variations in the plant's parameters, i.e., gain constant, plant poles, plant zeros.

This paper starts with a definition of "root sensitivity", relating the changes in plant's parameters to corresponding changes in the system's roots. Interesting properties of root-sensitivity are shown, then applied to the derivation of a laborless graphical method for obtaining the sensitivity of a given root. Finally a compensation design method is proposed, which not only secures a desired location for the system's dominant roots on the s-plane, but also simultaneously satisfied conditions concerning the sensitivity of these dominant roots to the varying plant parameter (ters). Examples are solved using the proposed method, and the results verified with the analog computer.

CHAPTER ONE

Root Sensitivity to Parameter Changes

I - 1: Introduction

It is well established that the reasons for wrong feedback in control systems can be classified as follows:

1 - To improve static and dynamic performance of the system. Feedback can stabilize an unstable system or increase the stability of a stable system. Feedback can shape the system response in to some desired pattern. Feedback can reduce the steady-state error of a class of control systems. This aspect of feedback has been dealt with abundantly in the past, and well known results may be found in the literature as well as textbooks.

2 - To minimize the effects of man's ignorance of the plant's environment. What man cannot predetermine in the plant's environment is commonly referred to as disturbance or noise. Feedback in fact reduces the effect of this ignorance on the system output to an acceptable value. This side of the problem also has been carefully investigated and the related results well established.

3 - To minimize the effects of man's ignorance of the plant itself. Man's ignorance of the plant, which he wishes to control, can be of various categories. Some plants cannot be readily analyzed and a mathematical model cannot be readily obtained. Such is the case for problems in biology, medicine or other natural sciences. Other plants are easier to analyze, but a rigid mathematical model is difficult to obtain due to the changing nature of the plant, resulting in varitions in the plant's parameters. Such is the case with problems in economy, industry, or management. Such is also the case with a large number of engineering problems, of which a few examples will be given in the next section. Plant para-

parameter variations result either from the basic nature of the plant itself (chemical processes), or from environmental changes (climatic and other ambient conditions).

Variations of system's response as related to plant parameter changes are expressed by the "sensitivity" of the system. The sensitivity-reduction aspect of feedback is the purpose of this report. This chapter introduces the notion of root-sensitivity and its properties. Chapter Two will make use of them in a design procedure.

At this point, it seems necessary to classify plant parameter changes into two kinds: incremental or small parameter changes, and large parameter changes. The technique for treating each class of problem is different and no extension from one class to the other seems to be possible nor recommendable. Examples for small parameter changes can be found in many situations: chemical processes where the speed of various chemical reactions changes with pressure, with ambient temperature, humidity; electronic circuitry where component values change with temperature or aging; rotating generators where small changes in the field resistance cause proportional changes in the voltage gain as well as in the time constant; pneumatic or hydraulic systems in which fluid properties change with temperature and aging; mechanical systems in which friction, spring characteristics, etc., are far from being constant.

Larger parameter changes, on the other hand, are common in a number of other problems, ranging from automatic steel rolling mills where the thickness of the slab varies within wide limits, paper mills where roll diameter starts from zero and ends up at its maximum value, to the more recent problems of missle and space technology, where the vehicles are called upon to function at extreme environmental conditions, with wild changes in mass due to the burning out of fuel.

This report will be concerned with analysis and synthesis methods for problems with small parameter changes.

I - 2: System sensitivity and root sensitivity:

Several definitions of sensitivity have been used in the past. The first one, as far as is known to the author, is by Bode[1], defining

sensitivity of the overall transfer function T to the gain constant K as:

$$S_K^T \triangleq \frac{\frac{dK}{K}}{\frac{dT}{T}} = \frac{\partial \ln K}{\partial \ln T}$$

Horowitz took the inverse of Bode's definition:

$$S_K^T \triangleq \frac{\frac{dT}{T}}{\frac{dK}{K}}$$

Defined one way or the other, S_K^T is generally known under the name of "classical sensitivity" or more suggestively "system sensitivity" since it relates the change in system transfer function to the change in parameter K.

Another kind of sensitivity is based on the location of system's dominant roots. Such a sensitivity relates the change in q_i (i^{th} dominant root of the closed-loop system) to the change in x; where x may be the gain constant, or an open-loop zero, or an open-loop pole of the plant. The sensitivity thus defined is known as "root-sensitivity".

Formal definitions of root-sensitivity vary from author to author. Horowitz[10] and Ur[2] defined the sensitivity of closed-loop root q_i with respect to parameter x(where x may be gain constant, or pole, or zero) as:

$$S_x^i \triangleq \frac{\partial q_i}{\frac{\partial x}{x}} \qquad (1)$$

Huang[3], on the other hand, used:

$$S_x^i \triangleq \frac{\frac{\partial q_i}{q_i}}{\frac{\partial x}{x}} \qquad (2)$$

More recently, McRuer and Stapleford[4] prefer different definitions for sensitivity to gain (K), and sensitivity to poles or zeros (x):

$$S_K^i \triangleq \frac{\partial q_i}{\frac{\partial K}{K}} \qquad (3)$$

$$S_x^i \triangleq \frac{\partial q_i}{\partial x} \qquad (4)$$

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It will be shown in Section I-5 that definitions (3) and (4) are most suitable for the work presented here, and therefore will be adopted.

1 - 3: Survey of previous works and scope of this chapter:

a) As far as large parameter variations are concerned, the most significant work known to the author is Horowitz's book[10] in which an extensive treatment of passive-adaptation is given, concerning systems with one or more parameters varying simultaneously and independently within wide ranges. Horowitz's methods are mainly based on frequency response, and since

$$\frac{T_o}{T_f} = \frac{L_o + \frac{P_o}{P_f}}{L_o + 1}$$

(L =loop transfer function. Subscript o means original value, f means final value), the problem is to select L_o $(j\omega)$ so as to achieve tolerances on $\frac{T_o}{T_f}$, despite the variations in $\frac{P_o}{P_f}$. This is called "loop shaping" of L_o.

Variations in P are represented on the polar plane as an area (section 3.5, reference 10). As a consequence, the method becomes impractical for more than 2 changing parameters. Horowitz's work extends well beyond the limits of the sensitivity problem alone, but in the treatment of the latter his certainly is one of the most valuable contributions up to the present time.

Along the same passive-adaptive line is the recent work of Liu, Han and Thaler[5]. For a second order system with tachometer feedback, the three parameters are gain K, open-loop pole p, and tachometer gain K_t. A graphical method is proposed to determine the optimal values for K and K_t, when p changes, in order to maintain the damping-ratio ζ within a certain limit. When K changes, p and K_t are similarly determined graphically. The procedure is also extended to third order systems. This is the economical way to solve the problem, using to its best the limited amount of passive-adaptation inherent to any feedback systems.

On the other hand, in many cases, passive-adaptation may not be sufficient and one must have recourse to active adaptation, which has been the subject of a profuse literature. Mention must be made of the APRACS tech-

nique, for "Amplitude and Phase Regulated Adaptive control systems", and the recent work of Horton and Elsner[6], who propose a method whereby the system's dominant poles are maintained fixed despite changes in the plant P(s). In order to do so, gain and phase of the controller C (s) must change in such a way as to compensate for similar changes in P. A test signal is injected into the system and the output measured. Amplitude and phase of such output are compared with the input test signal. Differences are used as driving force to adjust gain and phase of C(s) in order to null the affects of changes in P.

b) Turning next to small parameter changes, a great deal of work has been done in the recent past concerning the analysis of the problem but so far no significant effort has been spent on synthesis. Ur[2] derived interesting root-locus properties and proposed a graphical method for evaluation of S_k^i. Huang[3] showed by a number of examples the usefulness of root-sensitivity in a wide variety of analysis problems. McRuer and Stapleford[4] derived interesting properties of root sensitivity and worked out various graphical and analytical methods for computing S_x^i, not all of which are practical.

Considering what has been done in the past, the remainder of this chapter will be devoted to a study of root-sensitivity properties, and in the next chapter, use will be made of these results to formulate a design method.

In the literature mentioned above as well as in what follows, emphasis is laid on the location of dominant system roots. One may argue on the validity of such a philosophy when applied to synthesis, since nothing guarantees that dominant roots remain dominant after the system has been compensated. In practice, however, it usually happens that if any extra root is introduced by the compensation, either it is far away enough to be negligible, or it will be close enough to a system zero, so that its effect on the transient is thereby cancelled. In case of doubt, however, it is advisable to perform an analytical or analog-computer check after a solution has been obtained, in order to make sure it does satisfy the specifications.

1 - 4: Root-sensitivity: definition

In this section, it will be shown how a definition of root-sensitivity is arrived at. In the next three sections, some important properties of root-sensitivity are derived. Let P be the transfer function of the plant to be controlled, C that of the cascade controller, and F that of the feedback controller(fed back around C and P). We define G =PC as forward transfer function, and L =GF =PCF as loop transfer function of the system. Then the system characteristic equation is

$$1 + L = 0$$

and if q_i is a systems root, then

$$1 + L(s)\Big|_{s=-q_i} = 0$$

If K is the gain constant of L(s), and z_j, p_j its open-loop zeros and poles, then one can write:

$$L = L\,(s,k,z_j,p_j)$$

and take the total differential of L:

$$dL = \frac{\partial L}{\partial s}\,ds + \frac{\partial L}{\partial K}\,dK + \sum_{j=1}^{n} \frac{\partial L}{\partial z_j}\,dz_j + \sum_{j=1}^{m} \frac{\partial L}{\partial p_j}\,dp_j$$

On a root locus, L =-1 =constant, ie: the total differential dL is zero for $s = -q_i$. Let dL = 0 and $s = q_i$ in the above equation, this gives:

$$0 = -\frac{\partial L}{\partial s}\Big|_{s=-q_i} dq_i + \frac{\partial L}{\partial K}\Big|_{s=-q_i} dK + \sum \frac{\partial L}{\partial z_j}\Big|_{s=-q_i} dz_j + \sum \frac{\partial L}{\partial p_j}\Big|_{s=-q_i} dp_j$$

Rearranging:

$$dq_i = \frac{1}{\left(\partial L/\partial s\right)_{s=-q_i}} \left[\left(\frac{\partial L}{\partial K}\right)_{s=-q_i} dK + \sum \left(\frac{\partial L}{\partial z_j}\right)_{s=-q_i} dz_j + \sum \left(\frac{\partial L}{\partial p_j}\right)_{s=-q_i} dp_j \right] \qquad (5)$$

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But q_i itself is a function of K, z_j and p_j

$$q_i = q_i\,(K,\, z_j,\, p_j).$$

Taking the total differential of q_i:

$$dq_i = K\,\frac{\partial q_i}{\partial K}\,\frac{dK}{K} + \sum_j \frac{\partial q_i}{\partial z_j}\,dz_j + \sum_j \frac{\partial q_i}{\partial p_j}\,dp_j \qquad (6)$$

Equation (6) suggests that dq_i be written as:

$$dq_i = S_K^i\,\frac{dK}{K} + \sum S_{zj}^i\,dz_j + \sum S_{pj}^i\,dp_j \qquad (7)$$

Equations (6) and (7) thus define the sensitivity of root q_i to gain K as:

$$\boxed{S_k^i \triangleq \frac{\partial q_i}{\frac{\partial K}{K}}} \qquad (8)$$

and the sensitivity of root q_i to open-loop zero z_j as:

$$\boxed{S_{zj}^i \triangleq \frac{\partial q_i}{\partial z_j}} \qquad (9)$$

and the sensitivity of root q_i to open-loop pole p_j as:

$$\boxed{S_{pj}^i \triangleq \frac{\partial q_i}{\partial p_j}} \qquad (10)$$

Equations (8) through (10) are the same as definitions in equations (3) and (4) given earlier, used by McRuer and Stapleford. S_K^i relates the change in q_i with the corresponding <u>percent</u> change in gain K, while S_{zj}^i and S_{pj}^i relate the change in q_i with the <u>total</u> change in z_j or p_j. There is no reason why other definitions cannot be adopted. It is just a matter of convenience.

I - 5: Property 1: Relationships between S_K^i, S_z^i and S_p^i

Comparison of equations (5) and (7) yields:

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$$[illegible] \qquad (\ldots)$$

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$$S_K^i = \left(\frac{\partial L/\partial K}{\partial L/\partial K}\right)_{s=-q_i} \tag{11}$$

$$S_{zj}^i = \left(\frac{\partial L/\partial z_j}{\partial L/\partial s}\right)_{s=-q_i} \tag{12}$$

$$S_{pj}^i = \left(\frac{\partial L/\partial p_i}{\partial L/\partial s}\right)_{s=-q_i} \tag{13}$$

But

$$L = K \frac{\prod_1^n (s + z_j)}{\prod_1^m (s + p_j)}$$

Then

$$\left(\frac{\partial L}{\partial K}\right)_{s=-q_i} = \left.\frac{\prod_1^n (s + z_j)}{\prod_1^m (s + p_j)}\right|_{s=-q_i} = \left(\frac{L}{K}\right)_{s=-q_i} = \frac{-1}{K}$$

Then (11) becomes

$$S_K^i = \frac{-1}{\left(\frac{\partial L}{\partial s}\right)_{s=-q_i}} \tag{14}$$

Similar derivation for $\left(\frac{\partial L}{\partial z_j}\right)$ and $\left(\frac{\partial L}{\partial p_j}\right)$ leads to:

$$\boxed{S_{z_j}^i = \frac{S_K^i}{z_j - q_i}} \tag{15}$$

$$\boxed{S_{p_j}^i = \frac{S_K^i}{q_i - p_j}} \tag{16}$$

Equations (15) and (16) show the convenience of the definitions used. The sensitivities to all singularities are directly proportional to S_K^i, and inversely proportional to the distance between q_i and the singularity involved. Thus, whatever properties are found for S_K^i may be extended to S_z^i or S_p^i. A particular case of equation (16) is for $P_o = 0$, ie: the root-sensitivity to the pole at origin of the s-plane.

$$S^i_{P_o} = \frac{S^i_K}{q_i} \qquad (17)$$

$S^i_{P_o}$ is proportional to S^i_K, the constant of proportionality being $\frac{1}{q_i}$ (complex quantity).

1 - 6: Property 2: relationship between root-sensitivity of $-q_i$ and residue at $-q_i$.

In section I-5 it has been shown that S^i_K, S^i_z and S^i_p are all proportional. In this section it will be shown that, if q_i is a single system root, then

$$\boxed{S^i_K = F_{-q_i} Q_i} \qquad (18)$$

where $F(-q_i)$ is the feedback transfer function F evaluated at $s = -q_i$, and Q_i is the residue of the system transfer function at q_i. For unity feedback, (18) very simply becomes:

$$\boxed{S^i_K = Q_i} \qquad (19)$$

Finally it will be shown that when q_i is an N^{th} order pole, (18) becomes:

$$\boxed{S^i_K = (-1)^{N-1} F(-q_i) Q_{in}} \qquad (20)$$

For unity feedback it becomes:

$$\boxed{S^i_K = (-1)^{N-1} Q_{in}} \qquad (21)$$

The remainder of this section, is concerned with proofs of equations (18) and (20).

The overall transfer function is

$$T(s) = \frac{P(s)\,C(s)}{1 + L(s)} = \frac{L(s)}{F(s)[1 + L(s)]}$$

The residue of T(s) at q_i is

$$Q_i = (s + q_i)\,T(s)\Big|_{s=-q_i} = \frac{(s + q_i)\,L(s)}{F(s)\,[1 + L(s)]}\Bigg|_{s=-q_i}$$

Let the rightmost expression be denoted as R_i, ie, by definition

$$R_i\ (-q_i) = Q_i \tag{22}$$

$$F\ (1 + L)\ R_i = (s + q_i)\ L$$

Take the drivative with respect to s of both sides:

$$\frac{\partial F}{\partial s}\,[1 + L]\ R_i + FR_i\,\frac{\partial L}{\partial s} + F\,[1 + L]\,\frac{\partial R_i}{\partial s} = L + (s + q_i)\,\frac{\partial L}{\partial s}$$

At $s = -q_i$, ie: at a point on the root locus, L = -1 and the above equation reducees to:

$$FR_i\,\frac{\partial L}{\partial s}\bigg|_{s = -q_i} = -1$$

or:

$$R\ (-q_i) = \frac{-1}{F(-q_i)\left(\frac{\partial L}{\partial s}\right)_{s = -q_i}} \tag{23}$$

Compare (14) with (23) and obtain equation (18) which is thereby proved.

