NPS55-80-007

NAVAL POSTGRADUATE SCHOOL
Monterey, California

DIFFUSION APPROXIMATIONS FOR
THE COOPERATIVE SERVICE
OF VOICE AND DATA MESSAGES

by

J. P. Lehoczky
and
D. P. Gaver

February 1980

[illegible]ved for Public Release; Distribution Unlimited.

[illegible]ed for:

Naval Postgraduate School
Monterey, California 93940

NAVAL POSTGRADUATE SCHOOL
MONTEREY, CALIFORNIA

Rear Admiral J. J. Ekelund
Superintendent

J. R. Borsting
Provost

This report was prepared by:

UNCLASSIFIED

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

REPORT DOCUMENTATION PAGE		READ INSTRUCTIONS BEFORE COMPLETING FORM
1. REPORT NUMBER NPS55-80-007	2. GOVT ACCESSION NO.	3. RECIPIENT'S CATALOG NUMBER
4. TITLE (and Subtitle) Diffusion Approximations for the Cooperative Service of Voice and Data Messages		5. TYPE OF REPORT & PERIOD COVERED Technical
		6. PERFORMING ORG. REPORT NUMBER
7. AUTHOR(s) J. P. Lehoczky and D. P. Gaver		8. CONTRACT OR GRANT NUMBER(s)
9. PERFORMING ORGANIZATION NAME AND ADDRESS Naval Postgraduate School Monterey, CA 93940		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS
11. CONTROLLING OFFICE NAME AND ADDRESS Naval Postgraduate School Monterey, Ca. 93940		12. REPORT DATE February 1980
		13. NUMBER OF PAGES 39
14. MONITORING AGENCY NAME & ADDRESS(if different from Controlling Office)		15. SECURITY CLASS. (of this report) Unclassified
		15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

Approved for Public Release; Distribution Unlimited.

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

Queues
Communications
Data Transmission
Voice Transmission
Semigroup Theory
Probability Modeling

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

A probability model is presented for a set of communication channels that share the service of data and voice transmissions. A diffusion-theoretic approximation is derived, utilizing new results of Burman (1979). It is shown that the data queue (which is of low priority relative to voice) is approximated by a Wiener process.

Unclassified

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

DIFFUSION APPROXIMATIONS FOR THE COOPERATIVE SERVICE OF VOICE AND DATA MESSAGES

by

J. P. Lehoczky
Carnegie-Mellon University
Pittsburgh, PA

and

D. P. Gaver
Naval Postgraduate School
Monterey, CA

INTRODUCTION

In this paper we study the behavior of a queueing system which arises in the study of certain communication networks. Specifically we study a queueing phenomenon which arises with the SENET network, as described by Coviello and Vena (1975) or Barbacci and Oakley (1976). This network allows for both voice and data messages to be transmitted over the same channels by using a special type of integrated circuit and packet-switched multiplexor structure. The two classes of traffic have substantially different performance requirements. Voice messages tend to possess great redundancy, and hence not to be sensitive to channel error rates, while data is very sensitive to channel error, having essentially no redundancy. Voice messages on the other hand have critical timing requirements and cannot be queued, while data is relatively insensitive to timing and can be queued. Additionally, voice messages tend to be very long relative to data messages which can be broken up into small packets. These special requirements have led to the following queueing network. A node of the network consists of $c + v$ channels or servers. The voice messages are assigned to v channels and do not queue. Thus the voice messages operate as a loss system. Data messages may use c channels exclusively and any unused voice channels; however, voice preempts data using voice channels. Data messages are queued if necessary. Typical performance measures that one may wish to calculate include the loss rate of voice traffic and the mean data queue length.

We make standard probabilistic assumptions. Specifically, we assume voice traffic arrives according to a Poisson(λ) process and each voice message has an independent exponential(μ) service time. Data messages are assumed to have independent exponential(η) service times and arrive according to a Poisson(δ) process. With these assumptions voice is an M/M/v/v loss system, and data is an M/M/S system where $S = c + v - V(t)$ with $V(t)$ = number of voice messages in service. The stochastic process $\{(X(t),V(t)), t \geq 0\}$ is Markov with state space $Z^+ \times \{0,1,\ldots,v\}$ where $X(t)$ = data system size at time t. One can easily write the Kolmogorov forward equations appropriate for this system; however, these equations do not yield a closed form solution. To describe this system one must either numerically solve the forward equations or introduce approximations.

This system has been studied previously by a number of researchers including Halfin and Segal (1972), Halfin (1972), Fischer and Harris (1976), Bhat and Fischer (1976), Fischer (1977), Chang (1977), and Gaver and Lehoczky (1979a,b). The last two papers introduce a "fluid flow" and a diffusion approximation and derive explicit formulas for data queue behavior. These papers focus on the important case in which $\rho_d = \delta/\eta > c$. In such a situation the data messages must have access to voice channels for the system to be stable. Furthermore, it was assumed that η/μ was large, say 10^4. Under these circumstances the data flow could be treated deterministically. Suppose we define $\rho_V = \lambda/\mu$ and $q = (\rho_V^v/v!)/\sum_{j=0}^{v} \rho_V^j/j!$, the Erlang B blocking probability. The total traffic intensity on the c + v channels is given by $\rho_d + \rho_V(1-q)$, or we could define $\rho = (\rho_d + \rho_V(1-q))/(c+v)$. A heavy traffic approximation can be derived for this case $\rho \nearrow 1$. Such an approximation was derived in Gaver and Lehoczky (1979b) assuming η/μ was large; a Wiener process with reflecting boundary was found appropriate. In this paper we derive a heavy traffic approximation for the system without the fluid flow assumption that η/μ is large. The methodology is drawn heavily from the approach of Burman (1979). In this approach one characterizes a Markov process by its infinitesimal generator. One next suitably normalizes the process so that the generator converges to a limiting infinitesimal generator (in this case to that of a reflected Brownian motion). This convergence allows the conclusion that the finite dimensional distributions of the normalized Markov process converge. The diffusion approximation consists of treating the actual process through its limiting behavior. The details are somewhat complicated by the presence of a boundary.

2.

Let $\{(X(t),V(t)),\ t \geq 0\}$ be a bivariate Markov process with state space $S = Z^+ \times \{0,1,\ldots,v\}$. Here $\{V(t),\ t \geq 0\}$ is marginally an M/M/v/v loss system with arrival rate λ and service rate μ. Conditional on $V(t)$, $\{X(t),\ t \geq 0\}$ is an $M/M/(c + v - V(t))$ queueing system with arrival rate δ and service rate η. We say that the V process subordinates the X process. We let

$$\underset{\sim}{Q} = \underset{\sim}{\mu}\begin{pmatrix} -\rho_v & \rho_v & & & & & \\ 1 & -(1+\rho_v) & \rho_v & & & & \\ & 2 & -(2+\rho_v) & \rho_v & & & \\ & & \ddots & \ddots & \ddots & & \\ & & & \ddots & \ddots & \rho_v & \\ & & & & v-1 & -(v-1+\rho_v) & \rho_v \\ & & & & & v & -v \end{pmatrix}, \tag{2.1}$$

the infinitesimal generator of the V process.

