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MULTIDIMENSIONAL SPECTRAL ESTIMATION
USING
ITERATIVE METHODS

by

Roderick C. Wester

June 1990

Thesis Advisor:	Charles W. Therrien
Co-Advisor:	Murali Tummala

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This thesis treats the topic of multidimensional autoregressive (AR) spectral estimation. An iterative algorithm for the solution of toeplitz block-toeplitz matrix equations is presented. This leads to a fast solution of the 2-D normal equation compared with direct inversion of the autocorrelation matrix. The covariance method is used to estimate the autocorrelation function. Because the resulting matrix is not toeplitz block-toeplitz, a modified iterative algorithm is presented. Quarter-plane (QP) and nonsymmetric half-plane (NSHP) support are used, as well as combined quadrant (CQ) averaging. Results of computer simulation show that in some cases a single iteration is sufficient to produce an acceptable spectral estimate. Because the AR parameters are estimated from previous values, this suggests the possibility to estimate spectral densities of slowly varying random processes.

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Multidimensional Spectral Estimation Using Iterative Methods

by

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Lieutenant, United States Navy
B.S.M.E., Massachusetts Institute of Technology, 1984

Submitted in partial fulfillment
of the requirements for the degree of

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ABSTRACT

This thesis treats the topic of multidimensional autoregressive (AR) spectral estimation. An iterative algorithm for the solution of toeplitz block-toeplitz matrix equations is presented. This leads to a fast solution of the 2-D normal equation compared with direct inversion of the autocorrelation matrix.

The covariance method is used to estimate the autocorrelation function. Because the resulting autocorrelation matrix is not toeplitz block-toeplitz, a modified iterative algorithm is presented. Quarter-plane (QP) and nonsymmetric half-plane (NSHP) support are used, as well as combined quadrant (CQ) averaging.

Results of computer simulation show that in some cases a single iteration is sufficient to produce an acceptable spectral estimate. Because the AR parameters are estimated from previous values, this suggests the possibility to estimate spectral densities of slowly varying random processes.

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I. INTRODUCTION

Estimating the power spectrum associated with a 2-D random process is important in a number of applications. In digital image processing, for example, Wiener filtering may be used to solve image restoration problems in which a signal is degraded by additive random noise [Ref. 1]. The frequency response of a noncausal Wiener filter requires knowledge of the spectral contents of the signal and background noise. Another example is an array of sensors. The frequency wave number spectrum generated from such an array contains information about the signal source and direction of arrival.

Various methods of spectral estimation are discussed in the literature [Refs. 1, 2, 3]. This thesis treats the topic of autoregressive (AR) spectral estimation. Because this method is known to produce high resolution spectral estimates with a small data set, it has attracted considerable interest [Refs. 4, 5, 6, 7].

Determining estimates of AR parameters requires the solution of a set of normal equations. This in turn requires the inversion of an NxN autocorrelation matrix. Direct inversion of the autocorrelation matrix requires $O(N^3)$ multiplications, which is computationally intensive for large N. An iterative method is presented in this thesis which reduces the number of multiplications significantly.

This thesis is organized into five chapters. The remainder of this chapter is devoted to defining the power spectral density and the resulting forms of the normal equation when quarter-plane (QP) and nonsymmetric half-plane (NSHP) support are used. Chapter

II develops an iterative solution to the normal equation for both QP and NSHP support which assumes the autocorrelation matrix is toeplitz block-toeplitz. Chapter III discusses the use of the covariance method to estimate the autocorrelation matrix. Because the resulting autocorrelation matrix for the covariance method is not toeplitz block-toeplitz, a modification of the iterative solution presented in Chapter II is developed. Additionally, the technique of combined-quadrant (CQ) averaging is introduced in this chapter. Results obtained from using QP and NSHP support, as well as CQ averaging, to estimate spectral densities are presented in Chapter IV. Conclusions and recommendations for future study are given in Chapter V.

A. POWER SPECTRAL DENSITY

A 2-D random process is a discrete function $x(n_1,n_2)$ such that, for each coordinate pair (n_1,n_2), the value of $x(n_1,n_2)$ is a random variable. The power spectral density $P_x(\omega_1,\omega_2)$ of a wide-sense stationary random process is the Fourier transform of the autocorrelation function,

$$P_x(\omega_1,\omega_2) = \sum_{l_1=-\infty}^{\infty} \sum_{l_2=-\infty}^{\infty} R_x(l_1,l_2) e^{-j\omega_1 l_1} e^{-j\omega_2 l_2} \tag{1-1}$$

where

$$R_x(l_1,l_2) = E[x(n_1,n_2)x^*(n_1-l_1,n_2-l_2)], \tag{1-2}$$

and $E[\cdot]$ denotes the statistical expectation. Conventional methods of estimating the spectral density include the correlogram. In this method, an estimate of the autocorrelation function $\hat{R}_x(l_1,l_2)$ is substituted into (1-1), where $\hat{R}_x(l_1,l_2)$ is an estimate such as

$$\hat{R}_x(l_1,l_2) = \frac{1}{N_1 N_2} \sum_{l_1=0}^{N_1-l_1-1} \sum_{l_2=0}^{N_2-l_2-1} x(n_1,n_2)x^*(n_1-l_1,n_2-l_2), \tag{1-3}$$

and $N_1 N_2$ is the number of points in the random process. The modern methods of spectral estimation include representing the random process with an autoregressive model. The model parameters are then used to estimate the spectral density. This is the method employed in this thesis.