Turning now to the case of N^{th} order root at q_i, theory of Heavyside's partial fraction gives:

$$T\ (s) = \frac{Q_{i1}}{s + q_i} + \frac{Q_{i2}}{(s + q_i)^2} + \ldots + \frac{Q_{iN}}{(s + q_i)^N} + \text{terms from other roots} \tag{24}$$

where

$$Q_{ik} = \frac{1}{(N - k)!}\left[\frac{\partial^{N-k}}{\partial s^{n-k}}\left[\frac{(s + q_i)L}{F(1 + L)}\right]\right]_{s = -q_i} \tag{25}$$

Again defining the quantity inside the small bracket as R_i, then repeating the operations as for equation (22) above, one obtains after repeated differentiation:

$$Q_{iN} = \frac{-\ N!}{F(-q_i)\ \frac{\partial^N L}{\partial s^N}\Big|_{s = -q_i}} \tag{26}$$

The next step in the derivation of equation (20) is to obtain the equivalent of equation (14) for the case of N^{th} order root at $-q_i$. In the repeated differentiations leading to equation (26), it is found that the derivatives of L:

$$\left.\frac{\partial^k L(s)}{\partial s^k}\right|_{s=-q_i} = 0 \quad \text{for } 1 \leqslant k \leqslant N-1 \tag{27}$$

This means that for $N \geqslant 2$, $\left(\frac{\partial L}{\partial s}\right)_{s=-q_i} = 0$ and the original definition of root sensitivity as by equation (14) becomes infinite.

In order to avoid this difficulty, a more suitable definition is suggested by writing an expansion of the total differential dL to include higher order terms, then retain only the lowest order terms for each parameter and at the following equation, counterpart of equation (7).

$$dq_i = \left[S_K^i \frac{dK}{K} + \sum_j S_{z_j}^i \, dz_j + \sum_j S_{p_j}^i \, dp_j \right]_{s=-q_i}^{\frac{1}{N}} \tag{28}$$

from which:

$$S_K^i = \frac{(-1)^N N!}{\left(\frac{\partial^N L}{\partial s^N}\right)_{s=-q_i}} \tag{29}$$

which is the counterpart of equation (14). Complete derivation of the above may be found on Appendix 1. Combining equations (26) and (29) directly yields equation (20) which is thereby proved.

Appendix 2 shows that equations (15) and (16), which relate S_K^i with S_z^i and S_p^i, are still valid when q_i is a N^{th} order system root.

I - 7: Property 3: Sum of all $S_{z_j}^i$ and $S_{p_j}^i$ in a system.

When $-q_i$ is a single-order system root, the sum of the sensitivities

of $-q_i$ to all open-loop zeros and poles is equal to 1.

$$\boxed{\sum_j S^i_{z_j} + \sum_j S^i_{p_j} = 1} \qquad (30)$$

This is easily seen by referring to the construction of root loci. If all open-loop zeros and poles are displaced by the same amount δ, then all closed-loop roots are displaced by the same amount, i.e., if $dz_j = dp_j = \delta$ for all j, then $dq_i = \delta$ for all i. This interesting property will be of great utility later on.

When $-q_i$ is a N^{th} order system root, setting dK = 0 in equation (28), and using the same reasoning as above, i.e., shifting all open-loop zeros and poles by δ, one obtains

$$dq_i = \left[\left(\sum_j S^i_{z_j} + \sum_j S^i_{p_j}\right)\delta\right]^{\frac{1}{N}}$$

But closed-loop roots shift by the same amount δ. Then the above equation becomes:

$$\sum_j S^i_{z_j} + \sum_j S^i_{p_j} = \delta^{N-1}$$

which no longer has a universal character as equation (30) since it depends on the magnitude of shift δ.

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CHAPTER TWO

A Sensitivity Design Method

II - 1: Introduction

In chapter one, a number of properties of the root-sensitivity to gain, poles and zeros have been derived. In particular it was shown (equation (30)) that for any system, the sum of the sensitivities of a system root q_i to each and every open-loop singularity, is always equal to unity. It was also shown (equations 15,16) that the root-sensitivity to each singularity is directly proportional to the root-sensitivity to gain, and inversely proportional to the distance from the root to the singularity involved.

It is now desired to apply these results to a number of design problems where specifications include conditions on the sensitivity of the dominant roots. These specifications may be in the form of an upper limit for the magnitude of the sensitivity of the dominant root, or for the change of damping factor, or the change of natural frequency and bandwith, when gain and/or singularity (ties) of the plant vary with time.

This chapter will be presented in the following sequence. The practical aspect of problems with small parameter changes is discussed first. Then a graphical method to obtain root-sensitivity values is formulated and other properties of sensitivity are derived therefrom. Finally a design procedure is presented and applied to several examples.

II - 2: Practical aspects of problems with small parameter changes.

In section I - 1 a number of situations where small parameter changes frequently occur have been mentioned. A desirable quality of control systems is undoubtedly the reliability of their response under varying operating conditions, and perhaps one of the most objectionable shortcomings is the unpredictable variations in system response, variations due to the combined effects of small changes in the plant gain or time constants or both.

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The question then arises as to when the plant gain is affected and when the plant time constants are, and whether they affect each other mutually. There is no unique answer to this question, and for each individual problem, an analysis is needed to determine, from physical situations, what parameters are changed and what is the extent of the change.

A simple example may be found in the amplidyne whose transfer function is

$$\frac{e_o}{e_c} = \frac{\dfrac{k_q k_i}{L_q L_c}}{\left(s + \dfrac{r_q}{L_q}\right)\left(s + \dfrac{r_c}{L_c}\right)} \equiv \frac{K}{(s + p_1)(s + p_2)}$$

where e_o is output voltage, e_c the control voltage, subscript q refers to the quadrature field, subscript c refers to the control field. One can see that if r_q changes with temperature, only p_1 is changed. If r_c changes, only p_2 is changed proportionally. But if an inductance value changes, not only the corresponding pole varies, but K does so as well.

As another example, take a mechanical system with inertia and friction:

$$J\ddot{\theta} + f\dot{\theta} = KE$$

where E is the driving error signal.

$$\frac{\theta}{E} = \frac{K}{Js^2 + fs} = \frac{K/J}{s(s + \frac{f}{J})}$$

In this case, a change in the friction modifies the time constant alone, while a fluctuation in the value of the inertia causes both gain and time-constant to vary accordingly.

In some instances, even the open-loop pole at the origin of the s-plane varies. This is the case of the above mechanical system when a shaft, intended to be rigid, is twisted under load, or when a transmission belt, designed to be of fixed length, is elongated under tension. Then:

$$\frac{\theta}{E} = \frac{K}{Js^2 + fs + k} = \frac{K/J}{\left(s^2 + \frac{f}{J}s + \frac{k}{J}\right)}$$

$$= \frac{K/J}{(s + p_o)(s + p_1)}$$

where

$$\frac{f}{J} = p_o + p_1$$

and

$$\frac{k}{J} = p_o p_1$$

The last equation shows that in the ideal case, k = 0 giving $p_o = 0$, but if some k exists, then p_o exists. As long as inertia J is fixed, the gain constant $\frac{K}{J}$ does not vary. If k alone varies, then both p_o and p_1 are changed since the sum $p_o + p_1 = \frac{f}{J}$ is constant. If both torsion k and friction f vary, p_o may change alone, or change simultaneously with p_1. Finally if J varies, then both gain constant and poles p_o, p_1 are changed proportionally.

Another similar example of parameter change may be found in mechanical systems with springs the constant of which varies in use.

II - 3: Graphical method for determining root-sensitivities.

Equations (15) and (16) derived in chapter one suggest that to singularities which are close to system root $-q_i$, q_i is more sensitive, and for singularities which are farther away, q_i is less sensitive until it becomes insensitive to singularities at infinity. The above concerns the magnitude of sensitivities. But sensitivities are vector quantities, since the change of a parameter may move the roots in different directions. It is then helpful to make use of equation (30) together with equations (15) and (16).

Refer to equation (15), where z and p indicate open-loop singularities and q_i is the system root in question. Draw a vector from q_i toward each and every pole, and away from each and every zero. The length of each vector will be inversely proportional to the distance from q_i to the singularity concerned. Then construct the sum U of all these vectors, which is a vector itself. If U is taken as unity vector in magnitude and phase, then the other vectors measure the sensitivity of q_i to each singularity respectively. Hereafter, the vector

$$\vec{U} = 1e^{io}$$

will be baptized "unity vector for root-sensitivities to poles and zeros", or more conveniently "unit-sensitivity vector". The first lengthy name

emphasizes the fact that this unity scale, applies to sensitivity poles and zeros only, i.e., $S^i_{z_j}$ and $S^i_{p_j}$, since equation (30) only concerns these two quantities. This unity scale does <u>not</u> apply to S^i_K.

The diagram just described is from now on referred to as the "vector diagram", as compared to the "circle diagram" to be introduced later. The vector diagram offers a quick way to measure both magnitude and phase of the vectors $S^i_{z_j}$ or $S^i_{p_j}$ for any j. Only one little detail needs be kept in mind: phase of sensitivity vectors must be measured as <u>positive</u> in the <u>clockwise</u> sense starting from the U vector. This seemingly arbitrary sign convention in fact comes from equations (15) and (16) which are the basis of the vector diagram:

$$S^i_{p_j} = \frac{S^i_K}{q_i - p_j} = \frac{S^i_K}{(-p_j) - (-q_i)} \equiv \frac{S^i_K}{\vec{V}_j} \qquad (31)$$

The denominator is the vector from root $(-q_i)$ toward pole $(-p_j)$ as shown on figure 2. The phase relationship of the above equation is:

$$\widehat{S^i_{p_j}} = \widehat{S^i_K} - \widehat{V_j} \qquad (32)$$

where the hat sign reads "phase of". The above is true no matter what conventions are applied to the measurement of the angles. Since the U-vector has been found to be equal to $1e^{j0^\circ}$ as far as S^i_p and S^i_z are concerned, angle $\widehat{S^i_{p_j}}$ will be measured starting from U as zero phase. On the other hand, $\widehat{S^i_K}$ and $\widehat{V_j}$ are measured in the conventional way, i.e., starting from the positive real axis and counting positively counter-clockwise. In equation (32), the quantity $\widehat{S^i_K}$ is not dependent on j, i.e., it is the same for all j's. Thus, for each j:

$$\widehat{S^i_{p_j}} = \text{constant} - \widehat{V_j}$$

meaning that the larger the value of $\widehat{V_j}$, the smaller must be $\widehat{S^i_{p_j}}$. Since $\widehat{V_j}$ is measured conventionally (positively counter-clockwise) $\widehat{S^i_{p_j}}$ must be measured positively clockwise.

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As an example, on figure 1a:

$$\left| S^{i}_{p_2} \right| = \frac{.37}{1.05} = .352$$

$$\widehat{S^{i}_{p_2}} = + 120^{o}$$

This means that if pole p_2 moves by dp_2 = .1 (to the left) while other parameters remain constant, then q_i will move by dq_i = (.352 $\underline{/120^{o}}$) (.1 $\underline{/0^{o}}$) = .035 $\underline{/120^{o}}$. To say that q_i moves by .035 $\underline{/120^{o}}$ means that the root ($-q_i$) moves by .035 $\underline{/-60^{o}}$, since q_i and $-q_i$ are two opposite quantities and move in opposite directions.

Finally note that the difference in the ways the sensitivity vectors are drawn for poles and zeros (toward the former, away from the latter) comes from the different signs in the denominators of equations (16 and (15). A different sign corresponds to a rotation of 180^{o}.

Before this section is ended, another interesting feature of the vector diagram is presented. This concerns the root-sensitivity to gain, S^{i}_{K}, which so far has not been mentioned on the vector-diagram. One recalls, however, that at the end of section I-5 a relation between S^{i}_{K} and $S^{i}_{p_o}$ was given, p_o being the open-loop pole at the origin of the s-plane:

$$S^{i}_{p_o} = \frac{S^{i}_{K}}{q_i} \qquad (17)$$

Equation 17 shows that, for a particular system root under investigation, S^{i}_{K} is equal to $S^{i}_{p_o}$ x q_i the phase relationship is:

$$\widehat{S^{i}_{K}} = \widehat{S^{i}_{p_o}} + \widehat{q_i}$$

Figure 2 shows the angles $\widehat{q_i}$ and $\widehat{S^{i}_{p_o}} = \alpha_o$, the latter being measured from U. Since $\widehat{q_i} + \widehat{S^{i}_{p_o}}$ gives the direction of U on the s-plane, it is thus established that on the s-plane, $\underline{S^{i}_{K}\text{ always lies on U}}$. Since S^{i}_{K} indicates the direction in which the root moves when K varies, i.e., the direction of the root locus, the above result can be stated as follows:

"At any point on the root-locus, the U vector is tangent to the root-locus." *

Again note that the direction of U-vector indicates the direction in which q_i moves when K increases. The direction in which $(-q_i)$ moves when K increases differs by 180^o.

It seems worthwhile to state once more the results obtained in this section which the reader should keep in mind before going on: Phases of root-sensitivities to p and z are measured positively clockwise starting from the U vector. Phase of root-sensitivity to gain K is measured in the conventional fashion, that is from the horizontal and negatively clockwise. On the s-plane, S_K^i always lies on U which is tangent to the system root locus at the point of contact.

II - 4: An example of application of the vector diagram.

In order to show the practical character of the vector diagram, the following numerical example is taken from reference 4, but solved by use of the vector diagram. A look at the lengthy arithmetical and graphical methods of reference 4, some exact and others approximate by nature, will convince the reader of the rapidity of the vector diagram. Results obtained here and those obtained in reference 4 are compared to show the relative degrees of accuracy.

Given the open-loop transfer function

$$P = \frac{K}{s(s+1)(s+5)}$$

For K = 2.07, the closed-loop roots are at locations indicated on figure 3. It is desired to calculate the sensitivity of the root at $-q_2$ to the gain K: (other sensitivies can be readily obtained, it is merely a matter of measuring a length and a phase on the graph).

The vector diagram is constructed and U - vector is drawn on figure 3. From measurement of vectors and phases, it is found that

$$S_{P_o}^2 = \frac{1.56}{2.07} \underline{/-51^o} = .753 \underline{/-51^o}$$

*: this result may be very useful in the construction of root loci, since it readily gives the orientation of the root locus at any point.

(Minus sign because clockwise from U-vector)

Since the system root $-q2$ is at $0.64 \underline{/135^o}$, q^2 is at $0.64 \underline{/-45^o}$. Then

$$S_K^2 = S_{P_o}^2 \times q_2$$

$$= (0.735 \underline{/-51^o})(0.64 \underline{/-45^o}$$

$$= (0.481 \underline{/-96^o}$$

Note that on s-plane, the U-vector also lies in the direction -96^o. Hence, S_K^2 is a vector lying on U. Note the tangency of U to the root locus at point $-q_2$.

The same example is worked in 10 different ways in reference 4. Results of only 5 of the most accurate methods are reproduced here for the purpose of comparison (order of increasing accuracy).

	Method	Value obtained for S_K^2
(1)	Root locus method by gain perturbation	$0.356 \underline{/262^o}$
(2)	Closed loop Bode Asymptotes	$0.451 \underline{/270^o}$
(3)	Root locus method by phase perturbation	$0.457 \underline{/264^o}$
(4)	Open loop Bode and ζ plot	$0.491 \underline{/266^o}$
(5)	Direct calculation	$0.492 \underline{/264^o.47}$
(6)	Method of this report	$0.481 \underline{/264^o}$

The vector diagram method is in fact an exact method since no approximation of any kind was made in the derivation. The more accurately the diagram is constructed and measured, the better the results. In order to improve accuracy, one may choose a scale for the sensitivity vectors different from the scale used for the root-locus.

<u>II - 5: Particular case of open-loop singularities of N^{th} order.</u>

Looking at the construction of the vector diagram as illustrated on

figure 1a, intuitively one can see that in case of a double pole or double zero (or higher-order), the same construction still applies, providing two vectors be drawn toward the double pole (or away from the double zero). This can be seen by assuming that pole $-p_2$ moves toward and reaches pole $-p_1$. Simultaneously the sensitivity vector of $-p_2$ would move toward that of $-p_1$ and reach the same magnitude as that of $-p_1$. Note that in such case, the length of the sensitivity vector of each pole at $-p_1$ remains the same as before, but the actual value of the sensitivity of -pl is different, since the scale-vector U has changed.

The same result may be obtained analytically as follows.* Assume L has an n^{th} order pole at p_1, then

$$L = \frac{K(\ldots.)}{(\ldots)(s + p_1)^n} = \frac{L^*}{(s + p_1)^n}$$

$$\frac{\partial L}{\partial p_1} = 0 + L^* \frac{n(s + p_1)^{n-1}}{(s + p_1)^{2n}} = n \frac{L^*}{(s + p_1)^{2n}} = n \frac{L}{s + p_1}$$

But $\frac{L}{s + p_1}$ is equal to $\frac{\partial L}{\partial p_1}$ if p_1 were a single pole. Also, from equation (51), $S^i_{p_1}$ is proportional to $\frac{\partial L}{\partial p_1}$. Then the above result:

$$\left[\frac{\partial L}{\partial p_1}\right]_{p_1 \text{ is } n^{th} \text{ order pole}} = n \times \left[\frac{\partial L}{\partial p_1}\right]_{\text{if } p_1 \text{ were a single pole}}$$

is equivalent to saying that the root-sensitivity to a n^{th} order pole is n times that to the same pole assumed single.

The same reasoning applied to a n^{th} order zero leads to identical conclusion.

II - 6: A design philosophy

It has been shown from the vector diagram that the root-sensitivity of q with respect to each singularity is given by the vector associated with that singularity, measured with U-vector as unity scale, both in magnitude

*: Casual reader may skip.

and phase.