The generator of the (X,V) process is given by

$$Af(x,k) = \begin{cases} Qf(x,k) + \delta(f(x+1,k) - f(x,k)) \\ \qquad + \eta(c+v-k)(f(x-1,k)-f(x,k)) \\ \qquad\qquad \text{if } x \geq c+v-k \\ \\ Qf(x,k) + \delta(f(x+1,k) - f(x,k)) \\ \qquad + \eta x(f(x-1,k) - f(x,k)) \\ \qquad\qquad \text{if } x < c+v-k \end{cases} \tag{2.2}$$

for $f:S \to R$ continuous where

$$Qf(x,k) = \rho_v f(x,k+1) - (k+\rho_v) f(x,k) + kf(x,k-1) \qquad v \geq k \geq 0 \tag{2.3}$$

and $f(x,-1) = f(x,v+1) = 0$. Clearly $Qf(x) = 0$, that is Q annihilates functions of x alone. We next normalize the (X,V) process by defining $X_n(t) = X(nt)/\sqrt{n}$ and $V_n(t) = V(nt)$. One can calculate the generator of the Markov process $\{(X_n(t), V_n(t)), t \geq 0\}$ having state space $S_n = \{0, 1/\sqrt{n}, 2/\sqrt{n},\ldots\} \times \{0,1,\ldots,v\}$ to be

$$A_n f(x,k) = \begin{cases} nQf(x,k) + \delta n(f(x+1/\sqrt{n},k) - f(x,k)) \\ \qquad + \eta_n (f(x-1/\sqrt{n},k) - f(x,k)) \\ \qquad\qquad \text{if } x \geq \dfrac{c+v-k}{\sqrt{n}} \\[2ex] nQf(x,k) + \delta n(f(x+1/\sqrt{n},k) - f(x,k)) \\ \qquad + \eta n\sqrt{nx}\,(f(x+1/\sqrt{n},k) - f(x,k)) \\ \qquad\qquad \text{if } x = 0, 1/\sqrt{n},\ldots, (c+v-k)/\sqrt{n}\,. \end{cases} \tag{2.4}$$

We assume $f(x,k)$ has three bounded derivatives in x for each fixed k. With this assumption one can expand terms in (2.4) in a Taylor series and rewrite as

$$A_n f(x,k) = \begin{cases} nQf(x,k) + n^{1/2} f_x(x,k)(\delta - \eta(c+v-k)) \\ \qquad + \frac{1}{2} f_{xx}(x,k)(\delta + \eta(c+v-k)) \\ \qquad + O(n^{-1/2}) \\ \qquad \text{if } \; x \geq (c+v-k)/\sqrt{n} \\ \\ nQf(x,k) + n^{1/2} f_x(x,k)(\delta - \eta\sqrt{nx}) \\ \qquad + \frac{1}{2} f_{xx}(x,k)(\delta + \eta\sqrt{nx}) + O(n^{-1/2}) \\ \qquad \text{if } \; x = 0, 1/\sqrt{n}, \ldots, (c+v-k)/\sqrt{n} \end{cases} \tag{2.5}$$

with $f_x(x,k) = \frac{\partial}{\partial x} f(x,k)$ and $f_{xx}(x,k) = \frac{\partial^2}{\partial x^2} f(x,k)$.

We ultimately wish to prove that the finite dimensional distributions of $\{X_n(t), t \geq 0\}$ converge to those of a Wiener process with reflecting barrier at the origin. This can be restated in terms of semi-groups. We let $\{T_t^n, t \geq 0\}$ be the semi-group of operators associated with $\{(X_n(t), V_n(t)), t \geq 0\}$ and $\{T_t^\infty, t \geq 0\}$ be that associated with a Wiener process having reflecting barrier at 0. Let g be a continuous function $g: R' \to R'$. Knowledge of the semi-group is equivalent to knowledge of the transition functions by taking a sequence of g's which approximate indicator functions. We wish to prove $|T_t^n g(x,k) - T_t^\infty g(x)| \to 0$ as $n \to \infty$ for all (x,k). Here $T_t^n g(x,k) = E(g(X_t) | X_N(0) = x, V_n(0) = k)$. The presence of the variable k prevents this from being done directly. The method we use is to construct a convenient sequence of functions $\langle g_n \rangle_{n=1}^{\infty}$ which converge in some sense to g. We write

$$\|T_t^n g - T_t^\infty g\|_n \leq \|(T_t^\infty g)_n - T_t^\infty g\|_n + \|T_t^n g - T_t^n g_n\|_n$$
$$+ \|T_t^n g_n - (T_t^\infty g)_n\|_n \qquad (2.6)$$

where $\| \ \|_n$ refers to the sup norm over S_n. Both T_t^n and T_t^∞ are contraction semigroups.

$\langle (T_t^\infty g)_n \rangle_{n=1}^{\infty}$ is the sequence of functions constructed from $T_t^\infty g$. Our goal is to show that each of the three terms on the right side of (2.6) converges to 0. The first and second terms can be handled similarly. For any function g, we must guarantee that the constructed $\langle g_n \rangle_{n=1}^{\infty}$ sequence satisfies $\|g_n - g\|_n \to 0$. It will follow that $\|(T_t^\infty g)_n - T_t^\infty g\|_n \to 0$. Moreover, since $\{T_t^n, t \geq 0\}$ is a contraction semi-group $\|T_t^n g - T_t^n g_n\|_n \leq \|g - g_n\|_n$ which also converges to 0. The sequence $\langle g_n \rangle_{n=1}^{\infty}$ will be chosen in such a way that the third term converges to 0.

We focus on a convergence determining class of functions g, those which are bounded and have three bounded derivatives. For such a function $g(x)$ we define

$$g_n(x,k) = g(x) + \frac{1}{\sqrt{n}} u(x,k) + \frac{1}{n} v(x,k) \qquad (2.7)$$

where u and v have two bounded derivatives in x for each fixed k. The functions u and v will be determined explicitly later and are chosen to control the third term in (2.6). Clearly when g_n is defined by (2.7), $\|g_n - g\|_n = O(n^{-1/2})$ and therefore converges to 0 as required.

One can apply the generator A_n to g_n to derive

$$A_n g_n(x,k) = \begin{cases} nQg(x) + n^{1/2}[Qu(x,k) + g'(x)(\delta - \eta(c+v-k))] \\ + [Qv(x,k) + u_x(x,k)(\delta - \eta(c+v-k)) \\ \quad + \frac{1}{2} g''(x)(\delta - \eta(c+v-k))] + O(n^{-1/2}) \\ \qquad\qquad \text{if } \; x \geq (c+v-k)/\sqrt{n} \\ \\ nQg(x) + n^{1/2}[Qu(x,k) + g'(x)(\delta - \eta n^{1/2} x)] \\ \quad + [Qv(x,k) + u_k(x,k)(\delta - \eta n^{1/2} x) + \frac{1}{2} g''(x)(\delta + \eta n^{1/2} x)] \\ \quad + O(n^{-1/2}) \\ \qquad\qquad \text{if } \; x = 0,\, 1/\sqrt{n}, \ldots,\, (c+v-k)/\sqrt{n} \end{cases}$$

where $u_x(x,k) = \frac{\partial}{\partial x} u(x,k)$. Recall that Q annihilates functions of x alone, thus $nQg(x) \equiv 0$. We want to have $A_n g_n(x,k)$ converge to a finite limit and to have that limit be independent of k. For this to occur, the $n^{1/2}$ term must be controlled and the functions u and v must be chosen in such a way as to eliminate the variable k.

The $n^{1/2}$ coefficient in (2.8) can be rewritten by adding and subtracting

$$\sum_{k=0}^{v} \pi_k g'(x) \ (\delta - \eta(c+v-k)) = -\eta(c+v)(1-\rho) g'(x) \ .$$

We next pick $u(x,k)$ to be a solution of

$$\begin{aligned} \Omega u(x,k) &= -(g'(x)\ \eta(\rho_d-(c+v-k)) + g'(x)\ \eta(c+v)(1-\rho)) \\ &= -g'(x)\ \eta(k-\rho_v(1-q)) \end{aligned} \qquad (2.9)$$

When $u(x,k)$ is any solution of (2.9), the coefficient of the $n^{1/2}$ term in (2.8) becomes

$$\begin{cases} -g'(x)\ \eta(c+v)(1-\rho) & \text{if } x \geq \dfrac{c+v-k}{\sqrt{n}} \\ g'(x)\ \eta((c+v)\rho - n^{1/2}x - k) & \text{if } 0 \leq x < \dfrac{c+v-k}{\sqrt{n}} \end{cases}$$

Equation (2.9) can be solved explicitly. Define $a_k = -g'(x)\ \eta(k-\rho_v(1-q))/\mu$, so that (2.9) can be written as

$$\begin{aligned} -\rho_v(u(x,k)-u(x,k-1))-(k-1)(u(x,k-1)-u(x,k-2)) &= a_{k-1},\ k = 1,\ldots,v \\ -v(u(x,v)-u(x,v-1)) &= a_v \end{aligned} \qquad (2.10)$$

Equation (2.10) has a solution since $\sum_{k=0}^{v} \pi_k a_k = 0$, where $\langle \pi_k \rangle_{k=0}^{v}$ is the stationary distribution associated with Ω, or $\pi_k = (\rho_v^k/k!)/(\sum_{i=0}^{v} \rho_v^i/i!)$. The solution is given by

$$u(x,k) - u(x,k-1) = \frac{\sum_{i=0}^{k-1} \pi_i a_i}{\rho_v \pi_{k-1}} = \frac{-g'(x)\eta T_{k-1}}{\mu\rho_v \pi_{k-1}}$$

where

$$T_k = \sum_{i=0}^{k} \pi_i (1-\rho_v(1-q)) \quad \text{and} \quad T_v = 0.$$

Clearly

$$u(x,k) = u(x,0) - \frac{g'(x)\eta}{\mu\rho_v} \sum_{i=1}^{k} \frac{T_{i-1}}{\pi_{i-1}}, \quad 1 \leq k \leq v \tag{2.11}$$

where $u(x,0)$ is arbitrary. We let $u(x,0) = \frac{1}{2} g'(x)$ so

$$u(x,k) = g'(x)\left(\frac{1}{2} - \frac{\eta}{\mu\rho_v} \sum_{i=1}^{k} \frac{T_{i-1}}{\pi_{i-1}}\right), \quad 0 \leq k \leq v \tag{2.12}$$

For the choice of u specified by (2.12) we next wish to insure that the limiting generator is independent of the variable k. The function v is chosen to eliminate the dependence on k. The $O(1)$ term of (2.8) is given, for $x \geq (c+v-k)/\sqrt{n}$, by

$$Qv(x,k) + g''(x)\left[\frac{1}{2} - \frac{\eta}{\mu\rho_v}\sum_{i=1}^{k}\frac{T_{i-1}}{\pi_{i-1}}(\delta-\eta(c+v-k))\right]$$

$$+\frac{1}{2}g''(x)(\delta+\eta(c+v-k))$$

$$= Qv(x,k) + H(x,k).$$

Let $\bar{H}(x) = \sum_{k=0}^{v}\pi_k H(x,k)$ and consider $Qv(x,k) + (H(x,k)-\bar{H}(x))+\bar{H}(x)$. We now let $v(x,k)$ be any solution of

$$Qv(x,k) = -(H(x,k) - \bar{H}(x)) \ . \tag{2.13}$$

Equation (2.13) has a one-parameter family of solutions, since $\sum_{k=0}^{v}\pi_k(H(x,k) - \bar{H}(x)) = 0$. When $v(x,k)$ is chosen to be any solution of (2.13), the $O(1)$ term of (2.3), for $x \geq (c+v-k)/\sqrt{n}$, will become $\bar{H}(x)$ and will therefore be independent of k. It remains to calculate $\bar{H}(x)$.

$$\bar{H}(x) = g''(x)\left[\sum_{k=0}^{v}\pi_k\left\{\left(\frac{1}{2} - \frac{\eta}{\mu\rho_v}\sum_{i=1}^{k}\frac{T_{i-1}}{\pi_{i-1}}\right)(\delta-\eta(c+v-k))\right.\right.$$

$$\left.\left.+\frac{1}{2}(\delta+\eta(c+v-k))\right\}\right]$$

$$= g''(x)\left[\delta - \frac{\eta}{\mu\rho_v}\sum_{k=0}^{v}\pi_k(\delta-\eta(c+v-k))\sum_{i=1}^{k}\frac{T_{i-1}}{\pi_{i-1}}\right] \tag{2.14}$$

$$= g''(x)\left[\delta - \frac{\eta^2}{\mu\rho_v}\sum_{k=0}^{v}\pi_k(k-\rho_v(1-q))\sum_{i=1}^{k}\frac{T_{i-1}}{\pi_{i-1}}\right.$$

$$\left.+\frac{\eta(c+v)(1-\rho)}{\mu\rho_v}\sum_{k=0}^{v}\pi_k\sum_{i=1}^{k}\frac{T_{i-1}}{\pi_{i-1}}\right].$$

The second term can be rewritten by interchanging the order of summation. The third term is $O(1-\rho)$. We find

$$\bar{H}(x) = g''(x)\left[\delta - \frac{\eta^2}{\mu\rho_v}\sum_{i=0}^{v-1}\frac{T_i}{\pi_i}\sum_{k=i+1}^{v}\pi_k(k-\rho_v(1-q)) + O(1-\rho)\right]$$

$$= g''(x)\left[\delta - \frac{\eta^2}{\mu\rho_v}\sum_{i=0}^{v-1}\frac{T_i}{\pi_i}(T_v-T_i) + O(1-\rho)\right]$$

with $T_v = 0$ or

$$\bar{H}(x) = g''(x)\,\eta\left[\rho_d + \frac{\eta}{\mu\rho_v}\sum_{i=0}^{v-1}\frac{T_i^2}{\pi_i} + O(1-\rho)\right] . \tag{2.15}$$

For the functions u and v specified by (2.12) and (2.13), equation (2.8) can be rewritten as

$$A_n g_n(x,k) = \begin{cases} -n^{1/2}(1-\rho)(\sigma+v)\,\eta g'(x) \\ \quad + \eta\left[\rho_d + \frac{1}{\mu\rho_v}\sum_{i=0}^{v-1}\frac{T_i^2}{\pi_i} + O(1-\rho)\right] g''(x) + O(n^{-1/2}) \\ \qquad\qquad \text{for } x \geq (c+v-k)/\sqrt{n} \\ \\ n^{1/2}\,\eta[(c+v)\rho - n^{1/2}x-k]g'(x) \\ \quad + \eta g''(x)\left[\rho_d + \frac{\eta}{\mu\rho_v}\sum_{i=0}^{v-1}\frac{T_i^2}{\pi_i} + O(1-\rho)\right. \\ \qquad \left. - (c+v-k-n^{1/2}x)\frac{\eta}{\mu\rho_v}\sum_{i=0}^{k}\frac{T_i}{\pi_i}\right] + O(n^{-1/2}) \\ \qquad\qquad \text{for } x \leq (c+v-k)/\sqrt{n} \end{cases} \tag{2.16}$$