If the U-vector is changed, either in magnitude or in phase or both, then root-sensitivity changes. In particular, the larger the magnitude of U-vector, the smaller the root-sensitivity value, i.e., the more insensitive the system root.

This leads to a design philosophy whereby the U-vector would be modified in order to meet particular specifications or restrictions imposed on the sensitivity of the dominant root of the system under investigation. Intuitively one can see - and this will be shown to be true later - that in a given situation it is possible to maximize the magnitude of U-vector in order to minimize root-sensitivities; or to orient U in a certain fashion so to make the damping factor insensitive to the variations of a certain pole, or to make it insensitive to the variations of gain-constant, etc.

In a number of control problems, the system response specifications are expressed as a desired location on the s-plane for the dominant system roots. Such a location determines a damping factor ζ and a natural frequency ω_n for the system. Thus, fixing the location of the dominant roots and fixing the frequency response of the system with phase-and gain-margins and bandwith, are two ways of expressing the same conditions.

One can find, in the literature, a comprehensive treatment of the problem of compensating a given system in order to place the dominant roots where they are desired. The simplest way is to use lead or lag networks in cascade in single or multiple-stage. In fact, there exists an infinite number of solutions to the problem of forcing the root-locus of a system to go through a certain point on the s-plane. All what is needed is that the lead- or lag-network contribute the desired phase shift so to make the total phase at that point equal to 180°.

Hoever, if another contraint is placed on the system, the number of possible solutions decreases and eventually becomes unique. Such is the case of the problem in which the desired location of the dominant roots, <u>and</u> the steady-state accuracy of the system are to be satisfied simultaneously. The latter condition fixes the value of gain constant when the dominant roots are at their assigned location. This problem was solved in

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detail by Ross, Warren and Thaler[7], also by Pollak and Thaler[8], and recently Hsu[9] proposed a graphical method.

The following is concerned with two simultaneous conditions: location of dominant roots, and sensitivity of these dominant roots. The second condition on sensitivity may take a number of different forms, such as sensitivity of ζ, or sensitivity of ω_n, with respect to variations of gain or poles or zeros. In other situations, the dominant roots may be restricted to moving only within a certain area. As stated earlier, the philosophy of the design method is to modify U in magnitude and/or in phase. Thus the first step is to investigate what the possibilities are in modifying U using lead or lag networks; in other words, how the U-vector changes on the s-plane.

<u>II - 7: Locus of U on the s-plane</u>.

This section, as a preliminary to the design procedure to be presented next, is devoted to the determination of the geometric locus of the tip of U-vector on the s-plane. Knowledge of this locus tells the designer how and how much he can change magnitude and phase of the U-vector, and what good such change will do.

Consider the plant to be compensated:

$$G = \frac{K(s + z_1)}{s(s + p_1)(s + p_2)}$$

and the desired dominant-root location -q as indicated on figure 4a. A spirule measurement shows that an additional phase of $+ \phi$ is needed at location -q. Thus a lead network with a zero at Z and a pole at P is needed. The question is: how does the tip of the U-vector move on the s-plane, when Z and P take all possible values on the negative real axis? (avoid positive real axis to avoid possible conditional instability). Figures 4a, b and c illustrate the answer to the above question.

First, ignore the compensator irregularities Z and P, and draw the unity vector for the irregularities p_o, p_1, p_2, and z_1, alone. This "uncompensated" unity vector is labeled QI on figure 4a. Now if a pole P is added, it has associated with it a sensitivity vector Qm of magnitude $\frac{1}{QP}$. Then the unity vector QI is augmented by the vector quan-

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tity IM = Qm.

Similarly, when a zero Z is added, it has associated with it a sensitivity vector Qu, directed away from Z and of magnitude $\frac{1}{QZ}$. Then the unity vector QM is augmented by the vector quantity MU = Qu. The question is to find the geometric locus of U. Figure 4b shows that when P moves along the real axis, since $QM = \frac{1}{QP}$, m moves on a circle of radius $R = \frac{1}{2d}$, resulting from geometrical inversion of the real axis, inversion with center Q and ratio 1. Such a circle will be referred to as the (M) circle for convenience. The locus of point M is the (M) circle, of radius $R = \frac{1}{2d}$ and with I as its uppermost point.

Figure 4c shows that $IU = 2R \sin \phi = \frac{1}{d} \sin \phi$ which is a constant quantity for each problem, and hence U move on a circle centered at I and of radius $r = \frac{1}{d} \sin \phi$. The reasoning attached to figure 4c is as follows. Draw vector IN equal and opposite to MU. Since Z is a point on the real axis, N is on the (M) circle due to figure 4b. Then $MN = 2R \sin \phi$, and since IU = MN, $IU = 2R \sin \phi$ = constant. The circle on which point U moves will hereafter be known as the (U) circle.

The (M) circle, locus of point M, of radius $R = \frac{1}{2d}$ and the (U) circle, locus of point U, of radius $r = \frac{1}{d} \sin \phi$ are shown on figure 4a. For convenience in terminology, the diagram just drawn will be given the name of "circle diagram", as opposed to the "vector diagram" shown on figure 1.

For purpose of reference, the above result is restated below:

<u>The geometric locus of the tip of the unity vector is a circle, centered at I and of radius $r = \frac{1}{d} \sin \phi$;</u> where I is the tip of the "uncompensated" unity-vector, d the imaginary part of the dominant system root, and ϕ the phase shift to be introduced by the compensation.

II - 8: Limit of locus of U

It does not make sense to define a geometric locus without specifying its limits. This is the purpose of this section.

There must be 2 limit points on the locus of U, a right hand limit U_r, and a left hand limit U_ℓ. Point U_r is defined by the extreme condition where Z, the compensator's zero, would be at the origin, and P on its left, such that $\widehat{PQZ} = \phi$. Point U_ℓ is defined by the other extreme condition with

P at $-\infty$ and Z on its right, such that PQZ = ϕ. Any other possible case is between these two extremes.

Determination of U_r and U_ℓ may be done by first noting the following detail on figure 4a (or 4c): IN // QZ; IM // QP; if J is the midpoint of MN then IU is perpendicular to OJ

$$IU \perp OJ$$

This can be used to obtain U when P and Z are known: draw IN// OZ; draw IM //OP; take midpoint J; draw IU $\perp$ OJ. A quick way is to take the angle NOJ = ϕ, thus avoiding the trouble of obtaining the midpoint of an arc.

Conversely, and this is more important, one can start from any desired location of U and go back to obtain the corresponding Z and P, by doing the above construction in reverse order. This is the essence of this design method, whereby one changes the U-vector by proper compensation. When a particular location for U on its locus is chosen to satisfy some specification (next sections), obtain P and Z as follows:

Draw OJ $\perp$ IU which cut (M) circle at J. On (M) circle, measure arc JN = JM = ϕ. Then the direction of IN is the direction of QZ (thus one gets Z), and the direction of IM is the direction of QP (thus one gets P).

An example is shown on figure 6a. A desired U is given (purely as an example, for no particular reason). Perform the construction as indicated above and obtain α_P and α_Z as angles of the direction of P and Z with respect to the vertical. This determines the compensator pole and zero as on figure 6b. Obviously $\alpha_P + \alpha_Z = \phi$.

One may now go back to the problem of determining the limit points U_r and U_ℓ of the locus of U. This is merely a pair of problems similar to the one just solved.

For the extreme right case (subscript r), refer to figure 5a. A measure on figure 4a shows that when Z is at the origin, QZ makes -30° with the horizontal. Draw IN_r at -30° from the horizontal. N_r is the extreme-right position of N. Measure $N_rO\,J_r = \phi$ and draw $IU_r \perp OJ_r$. U_r is the extreme right limit of the locus of U.

For the extreme-left case (subscript ℓ), refer to figure 5b. Here P

is at $-\infty$. This calls for rotating angle $N_r I M_r$ of figure 5a, until it reaches the extreme-left position $N_\ell I M_\ell$ of figure 5b. Then M_ℓ is at I, and N_ℓ is at intersection of the two circles. The same construction $IU_\ell \perp OJ_\ell$ gives U_ℓ, the left limit of the locus of U.

Once the limits are found, one can see how far one can change the magnitude and phase of the unity vector, which is represented by QU on figure 6a. As an example, on figure 6a, the maximum magnitude that the U-vector can reach is QU_r, and the rightmost direction it can have is given by QU_ℓ.

Finally note that in the case of a lag network, the points M and N are simply interchanged and the determination of limit points U_r and U_ℓ is still the same, with P at origin in one limit case, and Z at infinity in the other case. Throughout the work, point M will be associated with pole P, point N will be associated with zero Z to help the reader follow the argument more easily.

II - 9: Design techniques.

In the above section it has been derived a method for finding P and Z, given the desired location of point U on its locus, i.e., the magnitude and phase of QU, the unity-vector. This method is re-stated below in a step-by-step form (refer figure 4a).

Step 1 - Considering Q as if it already were a point on the root locus, draw the vector diagram at Q for the uncompensated system and obtain QI, the "uncompensated" unity vector.

Step 2 - Draw the circle diagram, composed of: the (M) circle, of radius $R = \frac{1}{2d}$ and whose uppermost point is I.; and the (U) circle centered at I and of radius $r = \frac{1}{d} \sin \phi$. Fix the limits of the locus of U on the (U) circle.

Step 3 - Given QU as the desired unity-vector for the compensated system, draw $OJ \perp IU$ which cuts (M) circle at J. Draw angles JON = JON = ϕ (or arcs JN = JN = ϕ). Then Z and P are determined by drawing QZ // IN and QP // IM.

In this section, it will be seen how the U-vector (i.e., QU, i.e., the location of U on its locus) is selected to satisfy a particular condition.

a) Design for minimum root-sensitivity.

Since the U-vector is the scale used to measure the individual sensithe intermediate position [illegible] of Figure [illegible] as [illegible] of the two circles. [illegible]

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III.— [illegible]

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Lemma 1 — [illegible]

Lemma 2 — If the [illegible]

Lemma 3 — [illegible]

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tivity vectors, the larger the scale, the smaller the magnitude of the sensitivity measures. Thus one possibility is to design the compensation for maximum magnitude of U-vector, that is, minimum root sensitivity to open loop singularities. Figure 4a shows that maximum magnitude of QU is obtained by placing point U near the lowermost part of the (U) circle, or more exactly, on the extension of QI. If such point is not within the locus of U, then it cannot be a location for U, and one must select the lowest point which is on the locus. In figure 6a, this is point U_r. Thus, QU_r is the selected unity vector, and with this given, one can proceed to the 3 step procedure outlined in the beginning of this section. With such a design, $S^q_{p_i}$ and $S^q_{z_i}$ are all minimum, for all i's. For example, the minimum value of $S^q_{p_o}$ will be equal to the ratio of the sensitivity vector associated with p_o divided by vector QU_r, both in magnitude and phase. Note that these are minimum values of sensitivities obtained with only one filter stage. It will be seen later that by use of multiple-stage filters, results may be improved, but more often than not, improvements are small and do not justify the extra cost. Design example No. 1 given in the next section applies the above technique.

b) Design for constant damping when gain K varies (refer to figure 4a)

Another practical problem is to compensate a system in such a way that when gain K varies about its nominal value, the dynamic response of the system doesn't change. This calls for a constant ζ, i.e., a root-locus that remains tangent to the radial line OQ at the neighborhood of Q.

How can Z and P be found to obtain such a root-locus? It is now shown that this can be done by merely selecting point U so that the unity vector QU goes through the origin O of the s-plane. In other words, choose U so that Q, O, U be in line. (If the locus of U doesn't permit such a choice, this means it is not possible to obtain a constant ζ about Q for the given system. One can then choose the best solution available, by taking the U location that is closest to a straight line with QO)

The above statement can be proved very simply if one recalls

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equation (17) of Chapter One

$$S^{i}_{p_o} = \frac{S^{i}_{K}}{q_i} \qquad (17)$$

The specification here is to force S^{i}_{K} to have same direction as QO, thus making q_i move on a radial line when K varies. This means $\widehat{S^{i}_{K}}$ must be equal to the phase of QO (namely -30^{o} on figure 4a) which is also phase of q_i, or $\widehat{q_i}$. But from equation (17)

$$\widehat{S^{i}_{p_o}} = \widehat{S^{i}_{K}} - \widehat{q_i} = 0$$

The phase of $S^{i}_{p_o}$ must be 0, this means that the sensitivity vector $S^{i}_{p_o}$ must lie on the U-vector, or conversely, the U-vector must pass through p_o at the origin of the s-plane, qed.

A faster way to prove the above is to come back to section II - 3 where it has been shown that S^{i}_{K} always lies on the U vector. In order to keep ζ constant, S^{i}_{K} must be radial, thus U-vector must be radial.

Design example No. 2 of next section illustrates this part.

<u>c) Design for constant damping when a singularity varies.</u>

Again refer to figure 4a. Another practical design problem is the following: the nature of the plant is such that pole $-p_1$ varies more than the other singularities, (see section II - 2). It is desired to compensate the system in such a way that: 1-the dominant roots be at -q, <u>and</u> 2-that the sensitivity of the damping factor ζ with respect to changes in p_1, namely $S^{\zeta}_{p_1}$ be nullified, or at least made as small as possible.

<u>It is now shown that such problem is solved by simply making $\widehat{S^{i}_{p_1}}$ equal to $\widehat{q_i}$ or $\widehat{-q_i}$</u>, that is, forcing the U-vector to a postion such that the phase of S^{i}_{p}, be equal to the phase of q_i or of $-q_i$. The latter differ by Π, so the above underlined condition may be written as:

$$\boxed{\widehat{S^{i}_{p_1}} = \widehat{q_i} \oplus \Pi} \qquad (33)$$

where $\widehat{q_i} \oplus \Pi$ reads : "either $\widehat{q_i}$ alone, or $\widehat{q_i} + \Pi$ ". Equation (33) is the condition for ζ to be insensitive to p_1, that is for $S^{\zeta}_{p_1} = 0$.

Note that, as for previous design problems, it may happen that the limits of the locus of U do not permit a shaping of U-vector such that (33) be satisfied. In such case it is always possible to minimize $S^{\zeta}_{p_1}$ by making $\widehat{S^i_{p_1}}$ as close to $\widehat{q_i} \oplus \Pi$ as possible. Design example No.3 illustrates this method.

The following is proof of condition (33). The proof is based on this remark (see fig. 7a, d): for ζ to be kept constant, the change in q_i, that is dq_i, must be on the same radial line as q_i, i.e.,

$$\widehat{dq_i} = \widehat{q_i} \oplus \Pi \tag{34}$$

where the $\oplus$ sign has same meaning as in equation (33). If $\widehat{dq_i} = \widehat{q_i}$, the natural frequency ω_n increases. If $\widehat{dq_i} = \widehat{q_i} + \Pi$, ω_n decreases. From the definition of root-sensitivity, when p_1 changes by dp_1, root q_i changes by:

$$dq_i = S^i_{p_1}\, dp_1$$

or, phasewise:

$$\widehat{dq_i} = \widehat{S^i_{p_1}} + \widehat{dp_1} \tag{35}$$

Since p_1 varies on the negative real axis, $\widehat{dp_1} = 0 \oplus \Pi$. In more detail, if the pole at $-p_1$ moves to the left, $dp_1 > 0$, and $\widehat{dp_1} = 0$. If pole moves to the right, $dp_1 < 0$, and $\widehat{dp_1} = \Pi$. Combining equations (34) and (35), the following condition for $S^{\zeta}_{p_1} = 0$ obtains:

$$\widehat{S^i_{p_1}} \oplus \Pi = \widehat{q_i} \oplus \Pi$$

which is the same as equation (33), since an addition of Π to the right hand side is equivalent to the same addition to the left hand side.

d) Other possible sensitivity designs using the U locus

Three practical design problems have been discussed: a) design for minimum root sensitivity; b) design for constant damping when K is per-

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turbed, and c) design for constant damping when a plant singularity is perturbed. Still other problems may be solved using the same technique.

If it is desired to keep ω_n constant (constant bandwidth) when gain K or plant irregularities vary, then dq_i must be perpendicular to q_i, and condition (32) becomes:

$$\widehat{dq_i} = \widehat{q_i} \pm \frac{\Pi}{2} \qquad (36)$$

Combination of (20) and (21) yields:

$$\widehat{S^i_{p_1}} + \Pi = \widehat{q_i} \pm \frac{\Pi}{2}$$

that is

$$\widehat{S^i_{p_1}} = \widehat{q_i} \pm \frac{\Pi}{2} \qquad (37)$$

or

$$\widehat{S^i_{p_1}} = \widehat{q_i} \pm \frac{\Pi}{2} - \Pi = \widehat{q_i} \pm \frac{\Pi}{2} \quad \text{(same)}$$

which is the condition for keeping ω_n constant when p_1 is perturbed.

It is also possible to design the system in such a way that a particular plant-parameter pertubation has stabilizing effect (or destabilizing effect, if it is so wished!) on the system response. Fig. 7b shows that if $\frac{dq}{q}$ has a phase between 0 and Π, i.e., $0 < dq - q < \Pi$, the root variation has stabilizing effect. Figure 7c shows that if $\frac{dq}{q}$ has a phase between Π and 2Π, i.e., $\Pi < dq - q < 2 <$, then the root variation has destabilizing effect. A reasoning similar to that of part (c) of this section will yield the conditions for obtaining one or the other of the above effects.

II - 10: Design examples.

The preceding section shows that the proposed sensitivity design technique is an exact method, involving no approximation or cut-and-try. It is a reasonably quick method, all that is required as preliminary work is the construction of the U-locus. It is versatile, can be readily applied to various practical design problems involving small plant-parameter perturbation.

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In this section three design problems will be worked out in details in order to illustrate the techniques presented above, then analog computer simulations are done to check the results.

a) Design example No. 1:

The plant to be compensated has the transfer function:

$$P = \frac{K}{s\left(\frac{s}{0.7}+1\right)\left(\frac{s}{0.3}+1\right)}$$

Dynamic and bandwidth requirements lead to the desired location for dominant roots at $-q = -0.2 \pm j0.35 = 0.4 \angle 120^{\circ}$, that is, $q = 0.4 \angle -60^{\circ}$. All three plant poles are subject to fluctuations. It is desired to design a cascade compensator satisfying the above dominant root requirment, and in addition, guaranteeing a minimum value of sensitivity of q to the poles' fluctuations.

Plant singularities and desired root location are represented on figure 8.

Step 1 : Considering -q as if it already were a point on the root locus, the vector diagram is drawn and the "uncompensated" unity vector QI obtained (figure 8). Using a spirule, measure the phase shift ϕ necessary to make root-locus pass through -q. Found $\phi = -38^{\circ}$ (lag network needed).

Step 2 : Circle diagram (figure 9):

Draw the (M) circle, of radius $R = \frac{1}{2d} = \frac{1}{0.7} = 1.43$

Draw the (U) circle, of center I and radius

$$r = 2R \sin\phi = 2 \times 1.43 \times \sin 38^{\circ} = 1.76$$

The geometric locus of U is on the (U) circle.

The limits of this locus are found as explained in II - 8: For extreme right limit, filter pole P is at origin, then QP makes 30° with vertical. On circle diagram of figure 9, draw IM_r making 30° with vertical. On circle diagram of figure 9, draw IM_r making 30° with vertical. Draw angle $M_r O J_r = \phi = -38^{\circ}$, thus get J_r. Draw $IU_r \perp OJ_r$ (see explanation of section II - 8) thus get right limit-point U_r.

For extreme left limit, filter zero Z is at minus infinity, then QZ is horizontal. On circle diagram, draw IN_ℓ horizontal (N_ℓ coincides with I). Draw angle $N_\ell O J_\ell = 38^o$, thus get J_ℓ. Draw $IU_\ell \perp O J_\ell$, giving the left limit-point U_ℓ. Here U_ℓ is simply the intersections of the (M) circle and the (U) circle. This is true for all problems where a lag filter is needed. (Observe on figure 5b that for problems where a lead filter is needed, the left limit-point U_ℓ is diametrically opposite to the intersecting point of (M) and (U).

Step 3 : Select a location for U, and from this derive the necessary compensator. In this problem, it is desired to minimize the sensitivities, thus one must maximize the magnitude of U-vector. The maximum length that this vector can reach is QU_1, U_1 being on the extension of QI.

Now from U_1, find P and Z, using the construction presented in section II - 8. Draw $OJ_1 \perp IU_1$. Measure arcs $J_1N_1 = J_1M_1 = \phi$. Then the direction of IN_1 is the direction of QZ(thus get Z = 0.8 as shown on figure 9b. The direction of IM_1 is the direction of QP (thus get P = .36 on figure 9b). The complete compensated system's pole zero configuration and corresponding U vector are shown on figure 9b.

b) Design example No. 2.

It is desired to compensate the plant given in design example No. 1 in such a way that: 1 - the dominant roots be located at $-q = 0.4\underline{/120^o}$, and 2 - when plant gain K is perturbed, dominant root may move about the desired location but the system's damping factor will not change.

This amounts to designing for $S_K^\zeta = 0$, or to say the same thing differently, to force the system root-locus to follow a radial line in the vicinity of -q.

From part b) of section II - 9, it has been determined that this can be done by forcing the U-vector to go through the origin of the s-plane. This means, for this problem, that U-vector must make 30^o with the vertical direction; that is, it must occupy the position QU_2 indicated on

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figure 10a. Thus the location of U is fixed. (Subscript 2 used for this example)

The design is accomplished by performing the now familiar construction. Draw $OJ_2 \perp IU_2$, this gives J_2. Measure arcs $J_2M_2 = J_2N_2 = \phi$. Then P is given on figure 10b by drawing QP // IM_2, and Z is given by QZ // IN_2. The result is P = .14, Z = .39.

The complete compensated system's pole-zero configuration is drawn on figure 11 and U-vector is constructed thereon. As expected, it goes through the origin of the s-plane. As a check, the entire root locus is drawn on figure 11 and it does follow the radial line in the vicinity of -q.

The same problem is simulated on the analog computer and the results obtained are reported later in this same chapter.

c) Design example No. 3.

In the plant given in previous examples, it is observed that the pole at the origin, $-p_o = 0$, fluctuates most. Moreover, since $-p_o$ is the closest to Q among all plant poles, it has most effect on the location of Q (sensitivity relatively highest, at least in magnitude). The compensator must be designed such that the effect of the fluctuations of the pole at $-p_o = 0$, on the damping factor ζ, be nil or minimized.

From part c) of preceding section, it was found that, in order to make ζ insensitive to p_o, one must have:

$$\widehat{S^i_{p_o}} = \widehat{q_i} \oplus \Pi$$

In the present problem, $\widehat{q_i} = -60^o$, and the above condition becomes

$$\widehat{S^i_{p_o}} = -60^o \text{ or } + 120^o$$

Recalling that this phase is measured negative counter clockwise, the above condition calls for a U-vector making an angle of 60^o with QO and on the left of QO. A look at figure 12, however, shows that the leftmost position U can reach is U_ℓ, which gives a Z at minus infinity and a P at -0.65, as shown on figure 12b. This yields a U-vector making only 48^o with QO instead of the required 60^o. This is the best one can do to minimize $S^\zeta_{p_o}$, using a

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single stage compensator. By use of multiple stage, this result can be improved by making the angle UQO exactly 60^o, as will be seen later in section II - 12.

Another remark may be made on figure 12b. According to equation (37) if some pole p_j is so located that $\widehat{S^i_{p_j}} = \widehat{q_i} \pm \frac{\Pi}{2}$, then variations in p_j do not affect ω_n, but greatly affect ζ. In this example, $\widehat{q_i} \pm \frac{\Pi}{2} = 30^o$ or -150^o. There exists no pole p_j such that $\widehat{S^i_{p_j}} = 30$ or -150^o, but pole p_1 does have $\widehat{S^i_{p_1}} = 56^o$. One then can expect p_1 to have more effect on ζ than other singularities do, and the closer $\widehat{S^i_{p_1}}$ approaches 30^o (i.e., when p_1 moves to the right), then more effect p_1 will have on ζ.

This is found to be true when the system is simulated on the analog computer, the results of which are presented in the next section.

II - 11: Analog computer simulation.

The compensated systems as resulting from example 2 and 3 are simulated on the analog computer as shown on figure 13, and step responses for various values of gain and parameters are shown on figure 14 through 16.

The following is the equation for analog computer set-up: Let V(s) be the output from the compensator and Y(s) the system output. Then:

$$\frac{Y(s)}{V(s)} = \frac{K}{(s + p_o)(s + p_1)(s + p_2)}$$

where the nominal values of the poles are $p_o = 0$, $p_1 = 1.43$, $p_2 = 3.33$.

$$\frac{Y(s)}{V(s)} = \frac{K}{s^3 + (p_o + p_1 + p_2)s^2 + (p_op_1 + p_1p_2 + p_op_2)s + p_op_1p_2}$$

$$s^3Y + (p_o + p_1 + p_2)s^2Y + (p_op_1 + p_1p_2 + p_2p_3)sY + p_op_1p_2Y = KV$$

$$s^3Y = -(p_op_1 + p_2)s^2Y - (p_op_1 + p_1p_2 + p_2p_3)sY - p_op_1p_2Y + KV$$

$$s^2Y = -\int \left[(p_o + p_1 + p_2)s^2Y + (p_op_1 + p_1p_2 + p_2p_3)sY + p_op_1p_2Y - KV \right] dt$$

$$s^2Y = -\int \left[A_9\, s^2Y + A_8\, s\, Y + A_7Y - A_6V \right] dt$$

where $A_9 \equiv p_o + p_1 + p_2 = a_9\, g_9$

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$$A_8 \equiv P_oP_1 + P_1P_2 + P_2P_3 = a_8g_8$$

$$A_7 \equiv P_oP_1P_2 = a_7g_7$$

$$A_6 \equiv K = a_6g_6$$

The above equation leads to plant simulation set-up of figure 13. Compensatornis simulated separately, for example 2:

$$C = \frac{s + Z}{s + P}$$

where Z = .39 and P = .14

$$V = \frac{s + Z}{s + P} E$$

$$\text{Let } V = (s + Z) W \quad \text{where } W \triangleq \frac{E}{s + P}$$

$$\text{Then } V = sW + ZW$$

$$S = sW + PW$$

$$\text{or } sW = E - PW$$

$$\text{or } -W = \int \left[E - PW \right] dt$$

$$\text{now } V = E - PW + ZW$$

$$V = E + (Z-P) W$$

$$-V = -\left[E + (Z - P) W \right]$$

The above equations giving W and V lead to set-up of figure 13 for the compensator.

For example 3, $C = \frac{1}{s + P}$ where $P = 0.65$

$$V = \frac{E}{s + P}$$

$$sV = E - PV$$

$$-V = -\int \left[E - PV \right] dt$$

thus one integration with feedback will be needed (see figure 13).

For example 2:

$$a_3g_3 = P = .14 \longrightarrow \begin{cases} a_3 = .14 \\ g_3 = 1 \end{cases}$$

$$a_4g_4 = 1 \longrightarrow \begin{cases} a_4 = 1 \\ g_4 = 1 \end{cases}$$

$$a_5g_5 = Z - P = .25 \longrightarrow \begin{cases} a_5 = .25 \\ g_5 = 1 \end{cases}$$

$$a_6g_6 = K \text{ (variable)} \longrightarrow \begin{cases} a_6 \text{ variable} \\ g_6 = 10 \end{cases}$$

$$a_7g_7 = p_op_1p_2 = 0 \longrightarrow a_7 = 0$$

$$a_8g_8 = p_op_1 + p_1p_2 + p_2p_3 = 4.76 \longrightarrow \begin{cases} a_8 = .476 \\ g_8 = 10 \end{cases}$$

$$a_9g_9 = p_o + p_1 + p_2 = 4.76 \longrightarrow \begin{cases} a_9 = .476 \\ g_9 = 10 \end{cases}$$

For example 3:

$$a_3g_3 = P = .65 \longrightarrow \begin{cases} a_3 = .65 \\ g_3 = 1 \end{cases}$$

$$a_6g_6 = K = .935 \longrightarrow \begin{cases} a_6 = .0935 \\ g_6 = 10 \end{cases}$$

p_o varies:

$$p_o = 0 \begin{cases} a_7 = 0 \\ a_8 = .476 \\ a_9 = .476 \end{cases} \qquad p_o = .05 \begin{cases} a_7 = .238 \\ a_8 = .5 \\ a_9 = .481 \end{cases}$$

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$$p_o = .1 \begin{cases} a_7 = .476 \\ a_8 = .524 \\ a_9 = .486 \end{cases} \qquad p_o = -.1 \begin{cases} a_7 = -.476 \\ a_8 = .429 \\ a_9 = .466 \end{cases}$$

2-P_1 varies:

$$p_1 = 1.43 \begin{cases} a_7 = 0 \\ a_8 = .476 \\ a_9 = .476 \end{cases} \qquad p_1 = 1.33 \begin{cases} a_7 = 0 \\ a_8 = .444 \\ a_9 = .466 \end{cases}$$

$$p_1 = 1.23 \begin{cases} a_7 = 0 \\ a_8 = .41 \\ a_9 = .456 \end{cases} \qquad p_1 = 1.03 \begin{cases} a_7 = 0 \\ a_8 = .343 \\ a_9 = .436 \end{cases}$$

$$p_1 = .83 \begin{cases} a_7 = 0 \\ a_8 = .276 \\ a_9 = .416 \end{cases}$$

Discussion of results:

For example No. 2, step responses are displayed on figure 14 for different values of gain K, varying about the nominal value K = 1.45. It is found that there is essentially no change in damping for values of K between 1.2 and 1.6. Even beyond these values, damping change is rather slow. This agrees with root locus of figure 11.

For example No. 3, with fluctuations of the pole at origin, step responses for $-.1 < p_o < .1$ are presented on figure 15. Note that the damping is not changed for the above variations in p_o, the magnitude of which is not negligible considering the proximity of p_o to the dominant roots. Also note the faster rise time when p_o increases. This is due to an increase in ω_n, i.e., increase in bandwidth.

When p_o moves away from the origin, the system has some steady-state error. However, the purpose here being the study of $S^{\zeta}_{p_o}$, only changes in ζ are of interest.

For example No. 3 with fluctuations in pole-p_1, the step responses for various values of p_1 are presented on figure 16. It is unfortunate that in this example, $\widehat{S^i_{p_1}}$ is 56^o (figure 12b), while a sensitivity phase of 30^o is needed to make p_1 have maximum effect on ζ (last remark, section II - 10). However, even at 56^o, a small variation of .1 in the location of p_1 (1.43 to 1.33) changes M_{pt} from 1.15 to 1.20, i.e., ζ from 0.53 to 0.45, using 2nd order approximation.

Although the design was done on the basis of small parameter changes, p_1 was moved further to the right (toward the position where $\widehat{S^i_{p_1}}$ would approach the value 30^o), and as predicted by the theory, the change of damping is more and more violent as the condition $\widehat{S^i_{p_1}} = 30^o$ is approached. Therefore, the analog computer study has shown that, for the compensation scheme used, fluctuations of p_o have very little or no effect on ζ while those of p_1 change ζ appreciably, even for a small variation. The results would be still better if $\widehat{S^i_{p_o}}$ were equal to -60^o and $\widehat{S^i_{p_1}}$ were $+ 30^o$ as computed in section II - 10c.

II - 12: Single-stage or multiple-stage compensation?

a) The above designs have been done on the assumption that only one-stage compensators are to be used. The question arises as to whether any improvments can be obtained by using more than one stage.

When the magnitude of ϕ, phase shift needed to bring Q on the system root locus, is beyond a certain value, then one stage of compensator will not be sufficient. But even when ϕ is small enough so that one stage of compensator will do, one still has to ask the same question.

Further, in case of multiple stages, stages may be identical or different. Once the number of stages is decided, if identical stages are used, then the number of degrees of freedom remains the same as before, i.e., only one. But if different stages are used, then additional freedoms are introduced and possible improvements may come therefrom. "Improvement" here is used in the sensitivity-design sense, i.e. decrease in sensitivity i.e., increase in the magnitude of the scale vector U.

b) It will now be shown that in general, only negligible improvement is introduced by using n identical stages of compensation, (each giving a

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phase shift of $\frac{\phi}{n}$) instead of one stage (giving ϕ); and most of this negligible improvement is done by taking $n = 2$. Thus there is no reason to take $n > 2$. Besides, there is a risk of conditional instability involved.

If non-identical stages are used, conceptually it is possible that for some fortunate choices of compensator stage, an improvement is obtained. However, this involves cut-and try work, and no rules can be stated nor any definite results predicted.

c) A short remark is necessary before the proof of the above statement can be undertaken. It concerns the construction of the (U) circle and the U-locus when compensators have n- identical stages. Refer to figure 4a. assume that the compensator is double-staged. P is then a double pole. Two sensitivity vectors QM must be drawn, and when added to QI, they give a vector IM twice as long as the one in figure 4a. Thus, radius R of (M) has doubled. On the other hand, the radius r of (U), which was equal to $2R \sin \phi$ for the one-stage compensator, now becomes $\frac{2}{d} \sin \frac{\phi}{2}$ (the factor 2 comes from the fact that radius R has doubled, the angle $\frac{\phi}{2}$ is phase shift from each of the two stages.) More generally, for n - identical stages, radius $R = \frac{1}{2d}$ becomes $\frac{n}{2d}$, and radius $r = \frac{1}{d} \sin \phi$ becomes $r_n = \frac{n}{d} \sin \frac{\phi}{n}$.

d) To prove statement (b), the plant given in previous examples is used. See figure 8 for pole-zero configuration, figure 9a for the U-locus for 1 stage compensation. ϕ was measured to be -38°. Now, successively two stages giving $\frac{\phi}{2}$ each, then three stages giving $\frac{\phi}{3}$ each will be used. For each case, the U-locus is drawn, as indicated on figure 17. Values of R and r are given on figure 17 for each case. It can readily be seen that the increase in the magnitude of U-vector, which, at best, equals the increase in the radius r of the U-locus, is negligible, and thus is not worth the use of additional stages.