We now introduce the "heavy traffic approximation." In order for the generator to converge to a limiting generator we must have $1-\rho = O(n^{-1/2})$. Specifically, we assume $\rho = \rho_n = 1 - (\theta/\sqrt{n})$ for some $\theta \geq 0$. In this case, $n^{1/2}(1-\rho) = \theta$, and (2.16) becomes

$$A_n g_n(x,k) = \begin{cases} -\theta\eta(c+v)g'(x) + \eta\left[\rho_d + \dfrac{\eta}{\mu\rho_v}\displaystyle\sum_{i=0}^{v-1}\dfrac{T_i^2}{\pi_i}\right]g''(x) + O(n^{-1/2}) \\ \qquad\qquad \text{for } x \geq (c+v-k)/\sqrt{n} \\[2ex] \eta[(c+v)\rho - n^{1/2}x - k]n^{1/2}g'(x) \\ \quad + \eta g''(x)\left[\left\{\rho_d + \dfrac{\eta}{\mu\rho_v}\displaystyle\sum_{i=0}^{v-1}\dfrac{T_i^2}{\pi_i}\right\}\right. \\ \quad \left. - (c+v-k-n^{1/2}x)\dfrac{\eta}{\mu\rho_v}\displaystyle\sum_{i=0}^{k}\dfrac{T_i}{\pi_i}\right] \\ \qquad\qquad \text{for } x \leq (c+v-k)/\sqrt{n} \end{cases} \tag{2.17}$$

We now define a limiting generator A_∞ with domain consisting of all functions g having three bounded derivatives and $g'(0) = 0$. Let

$$A_\infty g(x) = -\theta\eta(c+v)\ g'(x) + \eta\left[\rho_d + \frac{\eta}{\mu\rho_v}\sum_{i=0}^{v-1}\frac{T_i^2}{\pi_i}\right]g''(x),$$

$$x > 0 \tag{2.18}$$

A_∞ is the generator of a Markov process which corresponds to a Wiener process with drift $-\theta\eta(c+v)$, scale

$$2\eta\left[\rho_d + \frac{\eta}{\mu\rho_v}\sum_{i=0}^{v-1}\frac{T_v^2}{\pi_i}\right],$$

and a reflecting barrier at 0. The $O(n^{-1/2})$ terms involve the first three derivatives of g which are bounded. It is clear from a direct comparison of (2.16) and (2.18) that $|A_n g_n(x,k) - A_\infty g| \to 0$ as $n \to \infty$ for all $x > 0$ and k arbitrary. In addition, $g'(0) = 0$ is necessary for the generator to converge at $x = 0$. Unfortunately even assuming $g'(0) = 0$,

$$|A_n g(0,k) - A_\infty g(0)| \to (c+v-k)\frac{\eta^2}{\mu\rho_v}\sum_{i=0}^{k}\frac{T_i^2}{\pi_i} g''(0) \quad \text{as} \quad n \to \infty$$

rather than to 0. One needs a special argument to handle this lack of convergence at the boundary.

We set out to prove the third term in (2.6) converges to 0. Standard semi-group results (see Burman, 1979, p. 33) give

$$(T_t^\infty g)_n - T_t^n g_n = \int_0^t T_{t-s}^n((A_\infty w)_n - A_n w_n)ds \tag{2.19}$$

where $w = w(t,x) = T_t^\infty g(x)$. Recall that $w_n = w + (1/\sqrt{n})u + (1/n)v$ with u and v defined by (2.12) and (2.13) with g replaced by w. It follows that

$$\|T_t^n g_n - (T_t^\infty g)_n\|_n$$

$$= \left\|\int_0^t T_{t-s}^n((A_\infty w)_n - A_\infty w + A_\infty w - A_n w_n)\,ds\right\|_n$$

$$\leq \int_0^t \|T_{t-s}^n((A_\infty w)_n - A_\infty w)\|_n\,ds + \left\|\int_0^t T_{t-s}^n(A_\infty w - A_n w_n)\,ds\right\|_n$$

$$\leq \int_0^t \|(A_\infty w)_n - A_\infty w\|_n\,ds + \left\|\int_0^t T_{t-s}^n(A_\infty w - A_n w_n)\,ds\right\|_n .$$

The first term is clearly $O(n^{-1/2})$. It remains to show that the second is $O(n^{-1/2})$ as well. We have shown $|A_\infty w - A_n w_n| = O(n^{-1/2})$ except at the boundary where it is $O(1)$. We split the integral into two parts, for one of which the process is away from the boundary, and for the other, near the boundary. The integral away from the boundary has an integrand which is $O(n^{-1/2})$. The integral near the boundary is also $O(n^{-1/2})$ since under a heavy traffic assumption the process is rarely near the boundary. The details are merely summarized here; they are based on the ideas of Burman (1979).

Let I_{on} be the indicator function of

$$\left[0, \frac{c+v-k}{\sqrt{n}}\right)$$

and I_{1n} be the indicator of

$$\left[\frac{c+v-k}{\sqrt{n}}, \infty\right) .$$

We have

$$\|\int_0^t T^n_{t-s}(A_\infty w - A_n w_n)\,ds\|_n$$

$$\leq \|\int_0^t T^n_{t-s}(A_\infty w - A_n w_n) I_{1n}\,ds\|_n + \|\int_0^t T^n_{t-s}(A_\infty w - A_n w_n) I_{0n}\,ds\|_n$$

$$\leq \|\int_0^t (A_\infty w - A_n w_n) I_{1n}\,ds\|_n + \|A_\infty w - A_n w_n\|_n \|\int_0^t T^n_{t-s} I_{0n}\,ds\|_n .$$

The first term is $O(n^{-1/2})$, since $|A_\infty w - A_n w_n| = O(n^{-1/2})$ off the boundary. The factor $\|A_\infty w - A_n w_n\| = O(1)$, thus it remains to show that

$$\|\int_0^t T^n_{t-s} I_{0n}\,ds\|_n = O(n^{-1/2}) .$$

This gives the total time in $[0,t]$ spent near the boundary.

We bound

$$\|\int_0^t T^n_{t-s} I_{0n}\,ds\|_n$$

by first introducing a function $h(x)$ not in the domain of A_∞. We let $h(x)$ have bounded support, be infinitely differentiable and be given by $h(x) = x$ for x near 0. One can construct $h_n(x)$ using (2.7) and apply A_n to h_n to find

$$A_n h_n = \begin{cases} O(1) & \text{if} \quad x \geq \dfrac{c+v-k}{\sqrt{n}} \\ n^{1/2}\eta((c+v)\rho - n^{1/2}x - k) + O(1) & \text{if} \quad x < \dfrac{c+v-k}{\sqrt{n}} \end{cases} \tag{2.20}$$

One has

$$T_t^n h_n - h_n = \int_0^t T_S^n A_n h_n dS$$

$$= \int_0^t T_S^n A_n h_n I_{1n} dS + \int_0^t T_S^n A_n h_n I_{0n} dS .$$

It follows that

$$\|\int_0^t T_S^n A_n h_n I_{0n} dS\|_n \leq \|T_t^n h_n - h_n\|_n + \|\int_0^t T_S^n A_n h_n I_{1n} dS\|_n$$

$$\leq 2\|h_n\|_n + O(1) .$$

We have shown $\|\int_0^t T_S^n A_n h_n I_{0n} dA\|_n$ to be bounded in n. An application of (2.2) shows

$$\|\int_0^t T_S^n A_n h_n I_{0n} dA\|_n = n^{1/2}\eta|(C+v)\rho - n^{1/2}x - k + O(1)|\,\|\int_0^t T_S I_{0n} ds\|_n$$

is bounded in n. It follows that $\|\int_0^t T_S^n I_{0n} dS\|_n = O(n^{-1/2})$.