It has been determined in the preceding paragraph, that for n - identical stages, the radius of (U) is $r_n = \frac{n}{d} \sin \frac{\phi}{n}$ as compared to $r_1 = \frac{1}{d} \sin \phi$ for single stage. Elementary trigonometry shows that for small ϕ, $\sin \phi \doteq n \sin \frac{\phi}{n}$, and $r_1 \doteq r_n$, that is, there is no improvement in increasing the

number of stages. For ϕ approaching 90°, $n \sin \frac{\phi}{n} > \sin \phi$ and some improvement is possible. For the very best situations where $\phi = 90^\circ$, $2 \sin \frac{\phi}{2}$ is 1.4 times $\sin \phi$, which gives an increase in r of 40%. However the corresponding percent increase in magnitude of U is less, since $\vec{U} = \vec{QI} + \vec{r}$, and QI doesn't change.

The curve of figure 18 shows the values of the ratio $\frac{n \sin \frac{90^\circ}{n}}{\sin 90^\circ}$ for n = 1,2,3 .... It shows that most of the improvement is negligible. For $\phi > 90^\circ$, $n \sin \frac{\phi}{n} < \sin \phi$ and there is no interest in using many stages if one stage can do the job.

In conclusion, unless $|\phi|$ comes out to be very close to 90° - in which case, use of two identical stages may lead to some improvement in sensitivity - it is not worth while to use unidentical stages when one stage can give the necessary phase shift. In addition to cost increase, the introduction of extra roots may be troublesome, while increase in the magnitude of U is insignificant.

I L L U S T R A T I O N S

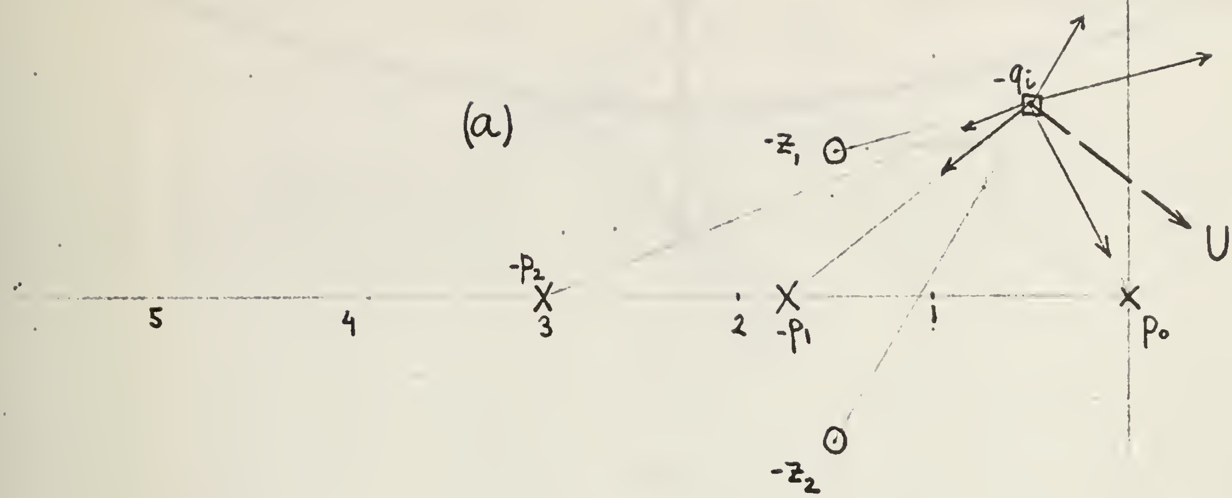


Fig 1 a- Construction of the vector diagram

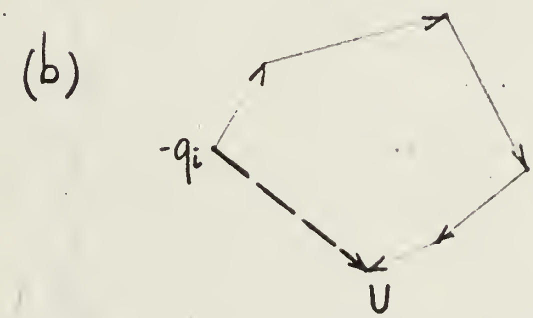


Fig 1 b- Vector addition to obtain U

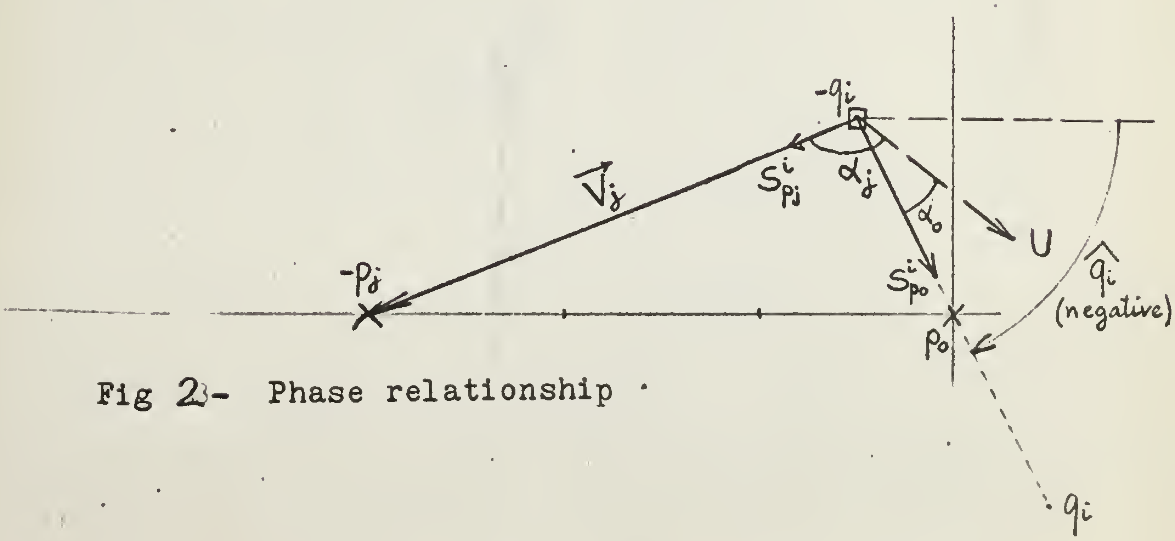


Fig 2- Phase relationship

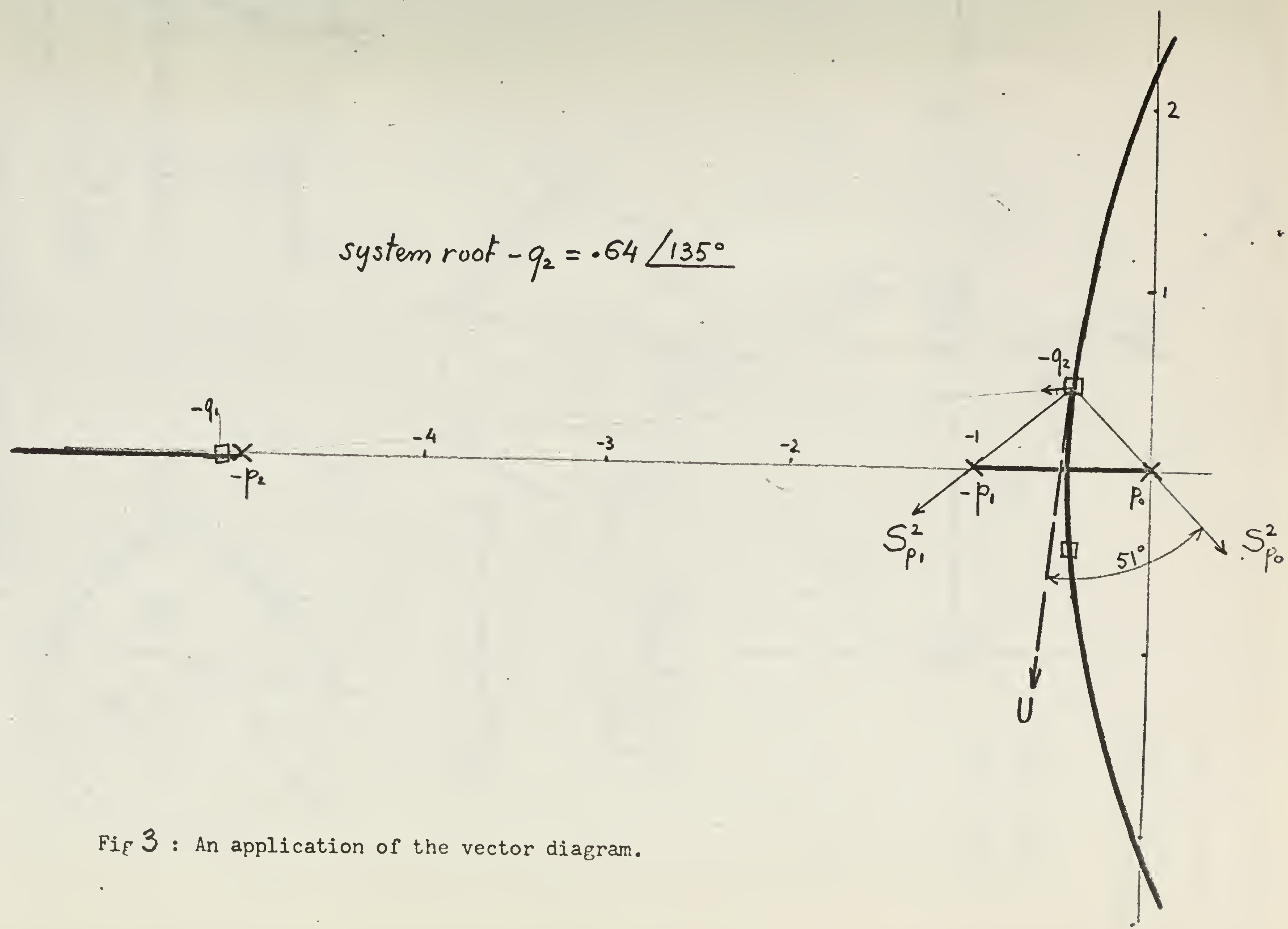


Fig 3 : An application of the vector diagram.

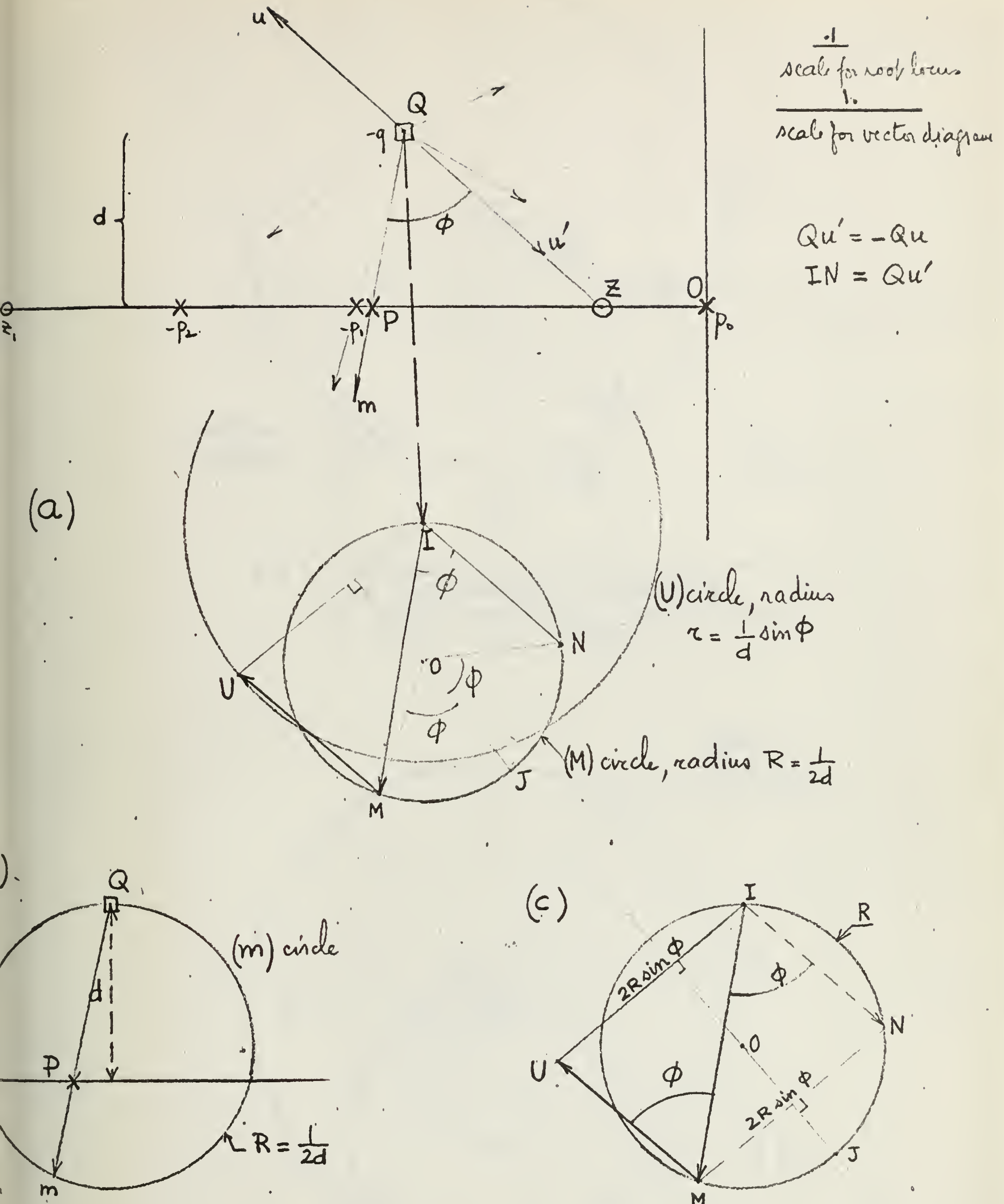


Fig 4 : Geometric locus of point U
a/ Vector diagram and Circle diagram
b/ the (m) circle, moved down by QI, gives (M) circle
c/ IU has constant length = 2Rsinφ

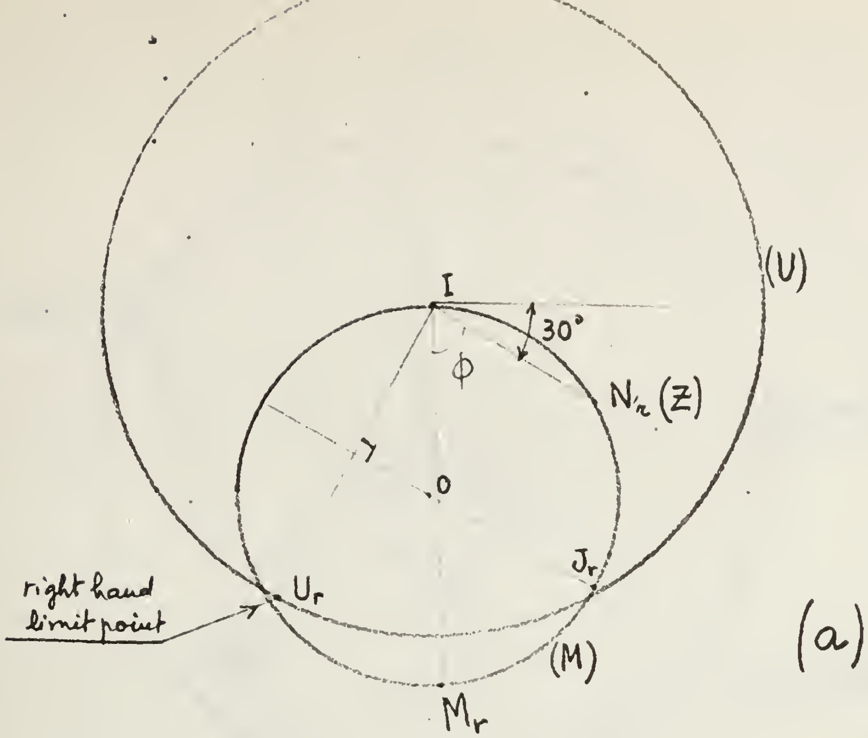
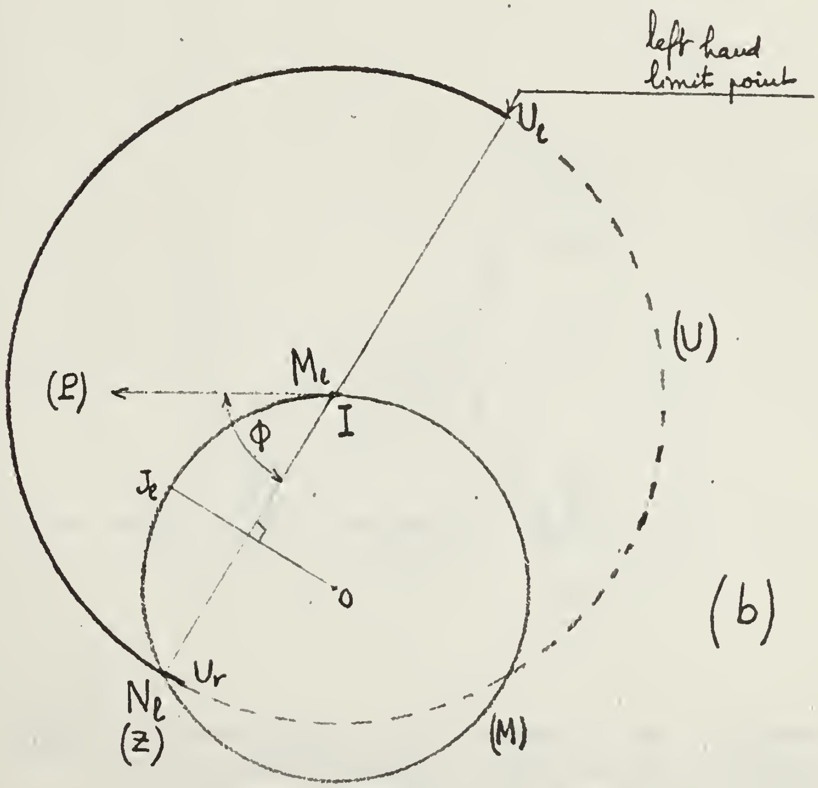


Fig 5: Determination of the limit points of the geometric locus of U.
a/ right hand limit: compensator's zero Z is at origin.



b/ left hand limit : compensator's pole P is at minus infinity.

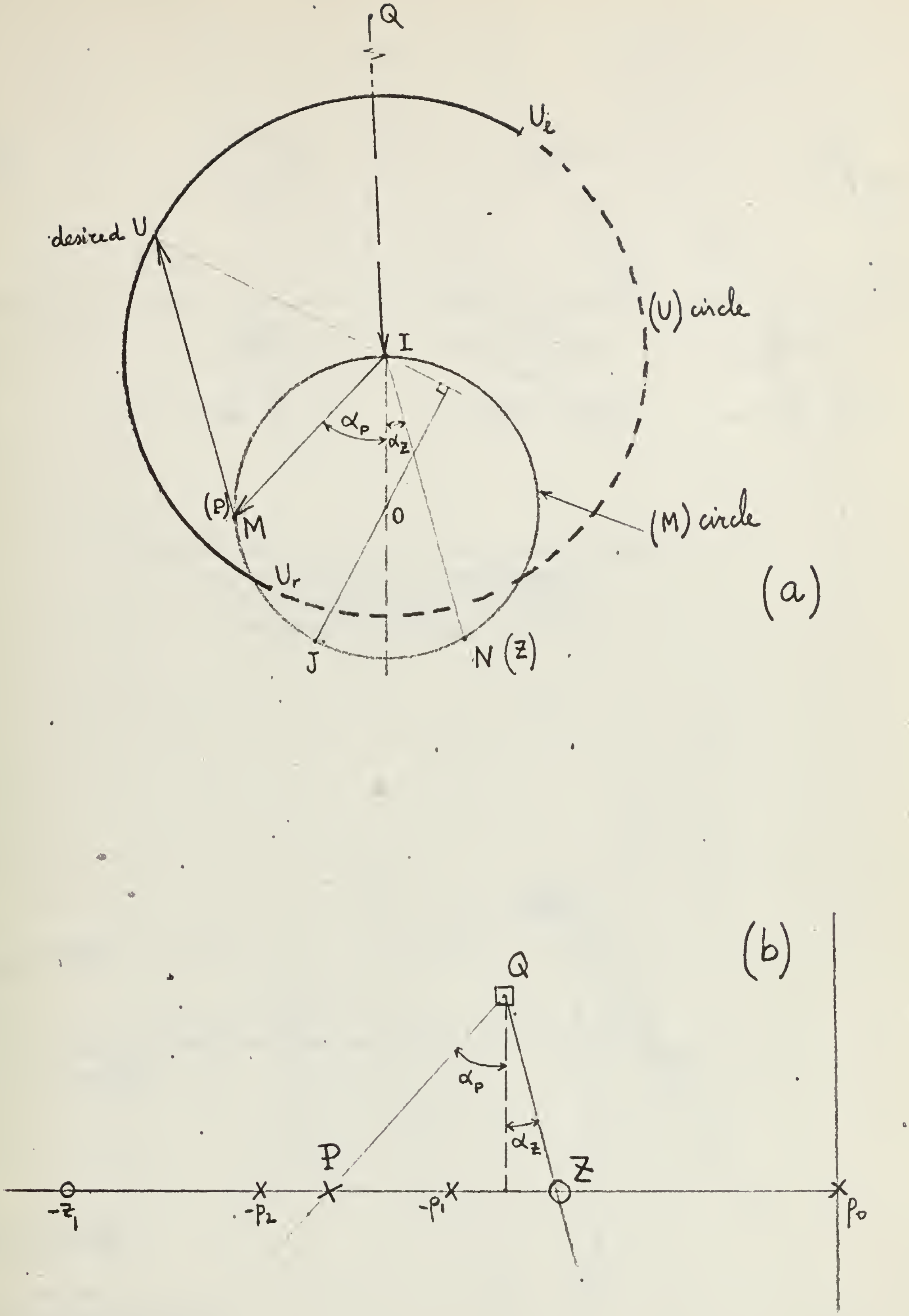


Fig 6: Determination of P and Z for a given U.
a/ obtain α_p and α_z on the circle diagram.
b/ obtain location of P and Z on root locus.

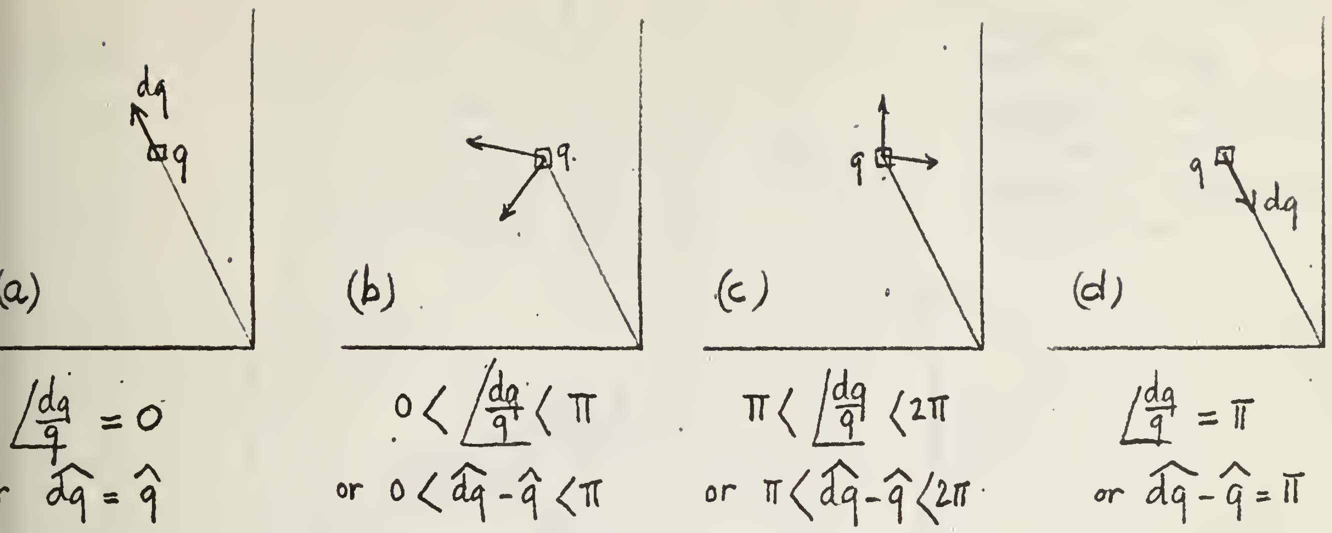


Fig 7: Variations in dominant poles.
a,d: stability unchanged.
b: stabilizing effect.
c: destabilizing effect.

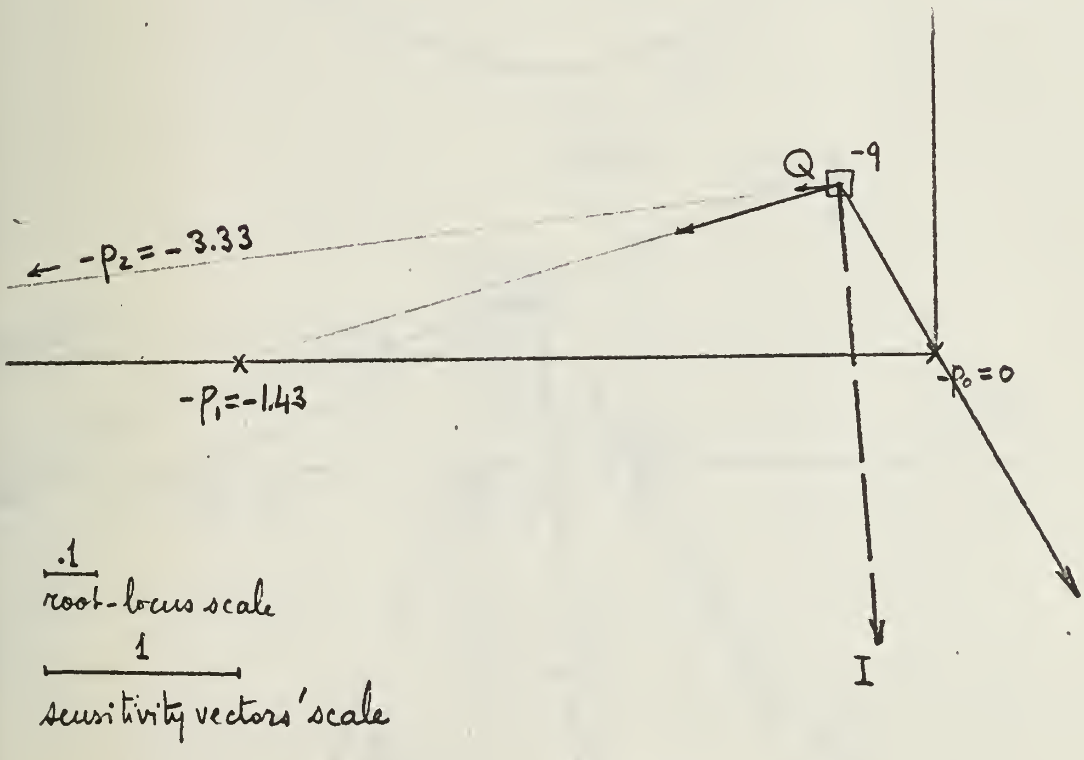


Fig 8: Design example no 1. (compensation for minimum sensitivities.) The "uncompensated" vector diagram.

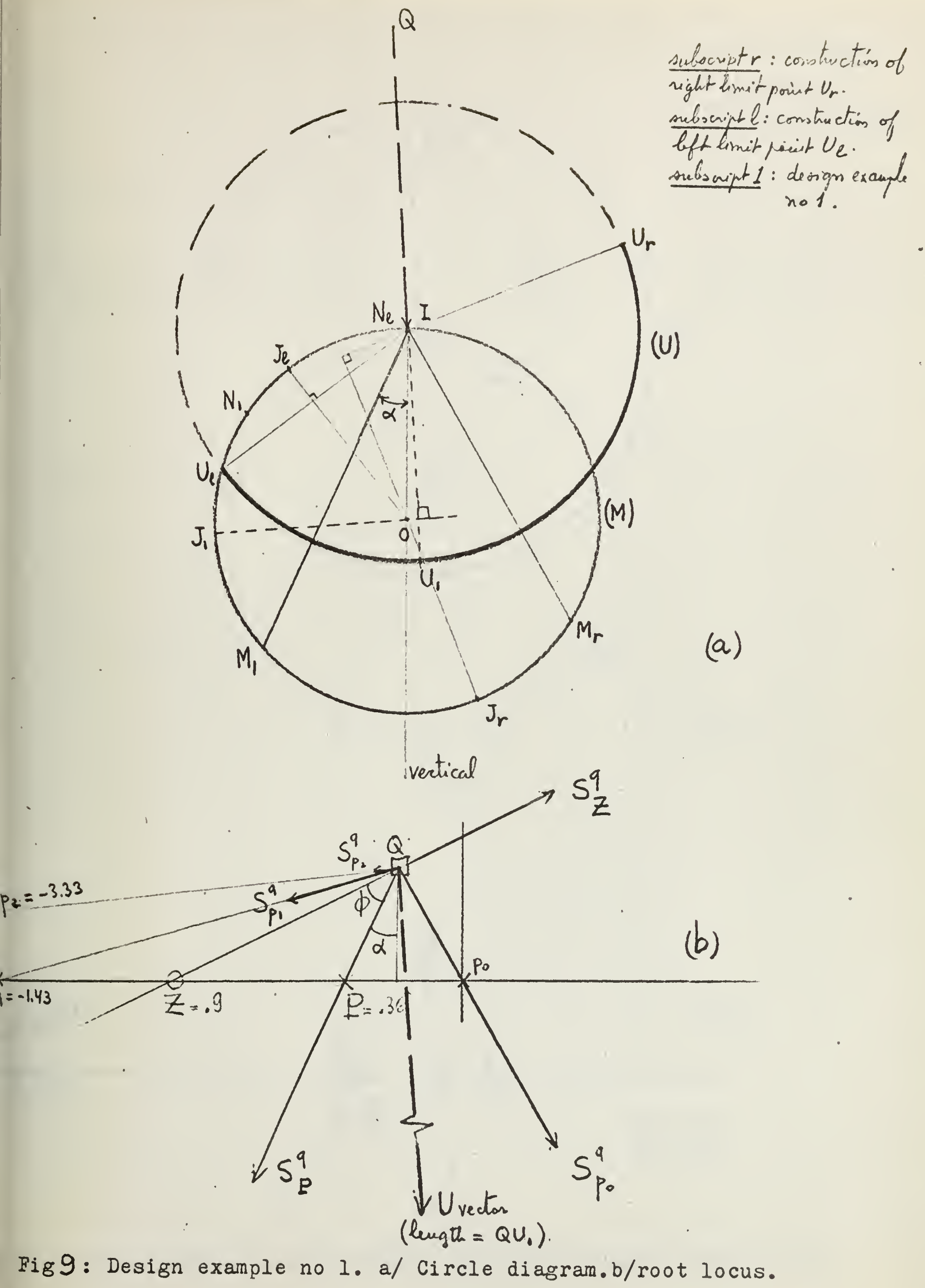
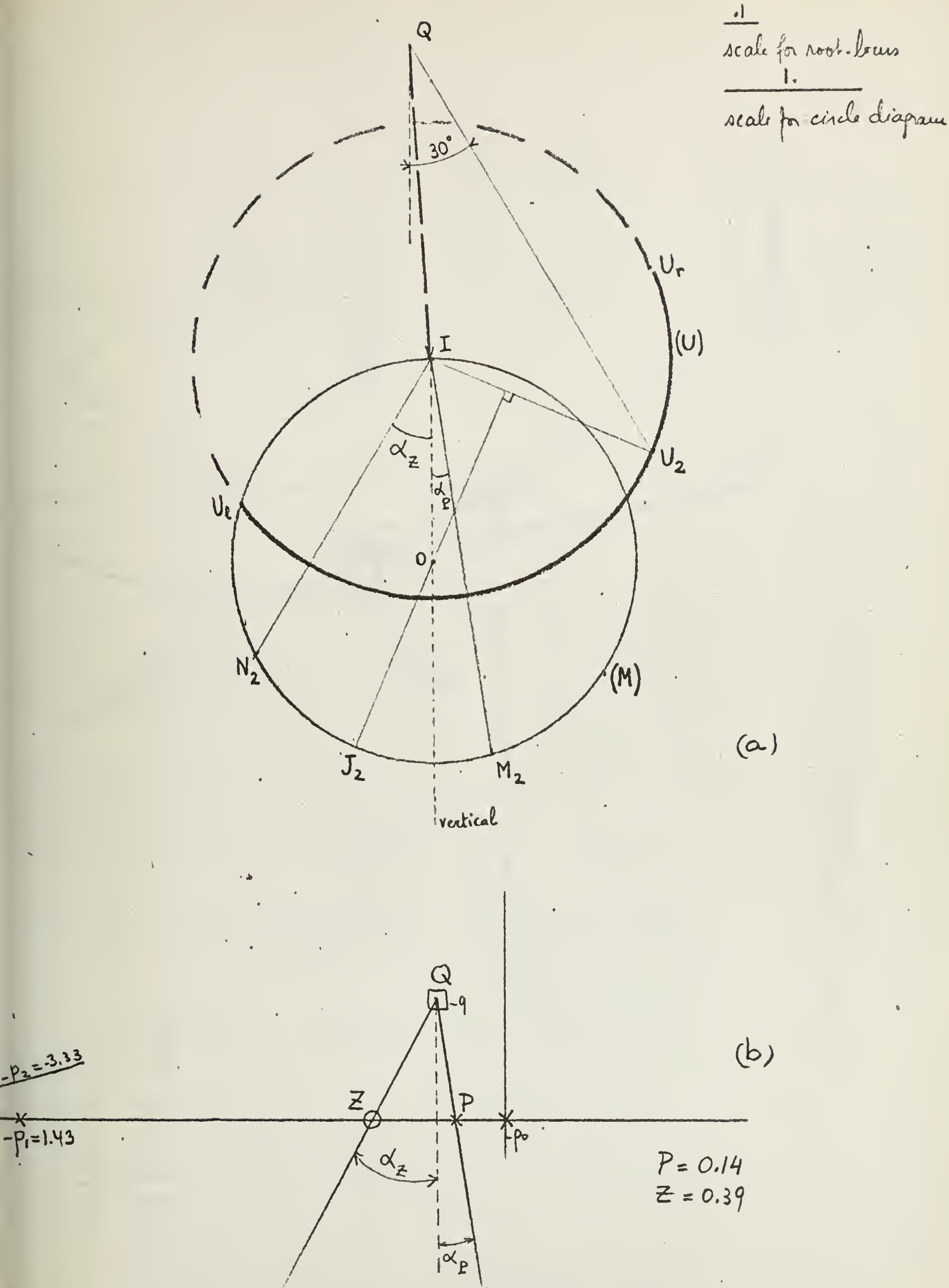


Fig 9: Design example no 1. a/ Circle diagram.b/root locus.



g10: Design example no 2.(for $S_K^s = 0$).a/Circle diagram.b/Root locus.