This finally concludes the argument which shows $\|T_t^n g_n - (T_t^\infty g)_n\|_n = O(n^{-1/2})$, hence by (2.6) $\|T_t^n g - T_t^\infty g\|_n = O(n^{-1/2})$.

We have thus shown that the finite-dimensional distributions of the $(X_n(t), V_n(t))$ process converge to those of a Wiener process with drift $-\theta\eta(c+v)$ scale

$$\eta\left(\rho_d + \frac{\eta}{\mu\rho_v}\sum_{i=0}^{v-1}\frac{T_i^2}{\pi_i}\right),$$

and reflection at 0. The diffusion approximation treats $X_n(t)$ as though it were such a Wiener process. For instance, the limiting Wiener process has a stationary exponential distribution with parameter

$$\frac{\theta(c+v)}{\rho_d + \frac{\eta}{\mu\rho_v}\sum_{i=0}^{v-1}(T_i^2/\pi_i)} .$$

This is a distribution for $X(nt)/\sqrt{n}$ and suggests $X(t)$ will have a steady state distribution given approximately by an exponential with parameter

$$(c+v)(1-\rho)\Bigg/\left(\rho_d + \frac{\eta}{\mu\rho_v}\sum_{i=0}^{v-1}\frac{T_i^2}{\pi_i}\right) .$$

The steady state mean data queue length would then be

$$E(X(t)) = \frac{\rho_d + \frac{\eta}{\mu\rho_v} \sum_{i=0}^{v-1} \frac{T_i^2}{\pi_i}}{(c + v)(1-\rho)} . \tag{2.21}$$

It is interesting to consider the special case $c = 0$, $v = 1$ where the two types of traffic use the same channel. Under heavy traffic $\rho = \rho_d + \rho_v/(1+\rho_v)$, so $\rho_d \approx (1+\rho_v)^{-1}$. The mean data queue length derived from the diffusion approximation (2.21) will be

$$\left(\rho_d + \frac{\eta}{\mu} \frac{\rho_v}{(1+\rho_v)^3}\right) \Big/ (1-\rho) \approx \frac{\rho_d}{1-\rho}\left(1 + \frac{\eta}{\mu} \frac{\rho_v}{(1+\rho_v)^2}\right) .$$

The latter is the exact expression derived by Fisher (1978) for this case. The expression (2.21) represents a generalization of the results of Gaver and Lehoczky (1979b). In this paper, a diffusion approximation is given based on the fluid flow assumption for the data. For this case the result is the same except that the scale is given by

$$\frac{\eta^2}{\mu\rho_v} \sum_{i=0}^{v-1} \frac{T_i^2}{\pi_i}$$

rather than

$$\eta \left(\rho_d + \frac{\eta}{\mu\rho_v} \sum_{i=0}^{v-1} \frac{T_i^2}{\pi_i}\right).$$

The results derived in this paper therefore definitely generalize the results of Gaver and Lehoczky (1979b) since the variability in the data queue is now included. When η/μ is large, the second term dominates, and the fluid flow approximation is satisfactory.

The Wiener process approximation for the $X(t)$ process provides a method for studying the dynamics of that process. For instance, suppose the data queue were at level x at time t where x is large. One might wish to study the time that elapses until the queue becomes empty. This is essentially the duration of the busy period under heavy traffic and corresponds to a first-passage time for a Wiener process. Let us denote it by T_x. Straightforward martingale arguments provide for its transform

$$E(e^{-sT_x}) = \exp\left[(\frac{x}{\sigma}) - (\frac{m}{\sigma}) - \sqrt{(\frac{m}{\sigma})^2 + 2s} \right] \tag{2.22}$$

where

$$m = \theta(c+v)\eta \approx n^{1/2}(1-\rho)(c+v)\eta$$

$$\frac{\sigma^2}{2} = \eta\left(\rho_d + \frac{\eta}{\mu\rho_v} \sum_{i=0}^{n-1} \frac{T_i^2}{\pi_i}\right).$$

It is also easy to find the mean first-passage time

$$E(T_x) = x/m \tag{2.23}$$

One might also be interested in the area beneath the sample path until emptiness occurs, since this area represents the total time waited by all data customers involved in the busy period. If A_x represents this area, simple backward equation arguments give

$$E(A_x) = \frac{x^2}{2m} + \frac{\sigma^2}{2m^2} x \tag{2.24}$$

where m and σ^2 are given in (2.22).

<u>Acknowledgment.</u> This research was supported in part by a contract from the Office of Naval Research.

BIBLIOGRAPHY

Barbacci, M. R. and Oakley, J. D. (1976). "The integration of Circuit and Packet Switching Networks Toward a SENET Implementation," 15th NBS-ACM Annual Technique Symposium.

Bhat, U. N. and Fischer, M. J. (1976). "Multichannel Queueing Systems with Heterogeneous Classes of Arrivals," Naval Research Logistics Quarter 23

Burman, David Y. (1979). "An Analytic Approach to Diffusion Approximations in Queueing," Ph.D. Dissertation, New York University, Courant Institute of Mathematics.

Chang, Lih-Hsing (1977). "Analysis of Integrated Voice and Data Communication Network," Ph.D. Dissertation, Department of Electrical Engineering, Carnegie-Mellon University, November.

Coviello, G. and Vena, P. A. (1975). "Integration of Circuit/ Packet Switching in a SENET (Slotted Envelop NETwork) Concept," National Telecommunications Conference, New Orleans, December, pp. 42-12 to 42-17.

Fischer, M. J. (1977a). "A Queueing Analysis of an Integrated Telecommunications System with Priorities," INFOR 15

Fischer, M. J. (1977b). "Performance of Data Traffic in an Integrated Circuit- and Packet-Switched Multiplex Structure, DCA Technical Report.

Fischer, M. J. and Harris, T. C. (1976). "A Model for Evaluating the Performance of an Integrated Circuit- and Packet-Switched Multiplex Strucutre," IEEE Trans. on Comm., Com-24, February.

Gaver, D. P. and Lehoczky, J. P. (1979a). "Channels that cooperatively service a data stream and voice messages," Technical Report, Naval Postgraduate School, Department of Operations Research.

Halfin, S. (1972). "Steady-state Distribution for the Buffer Content of an M/G/1 Queue with Varying Service Rate," SIAM J. Appl. Math., 356-363.

Halfin, S. and Segal, M. (1972). "A Priority Queueing Model for a Mixture of Two Types of Customers," SIAM J. Appl. Math., 369-379.

Lehoczky, J. P. and Gaver, D. P. (1979b). "Channels that Cooperatively Service a Data Stream and Voice Messages, II: Diffusion Approximations," Technical Report, Naval Postgraduate School, Department of Operations Research.