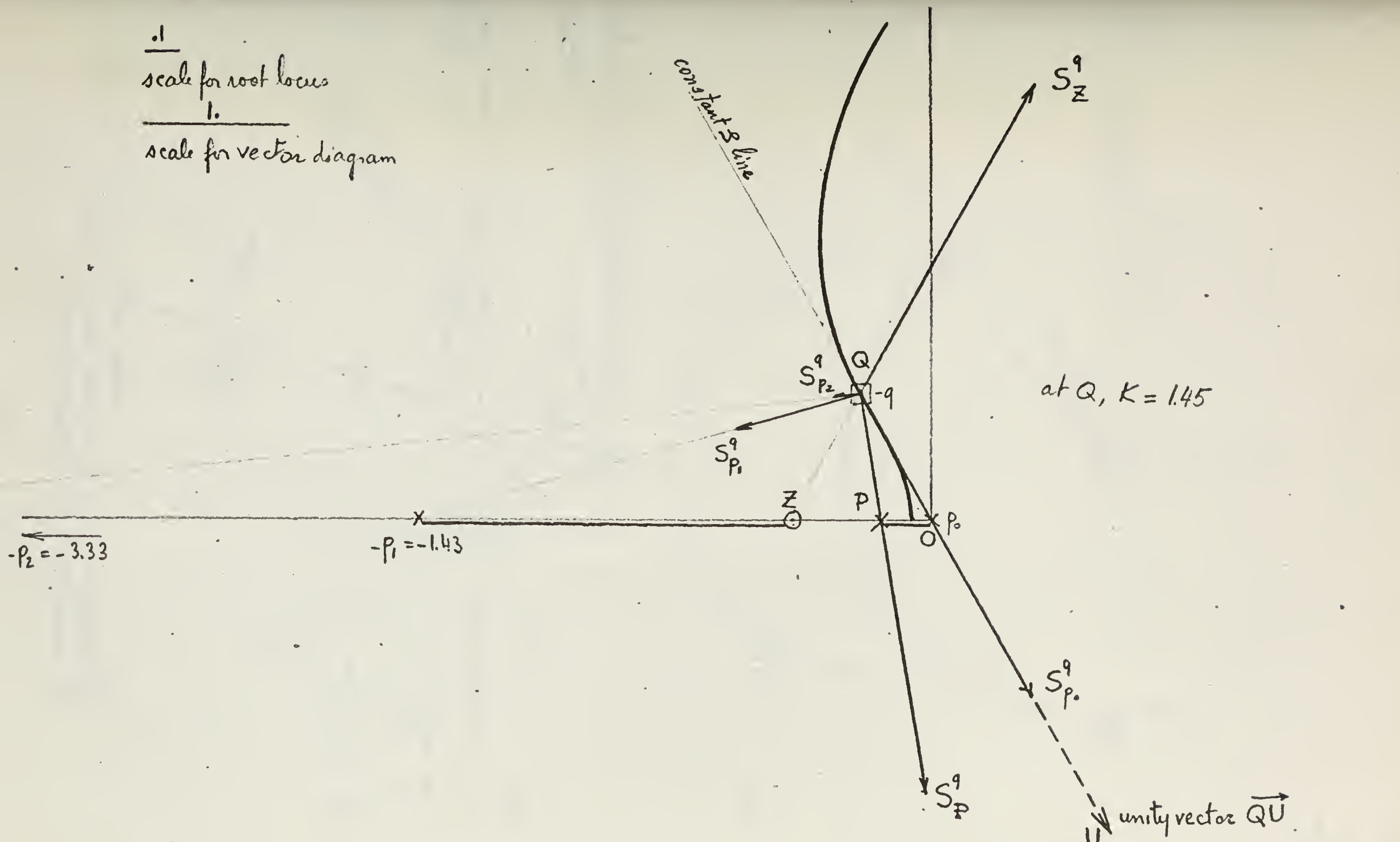


Fig 11 : Design example no.2:
The compensated root locus.

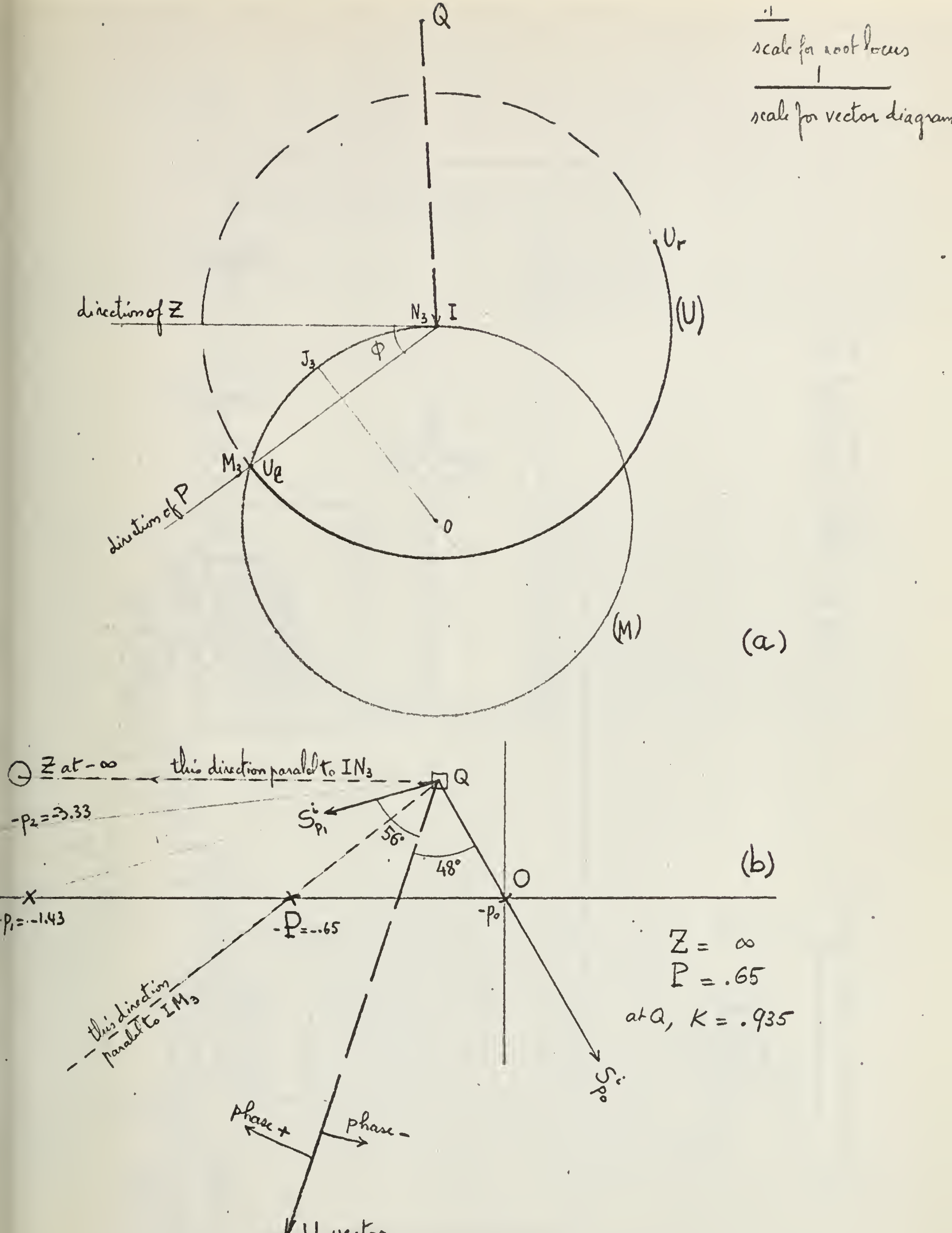
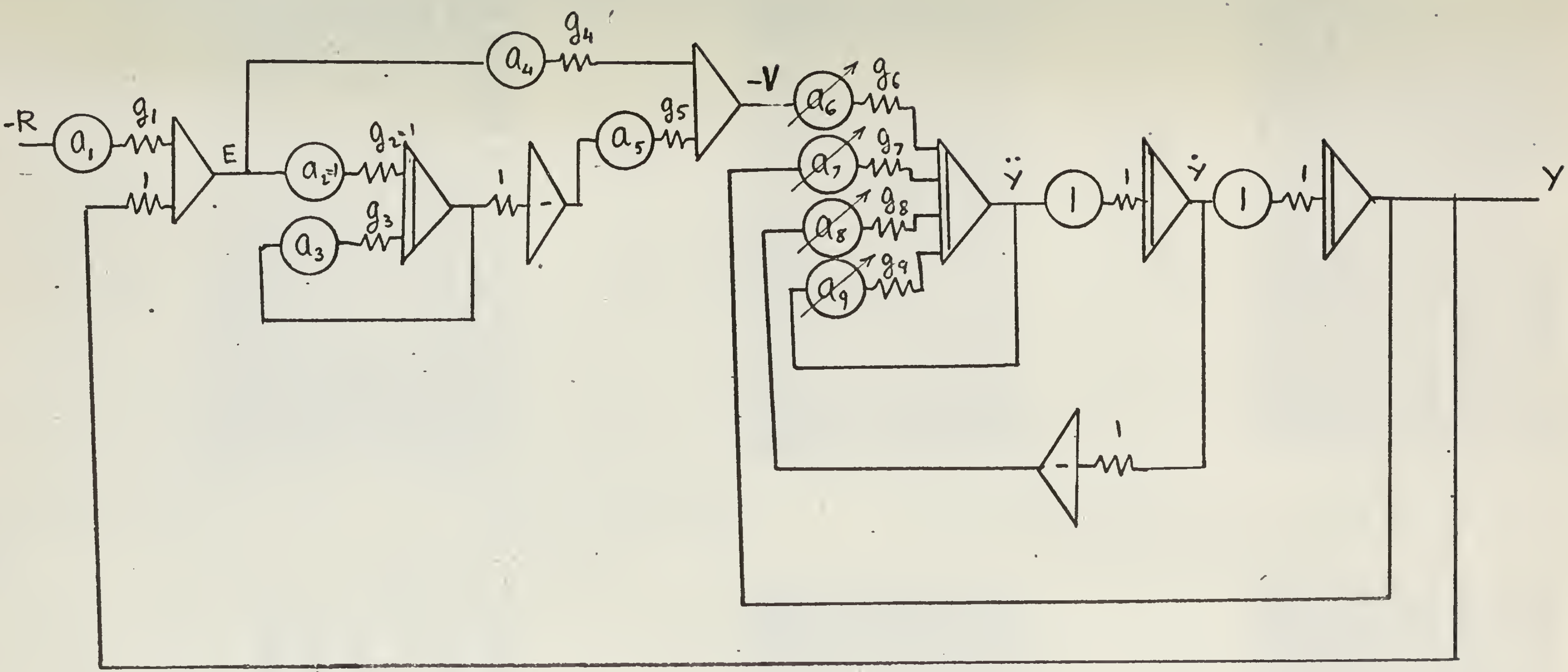
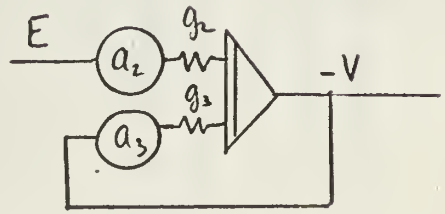


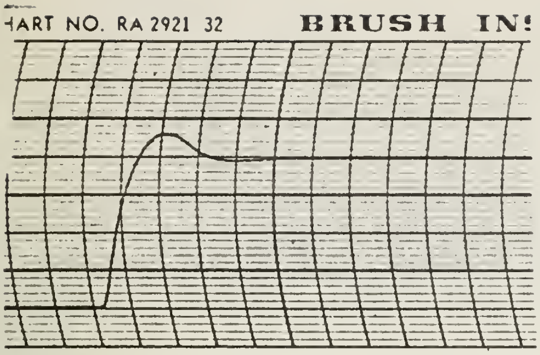
Fig 12 : Design example no 3 (for minimum $S_{p_0}^{s}$)
a/ Circle diagram. b/ Root locus.



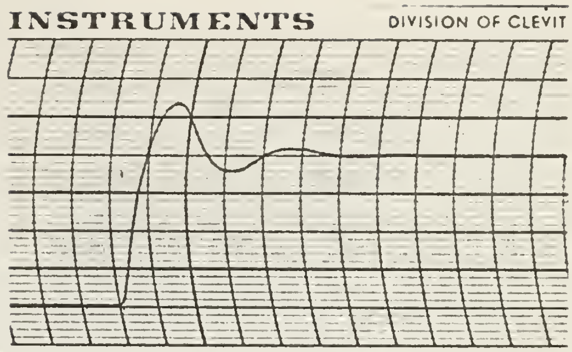


(Compensator with finite P only)

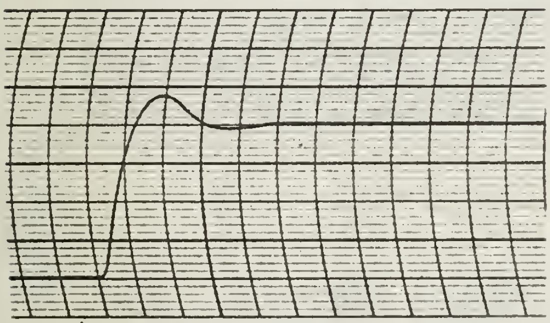
Fig 13: Analog simulation for design examples 2 and 3.



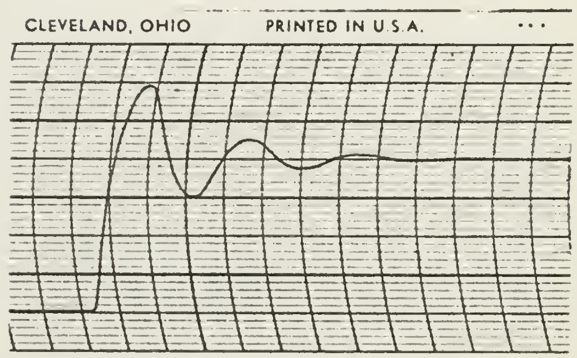
p_1 : 1.43



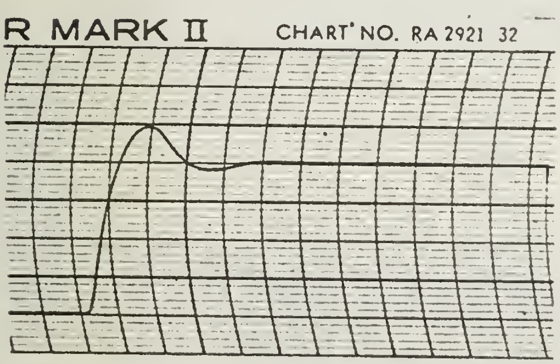
p_1 : 1.03



p_1 : 1.33



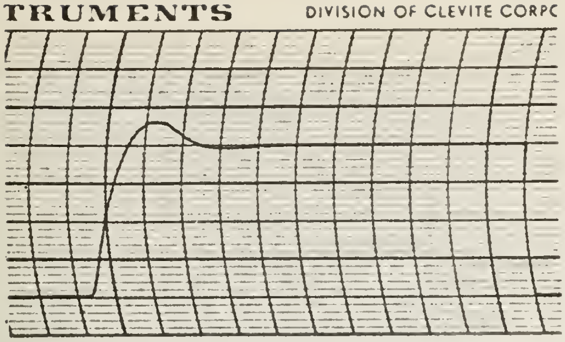
p_1 : 0.83



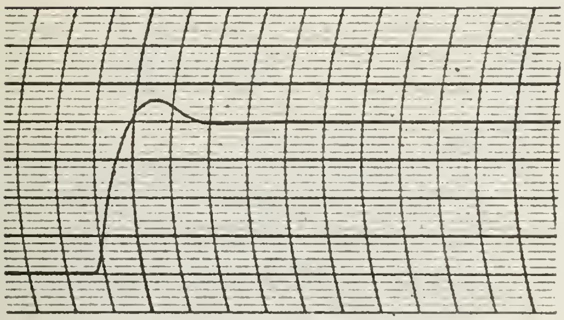
p_1 : 1.23

Fig 14 : Step response of compensated system of exam_ ple 3, when plant pole $-p_1$ varies. Note the rapid change of damping as $\widehat{S_{p_1}^{i}}$ approaches 30°.