INITIAL DISTRIBUTION LIST

	Number of Copies
Defense Technical Information Center Cameron Station Alexandria, VA 22314	2
Library Code Code 0142 Naval Postgraduate School Monterey, CA 93940	2
Library Code 55 Naval Postgraduate School Monterey, Ca. 93940	1
Dean of Research Code 012A Naval Postgraduate School Monterey, Ca. 93940	1
Attn: A. Andrus, Code 55	1
D. Gaver, Code 55	25
D. Barr, Code 55	1
P. A. Jacobs, Code 55	1
P. A. W. Lewis, Code 55	1
P. Milch, Code 55	1
R. Richards, Code 55	1
M. G. Sovereign, Code 55	1
R. J. Stampfel, Code 55	1
R. R. Read, Code 55	1
J. Wozencraft, Code 74	1
Mr. Peter Badgley ONR Headquarters, Code 102B 800 N. Quincy Street Arlington, VA 22217	1
Dr. James S. Bailey, Director Geography Programs, Department of the Navy ONR Arlington, VA 93940	1
Prof. J. Lehoczky Dept. of Statistics Carnegie Mellon University Pittsburgh, PA. 15213	10

DISTRIBUTION LIST

	No. of Copies
STATISTICS AND PROBABILITY PROGRAM OFFICE OF NAVAL RESEARCH CODE 436 ARLINGTON VA 22217	1
OFFICE OF NAVAL RESEARCH NEW YORK AREA OFFICE 715 BROADWAY - 5TH FLOOR ATTN: DR. ROGER GRAFTON NEW YORK, NY 10003	1
DIRECTOR OFFICE OF NAVAL RESEARCH BRANCH OFF 536 SOUTH CLARK STREET ATTN: DEPUTY AND CHIEF SCIENTIST CHICAGO, IL 60605	1
LIBRARY NAVAL OCEAN SYSTEMS CENTER SAN DIEGO CA 92152	1
NAVY LIBRARY NATIONAL SPACE TECHNOLOGY LAB ATTN: NAVY LIBRARIAN BAY ST. LOUIS MS 39522	1
NAVAL ELECTRONIC SYSTEMS COMMAND NAVELEX 320 NATIONAL CENTER NO. 1 ARLINGTON VA 20360	1
DIRECTOR NAVAL REAEARCH LABORATORY ATTN: LIBRARY (ONRL) CODE 2029 WASHINGTON, D.C. 20375	1
TECHNICAL INFORMATION DIVISION NAVAL RESEARCH LABORATORY WASHINGTON, D. C. 20375	1

DISTRIBUTION LIST

	No. of Copies
PRCF. C. R. BAKER DEPARTMENT CF STATISTICS UNIVERSITY CF NCTRH CARCLINA CHAPEL HILL, NCRTH CARGLINA 27514	1
PRCF. R. E. BECHHOFER DEPARTMENT CF CPERATICNS RESEARCH CCRNELL UNIVERSITY ITHACA NEW YORK 14850	1
PRCF. N. J. BERSHAC SCHOOL OF ENGINEERING UNIVERSITY CF CALIFCRNIA IRVINE , CALIFCRNIA 92664	1
P. J. BICKEL DEPARTMENT OF STATISTICS UNIVERSITY CF CALIFCRNIA BERKELEY , CALIFCRNIA 94720	1
PROF. H. W. BLOCK DEPARTMENT CF MATHEMATICS UNIVERSITY CF PITTSBURGH PITTSBURGH PA 15260	1
PROF. JCSEPH BLUM DEPT. OF MATHEMATICS, STATISTICS AND COMPUTER SCIENCE THE AMERICAN UNIVERSITY WASHINGTON DC 20016	1
PROF. R. A. BRADLEY DEPARTMENT CF STATISTICS FLCRIDA STATE UNIVERSITY TALLAHASSEE , FLCRIDA 32306	1
PROF. R. E. BARLOW OPERATICNS RESEARCH CENTER CCLLEGE CF ENGINEERING UNIVERSITY CF CALIFORNIA BERKLEY CALIFCRNIA 94720	1
MR. C. N BENNETT NAVAL CCASTAL SYSTEMS LABCRATORY CCDE P761 PANAMA CITY, FLCRIDA 32401	1

DISTRIBUTION LIST	No. of Copies
PROF. U. N. BHAT COMPUTER SCIENCE / OPERATIONS RESEARCH CENTER SOUTHERN METHODIST UNIVERSITY DALLAS , TEXAS 75275	1
PROF. W. R. BLISCHKE DEPT. OF QUANTITATIVE BUSINESS ANALYSIS UNIVERSITY OF SOUTHERN CALIFORNIA LOS ANGELES, CALIFORNIA 90007	1
DR. DERRILL J. BORDELON NAVAL UNDERWATER SYSTEMS CENTER CODE 21 NEWPORT RI 02840	1
J. E. BOYER JR DEPT. OF STATISTICS SOUTHERN METHODIST UNIVERSITY DALLAS TX 75275	1
DR. J. CHANDRA U. S. ARMY RESEARCH P. O. BOX 12211 RESEARCH TRIANGLE PARK NORTH CAROLINA 27706	1
PROF. H. CHERNOFF DEPT. OF MATHEMATICS MASS INSTITUTE OF TECHNOLOGY CAMBRIDGE, MASSACHUSETTS 02139	1
PROF. C. DERMAN DEPARTMENT OF CIVIL ENGINEERING AND ENGINEERING MECHANICS COLUMBIA UNIVERSITY NEW YORK NEW YORK 10027	1
PROF. R. L. DISNEY VIRGINIA POLYTECHNIC INSTITUTE AND STATE UNIVERSITY DEPT. OF INDUSTRIAL ENGINEERING AND OPERATIONS RESEARCH BLACKSBURG, VA 24061	1

DISTRIBUTION LIST

MR. GENE H. GLEISSNER 1
APPLIED MATHEMATICS LABORATORY
DAVID TAYLOR NAVAL SHIP RESEARCH
AND DEVELOPMENT CENTER
BETHESDA
MD 20084

PROF. S. S. GUPTA 1
DEPARTMENT OF STATISTICS
PURDUE UNIVERSITY
LAFAYETTE
INDIANA 47907

PROF. C. L. HANSON 1
DEPT OF MATH. SCIENCES
STATE UNIVERSITY OF NEW YORK,
BINGHAMTON
BINGHAMTON
NY 13901

Prof. M. J. Hinich 1
Dept. of Economics
Virginia Polytechnica Institute
and State University
Blacksburg, VA 24061

Dr. D. Depriest, 1
ONR, Code 102B
800 N. Quincy Street
Arlington, VA 22217

Prof. G. E. Whitehouse 1
Dept. of Industrial Engineering
Lehigh University
Bethlehem, PA 18015

Prof. M. Zia-Hassan 1
Dept. of Ind. & Sys. Eng.
Illinois Institute of Technology
Chicago, IL 60616

Prof. S. Zacks 1
Statistics Dept.
Virginia Polytechnic Inst.
Blacksburg, VA 24061

Head, Math. Sci Section 1
National Science Foundation
1800 G Street, N.W.
Washington, D.C. 20550

No. of Copies

Dr. H. Sittrop 1
Physics Lab., TNO
P.O. Box 96964
2509 JG, The Hague
The Netherlands

DR. R. ELASHOFF 1
BIOMATHEMATICS
UNIV. OF CALIF.
LOS ANGELES
CALIFORNIA
90024

PROF. GEORGE S. FISHMAN 1
UNIV. OF NORTH CAROLINA
CUR. IN OR AND SYS. ANALYSIS
PHILLIPS ANNEX
CHAPEL HILL, NORTH CAROLINA
20742

DR. R. GNANADESIKAN 1
BELL TELEPHONE LAB
HOLMDEL, N. J. 07733

DR. A. J. GOLDMAN 1
CHIEF, OR
DIV. 205.02, ADMIN. A428
U.S. DEPT. OF COMMERCE
WASHINGTON, D.C.
20234

DR. H. HIGGINS 1
53 BONN 1, POSTFACH 589
NASSESTRASSE 2

WEST GERMANY

DR. P. T. HOLMES 1
DEPT. OF MATH.
CLEMSON UNIV.
CLEMSON
SOUTH CAROLINA
29631

Dr. J. A. Hocke 1
Bell Telephone Labs
Whippany, New Jersey 07733

Dr. RobertHooke 1
Box 1982
Pinehurst, No. Carolina 28374

DR. D. L. IGLEHART 1
DEPT. OF O.R.
STANFORD UNIV.
STANFORD
CALIFORNIA
94305

Dr. D. Trizna, Mail Code 5323 1
Naval Research Lab
Washington, D.C. 20375

Dr. E. J. Wegman, 1
ONR, Code 436
Arlington, VA 22217

DR. H. KOBAYASHI 1
IBM
YORKTOWN HEIGHTS
NEW YORK
10598

DR. A. LEMOINE 1
1020 GUINDA ST.
PALO ALTO,
CALIFORNIA
94301

DR. J. MACQUEEN 1
UNIV. OF CALIF.
LOS ANGELES
CALIFORNIA
90024

Prof Kneale Marshall 1
Scientific Advisor to DCNO (MPT)
Code Op-01T, Room 2705
Arlington Annex
Washington, D.C. 20370

DR. M. MAZUMDAR 1
MATH. DEPT.
ESTINGHOUSE RES. LABS
CHURCHILL BORO
PITTSBURGH
PENNSYLVANIA 15235

DISTRIBUTION LIST

No. of Copies

PROF. W. M. HIRSCH — 1
INSTITUTE OF MATHEMATICAL SCIENCES
NEW YORK UNIVERSITY
NEW YORK
NEW YORK 10453

PROF. J. B. KADANE — 1
DEPARTMENT OF STATISTICS
CARNEGIE-MELLON
PITTSBURGH ,
PENNSYLVANIA
15213

DR. RICHARD LAU — 1
DIRECTOR
OFFICE OF NAVAL RESEARCH BRANCH OFF
1030 EAST GREEN STREET
PASADENA
CA 91101

DR. A. R. LAUFER — 1
DIRECTOR
OFFICE OF NAVAL RESEARCH BRANCH OFF
1030 EAST GREEN STREET
PASADENA
CA 91101

PROF. M. LEADBETTER — 1
DEPARTMENT OF STATISTICS
UNIVERSITY OF NORTH CAROLINA
CHAPEL HILL
NORTH CAROLINA 27514

DR. J. S. LEE — 1
J. S. LEE ASSOCIATES, INC.
2001 JEFFERSON DAVIS HIGHWAY
SUITE 802
ARLINGTON
VA 22202

PROF. L. C. LEE — 1
DEPARTMENT OF STATISTICS
VIRGINIA POLYTECHNIC INSTITUTE
AND STATE UNIVERSITY
BLACKSBURG
VA 24061

PROF. R. S. LEVENWORTH — 1
DEPT. OF INDUSTRIAL AND SYSTEMS
ENGINEERING
UNIVERSITY OF FLORIDA
GAINSVILLE ,
FLORIDA 32611

DISTRIBUTION LIST

	No. of copies
PROF G. LIEBERMAN STANFORD UNIVERSITY DEPARTMENT OF OPERATIONS RESEARCH STANFORD CALIFORNIA 94305	1
DR. JAMES R. MAAR NATIONAL SECURITY AGENCY FORT MEADE , MARYLAND 20755	1
PROF. R. W. MADSEN DEPARTMENT OF STATISTICS UNIVERSITY OF MISSOURI COLUMBIA MO 65201	1
DR. N. R. MANN SCIENCE CENTER ROCKWELL INTERNATIONAL CORPORATION P.O. BOX 1085 THOUSAND OAKS , CALIFORNIA 91360	1
DR. W. H. MARLOW PROGRAM IN LOGISTICS THE GEORGE WASHINGTON UNIVERSITY 707 22ND STREET , N. W. WASHINGTON , D. C. 20037	1
PROF. E. MASRY DEPT. APPLIED PHYSICS AND INFORMATION SERVICE UNIVERSITY OF CALIFORNIA LA JOLLA CALIFORNIA 92093	1
DR. BRUCE J. MCDONALD SCIENTIFIC DIRECTOR SCIENTIFIC LIAISON GROUP OFFICE OF NAVAL RESEARCH AMERICAN EMBASSY - TOKYO APO SAN FRANCISCO 96503	1

No. of copies

Dr. Leon F. McGinnis 1
School of Ind. And Sys. Eng.
Georgia Inst. of Tech.
Atlanta, GA 30332

DR. D. R. MCNEIL 1
DEPT. OF STATISTICS
PRINCETON UNIV.
PRINCETON
NEW JERSEY 08540

DR. F. MOSTELLER 1
STAT. DEPT.
HARVARD UNIV.
CAMBRIDGE
MASSACHUSETTS 02139

DR. M. REISER 1
IBM
THOMAS J. WATSON RES. CTR.
YORKTOWN HEIGHTS
NEW YORK 10598

DR. J. RIORDAN 1
DEPT. OF MATHEMATICS
ROCKEFELLER UNIV.
NEW YORK
NEW YORK 10021

DR. LINUS SCHRAGE 1
UNIV. OF CHICAGO
GRAD. SCHOOL OF BUS.
5836 GREENWOOD AVE.
CHICAGO, ILLINOIS 60637

Dr. Paul Schweitzer 1
University of Rochester
Rochester, N.Y. 14627

Dr. V. Srinivasan 1
Graduate School of Business
Stanford University
Stanford, CA. 94305

Dr. Roy Welsch 1
M.I.T. Sloan School
Cambridge, MA 02139

DISTRIBUTION LIST

	No. of Copies
DR. JANET M. MYHRE THE INSTITUTE OF DECISION SCIENCE FOR BUSINESS AND PUBLIC POLICY CLAREMONT MEN'S COLLEGE CLAREMONT CA 91711	1
MR. H. NISSELSON BUREAU OF THE CENSUS ROOM 2025 FEDERAL BUILDING 3 WASHINGTON , D. C. 20233	1
MISS B. S. ORLEANS NAVAL SEA SYSTEMS COMMAND (SEA 03F) RM 10S08 ARLINGTON VIRGINIA 20360	1
PROF. D. B OWEN DEPARTMENT OF STATISTICS SOUTHERN METHODIST UNIVERSITY DALLAS TEXAS 75222	1
Prof. E. Parzen Statistical Sceince Division Texas A & M University College Station TX 77843	1
DR. A. PETRASOVITS ROOM 207B , FOOD AND DRUG BLDG. TUNNEY'S PASTURE OTTOWA , ONTARIO K1A-OL2 , CANADA	1
PROF. S. L. PHOENIX SIBLEY SCHOOL OF MECHANICAL AND AEROSPACE ENGINEERING CORNELL UNIVERSITY ITHACA NY 14850	1
DR. A. L. POWELL DIRECTOR OFFICE OF NAVAL RESEARCH BRANCH OFF 495 SUMMER STREET BOSTON MA 02210	1
MR. F. R. FRICFI CODE 224 OPERATIONSL TEST AND ONRS EVALUATION FORCE (OPTEVFOR) NORFOLK , VIRGINIA 20360	1