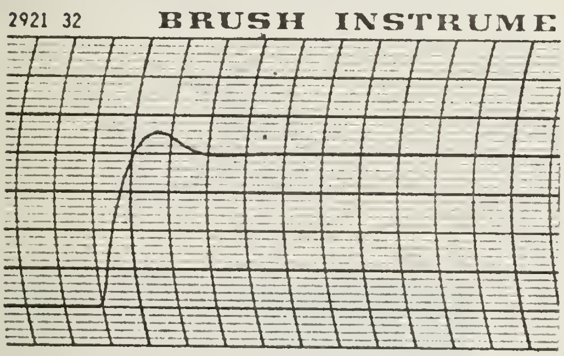
(1)



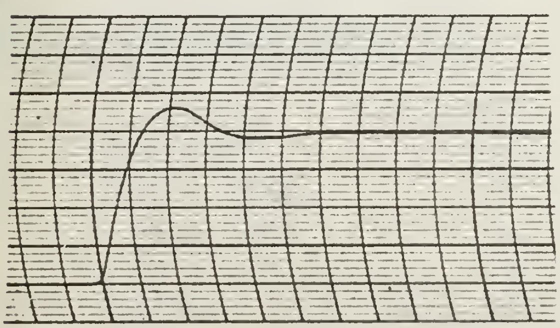
(2)



(3)

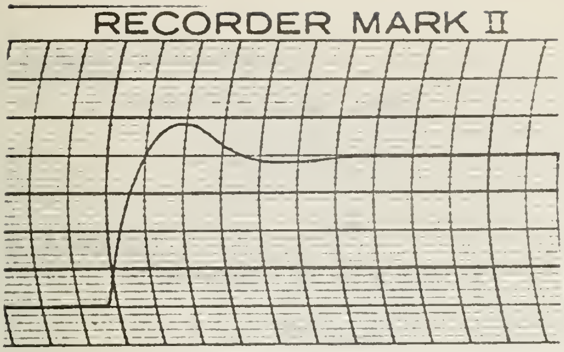


(4)



(1) p_o : 0
(2) p_o : 0.05
(3) p_o : 0.10
(4) p_o :-0.10

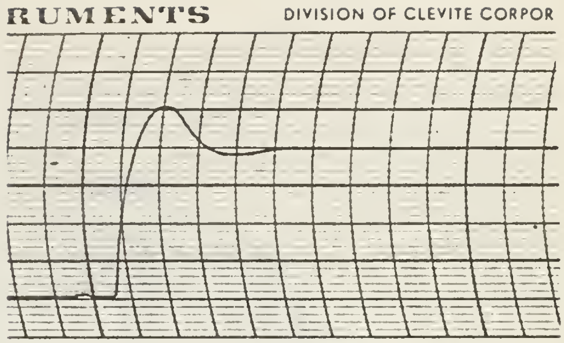
Fig 15 : Step response of compensated system of example 3, when plant pole $-p_o$ varies about the origin. Note that damping is unchanged.



K : 0.85

M_{pt} :1.2

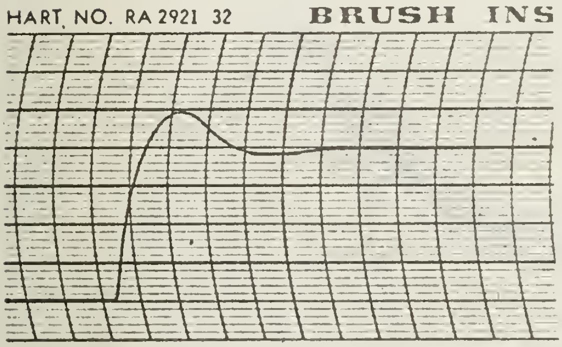
$\zeta \approx 0.45$



K : 1.45 (nominal)

M_{pt} : 1.26

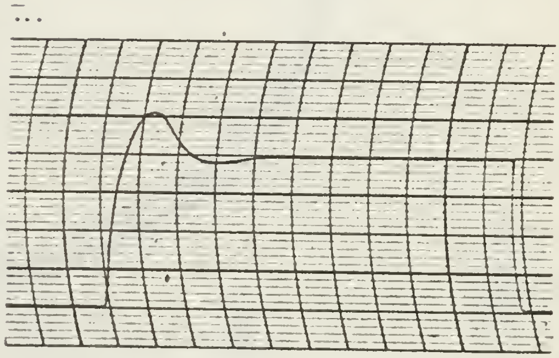
$\zeta \approx 0.40$



K : 1.00

M_{pt} : 1.23

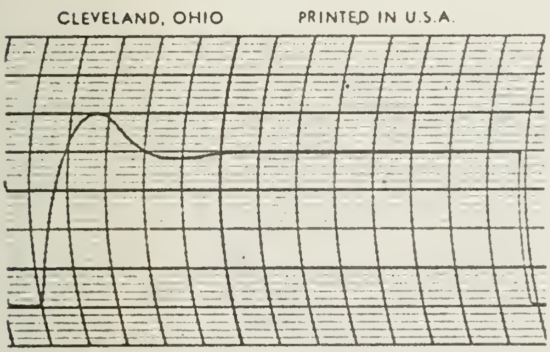
$\zeta \approx 0.42$



K : 1.60

M_{pt} : 1.26

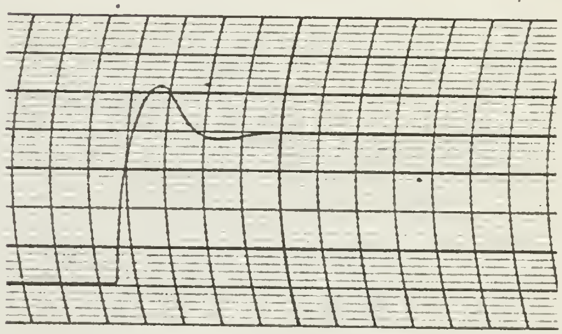
$\zeta \approx 0.40$



K ; 1.20

M_{pt} : 1.25

$\zeta \approx 0.41$



K : 1.75

M_{pt} : 1.29

$\zeta \approx 0.37$

Fig 16 : Step response of compensated system of example 2 when K varies about its nominal value.

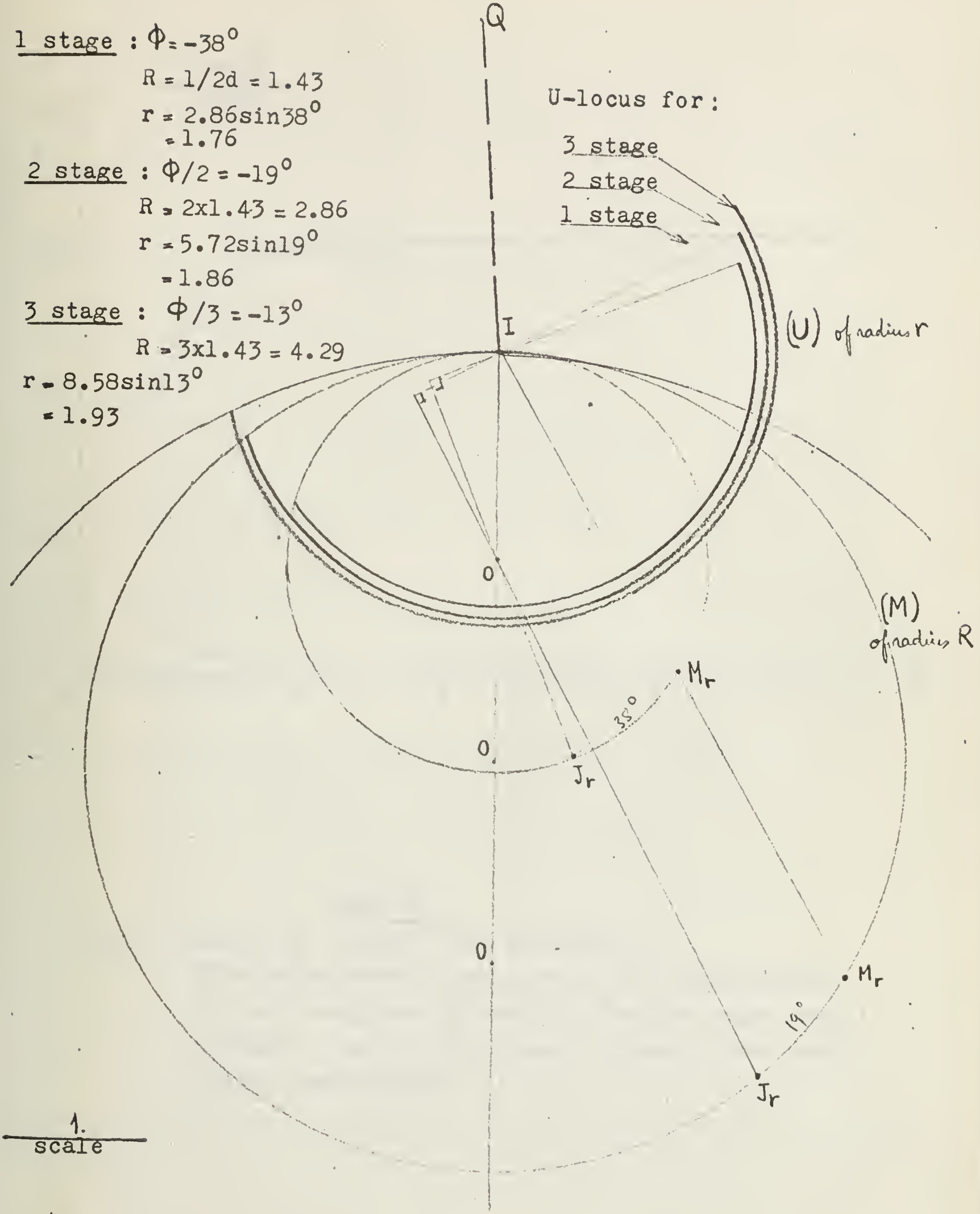
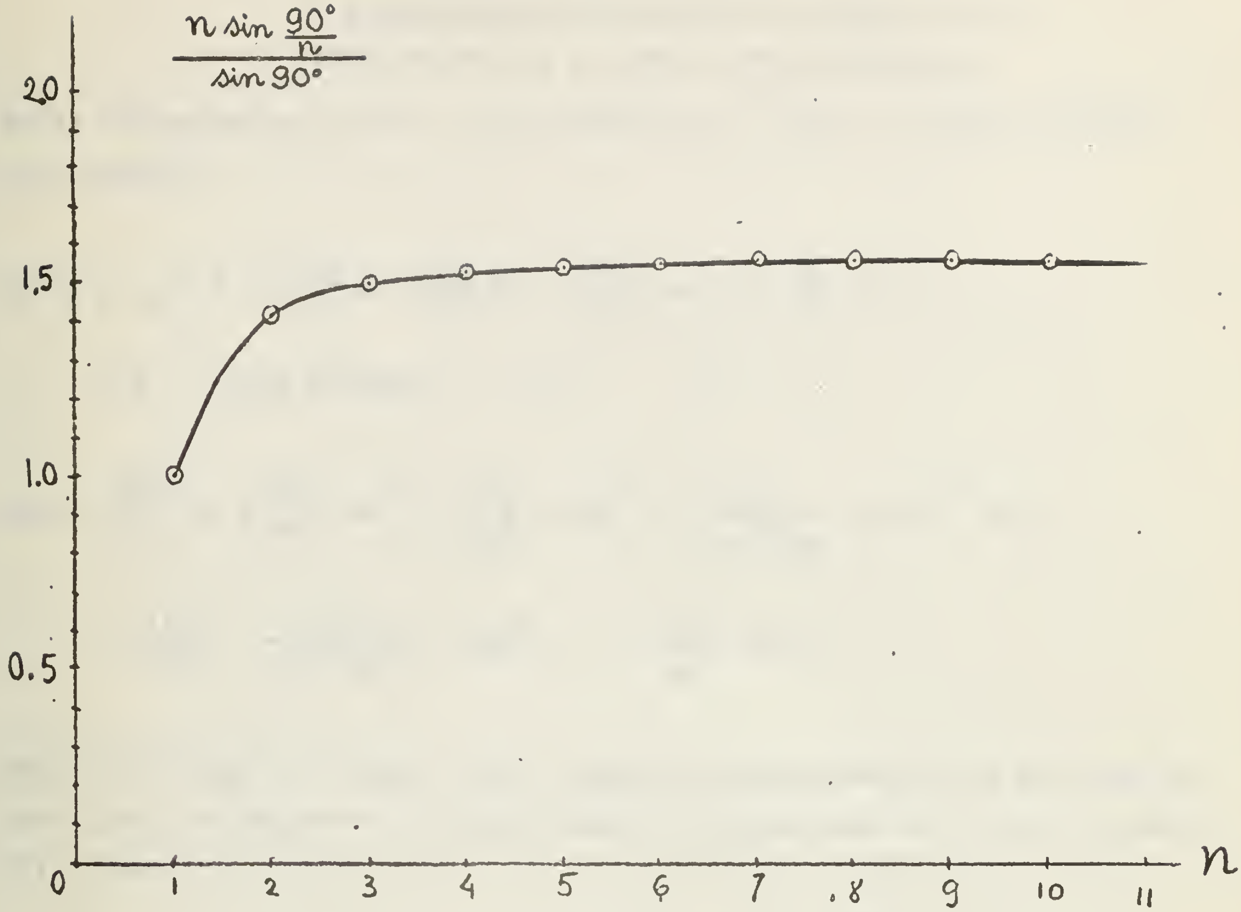


Fig 17 : U-locus for 1-stage, 2-stage, and 3-stage compensators. Improvement is negligible.



$$\frac{n \sin \frac{90^0}{n}}{\sin 90^0}$$

Fig 18 : Value of $\frac{n\sin\frac{90^0}{n}}{\sin 90^0}$ for different n.
This curve shows that most of the improvement, if any, is given by use of a double stage comp-ensator. Use of more than 2 stages gives negli-gible improvement.

APPENDIX 1

Derivation of an expression for the sensitivity of a multiple-order system root (equation 29)
(This derivation is taken from reference 4)

Write the expansion of the total differential of L(s) to include higher-order terms:

$$(dL)_{s=-q_i} = 0 = \left[\frac{\partial L}{\partial s} ds + \frac{\partial L}{\partial K} dK + \sum_j \frac{\partial L}{\partial z_j} dz_j + \sum_j \frac{\partial L}{\partial p_j} dp_j \right]_{s=-q_i}$$
$$+ \frac{1}{2!} \left[\text{same bracket} \right]^2 + \frac{1}{3!} \left[\quad \right]^3 + \ldots .$$

$$\text{where } \left[\frac{\partial L}{\partial s} ds + \frac{\partial L}{\partial K} dK \right]^i = \frac{\partial^i L}{\partial s^i} (ds)^i + i \frac{\partial^i L}{\partial s^{i-1} \partial K} (ds)^{i-1} dK +$$
$$+ \frac{i(i-1)}{2!} \frac{\partial^i L}{\partial s^{i-2} \partial K^2} (dK)^2 + \ldots . + \frac{\partial^i L}{\partial K^i} (dK)^i$$

Next retain only the lowest order terms for each parameter and not that the first (N-1) derivatives of L with respect to s are zero at $s = -q_i$ (equation 27). Then

$$dq_i = \left[\frac{(-1)^{N+1} N! \left[\frac{\partial L}{\partial K} dK + \sum_j \frac{\partial L}{\partial z_j} dz_j + \sum_j \frac{\partial L}{\partial p_j} dp_j \right]}{\frac{\partial^N L}{\partial s^N}} \right]^{\frac{1}{N}}_{s=-q_i}$$

This suggests the following notation:

$$dq_i = \left[S^i_K \frac{dK}{K} + \sum_j S^i_{z_j} dz_j + \sum_j S^i_{p_j} dp_j \right]^{\frac{1}{N}}$$

Comparison of the above 2 equations yields equation (29) of text, which is thus proved.

APPENDIX II

Relationship between S_K^i, S_z^j, S_p^i for
the multiple-order system root-case.

Prove that equations (15) and (16) are still valid for the case of multiple-order root at $-q_i$:

From comparison of the last two equation of appendix I, one obtains:

$$S_{z_j}^i \triangleq \frac{(-1)^{N+1} N!}{\frac{\partial^N L}{\partial s^N}} \left[\frac{\partial L}{\partial z_j} \right]_{s = -q_i}$$

But

$$\frac{\partial L}{\partial z_j} = \left. \frac{L}{s + z_j} \right|_{s = -q_i} = \frac{-1}{z_j - q_i}$$

Thus: $$S_{z_j}^i = \frac{S_K^i}{z_j - q_i}$$

Similar proof shows that

$$S_{p_j}^i = \frac{S_K^i}{q_i - p_j}$$

that is, equations (15) and (16) are valid for multiple-order root at $-q_i$ as well as for single-order root.

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