DISTRIBUTION LIST

No. of Copies

PROF. M. L. PURI
DEPT. OF MATHEMATICS
P.O. BOX F
INDIANA UNIVERSITY FOUNDATION
BLOOMINGTON
IN 47401 — 1

PROF. H ROBBINS
DEPARTMENT OF MATHEMATICS
COLUMBIA UNIVERSITY
NEW YORK,
NEW YORK 10027 — 1

PROF. M ROSENBLATT
DEPARTMENT OF MATHEMATICS
UNIVERSITY OF CALIFORNIA SAN DIEGO
LA JOLLA
CALIFORNIA
92093 — 1

PROF. S. M. ROSS
COLLEGE OF ENGINEERING
UNIVERSITY OF CALIFORNIA
BERKELEY
CA
94720 — 1

PROF. I RUBIN
SCHOOL OF ENGINEERING AND APPLIED SCIENCE
UNIVERSITY OF CALIFORNIA
LOS ANGELES ,
CALIFORNIA 90024 — 1

PROF. I. R. SAVAGE
DEPARTMENT OF STATISTICS
YALE UNIVERSITY
NEW HAVEN,
CONNECTICUT
06520 — 1

PROF. L. L. , SCHARF JR
DEPARTMENT OF ELECRICAL ENGINEERING
COLORADO STATE UNIVERSITY
FT. COLLINS ,
COLORADO
80521 — 1

PROF. R. SERFLING
DEPARTMENT OF STATISTICS
FLORIDA STATE UNIVERSITY

TALLAHASSEE FLORIDA 32306 — 1

PROF. W. R. SCHUCANY
DEPARTMENT OF STATISTICS
SOUTHERN METHODIST UNIVERSITY
DALLAS ,
TEXAS
75222 — 1

DISTRIBUTION LIST

	No. of Copies
PROF. C. C. SIEGMUND DEPT. OF STATISTICS STANFORD UNIVERSITY STANFORD CA 94305	1
PROF. M. L. SHOOMAN DEPT. OF ELECTRICAL ENGINEERING POLYTECHNIC INSTITUTE OF NEW YORK BROOKLYN, NEW YORK 11201	1
DR. A. L. SLAFKOSKY SCIENTIFIC ADVISOR COMMANDANT OF THE MARINE CORPS WASHINGTON , D. C. 20380	1
DR. D. E. SMITH DESMATICS INC. P.O. BOX 618 STATE COLLEGE PENNSYLVANIA 16801	1
PROF. W. L. SMITH DEPARTMENT OF STATISTICS UNIVERSITY OF NORTH CAROLINA CHAPEL HILL NORTH CAROLINA 27514	1
Dr. H. J. Solomon ONR 223/231 Old Marylebone Rd London NW1 5TH, ENGLAND	1
MR. GLENN F. STAHLY NATIONAL SECURITY AGENCY FORT MEADE , MARYLAND 20755	1
Mr. J. Gallagher Naval Underwater Systems Center New London, CT	1
Dr. E. C. Monahan Dept. of Oceanography University College Galway, Ireland	1

DISTRIBUTION LIST

No. of Copies

DR. R. M. STARK
STATISTICS AND COMPUTER SCI.
UNIV. OF DELAWARE
NEWARK
DELAWARE
19711 — 1

PROF. JOHN W. TUKEY
FINE HALL
PRINCETON UNIV.
PRINCETON
NEW JERSEY
08540 — 1

DR. THOMAS C. VARLEY
OFFICE OF NAVAL RESEARCH
CODE 434
ARLINGTON
VA
22217 — 1

PROF. G. WATSON
FINE HALL
PRINCETON UNIV.
PRINCETON
NEW JERSEY
08540 — 1

MR. DAVID A. SWICK
ADVANCED PROJECTS GROUP
CODE 8103
NAVAL RESEARCH LAB.
WASHINGTON
DC
20375 — 1

MR. WENDELL G. SYKES
ARTHUR D. LITTLE, INC.
ACORN PARK
CAMBRIDGE
MA
02140 — 1

PROF. J. R. THOMPSON
DEPARTMENT OF MATHEMATICAL SCIENCE
RICE UNIVERSITY
HOUSTON,
TEXAS
77001 — 1

PROF. W. A. THOMPSON
DEPARTMENT OF STATISTICS
UNIVERSITY OF MISSOURI
COLUMBIA ,
MISSOURI
65201 — 1

DISTRIBUTION LIST

No. of Copies

PROF. F. A. TILLMAN 1
DEPT. OF INDUSTRIAL ENGINEERING
KANSAS STATE UNIVERSITY
MANHATTAN
KS
66506

PROF. A . F . VEINOTT 1
DEPARTMENT OF OPERATIONS RESEARCH
STANFORD UNIVERSTITY
STANFORD
CALIFORNIA
94305

DANIEL H. WAGNER 1
STATION SQUARE ONE
PAOLI , PENNSYLVANIA
19301

PROF. GRACE WAHBA 1
DEPT. OF STATISTICS
UNIVERSITY OF WISCONSIN
MADISON
WI
53706

PROF. K. T. WALLENIUS 1
DEPARTMENT OF MATHEMATICAL SCIENCES.
CLEMSON UNIVERSITY
CLEMSON,
SOUTH CAROLINA 29631

PROF. BERNARD WIDROW 1
STANFORD ELECTRONICS LAB
STANFORD UNIVERSITY
STANFORD
CA
94305

DISTRIBUTION LIST	No. of Copies
OFFICE OF NAVAL RESEARCH SAN FRANCISCO AREA OFFICE 760 MARKET STREET SAN FRANCISCO CALIFORNIA 94102	1
TECHNICAL LIBRARY NAVAL ORDNANCE STATION INDIAN HEAD MARYLAND 20640	1
NAVAL SHIP ENGINEERING CENTER PHILADELPHIA DIVISION TECHNICAL LIBRARY PHILADELPHIA PENNSYLVANIA 19112	1
BUREAU OF NAVAL PERSONNEL DEPARTMENT OF THE NAVY TECHNICAL LIBRARY WASHINGTON D. C. 20370	1
PROF. M. ABDEL-HAMEED DEPARTMENT OF MATHEMATICS UNIVERSITY OF NORTH CAROLINA CHARLOTTE NC 28223	1
PROF. T. W. ANDERSON DEPARTMENT OF STATISTICS STANFORD UNIVERSITY STANFORD , CALIFORNIA 94305	1
PROF. F. J. ANSCOMBE DEPARTMENT OF STATISTICS YALE UNIVERSITY NEW HAVEN CONNECTICUT 06520	1
PROF. L. A. AROIAN INSTITUTE OF INDUSTRIAL ADMINISTRATION UNION COLLEGE SCHENECTADY , NEW YORK 12308	1