B. AUTOREGRESSIVE SPECTRAL ESTIMATION

Autoregressive spectral estimation assumes the random process $x(n_1,n_2)$ is the response of an AR model excited by white noise $w(n_1,n_2)$ with variance σ^2, as shown in Figure 1. To estimate the AR parameters $a(n_1,n_2)$, the system is expressed as a recursive difference equation given by

$$x(n_1,n_2) = -\sum\sum_{(i,j)\in A} a(i,j)^T x(n_1-i,n_2-j) + w(n_1,n_2) \tag{1-4}$$

where A is the region of support over which $a(i,j)$ have non-zero values. The frequency-response function of the AR model is expressed as

$$H(\omega_1,\omega_2) = \frac{1}{1+\sum\sum_{(k_1,k_2)\in A} a(k_1,k_2)e^{-j\omega_1 k_1}e^{-j\omega_2 k_2}} . \tag{1-5}$$

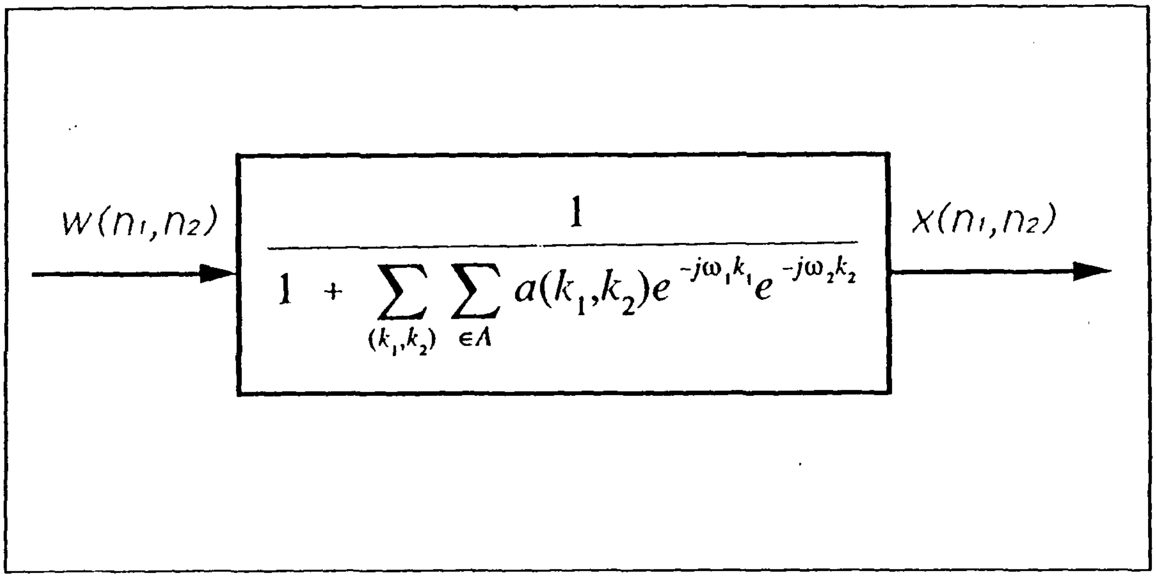

Figure 1. AR model excited by white noise

The AR power spectral estimate $\hat{P}_x(\omega_1,\omega_2)$ is given by

$$\hat{P}_x(\omega_1,\omega_2) = |H(\omega_1,\omega_2)|^2 P_w \tag{1-6}$$

where P_w is the spectral density of the white noise input. The input has a constant power spectrum of amplitude σ^2. Therefore (1-6) can be written as

$$\hat{P}_x(\omega_1,\omega_2) = \frac{\sigma^2}{\left|1 + \sum\sum_{(n_1,n_2)\in A} a(n_1,n_2)e^{-j\omega_1 n_1}e^{-j\omega_2 n_2}\right|^2} . \tag{1-7}$$

To estimate the AR parameters, multiply both sides of (1-4) by $x^*(n_1-l_1,n_2-l_2)$ and take the statistical expectation [Ref. 2]. This leads to the following normal equation

$$\sum_{(i,j)}\sum_{\in A} a(i,j)R_x(l_1-i,l_2-j) = R_x(l_1,l_2). \tag{1-8}$$

The structure of the normal equation depends upon the shape of A. Two regions of support will be considered in the following, quarter-plane and nonsymmetric half-plane support.

1. Quarter-Plane Support

The region A is said to have quarter-plane support when $a(i,j)$ has non-zero values in one quadrant only, as shown in Figure 2, which illustrates quarter plane support in the first quadrant. In this case the normal equation becomes

$$\sum_{\substack{i=0 \\ (i,j)\neq(0,0)}}^{P_1-1}\sum_{j=0}^{P_2-1} a(i,j)R_x(l_1-i,l_2-j) = R_x(l_1,l_2) \tag{1-9}$$

where l_1=0,1,2,...,P_1-1, l_2=0,1,2,...,P_2-1, and P_1 and P_2 are the dimensions of A as shown in Figure 2. If it is assumed that $a(0,0)$ = 1, then (1.9) can be written in block-matrix form as

$$\begin{bmatrix} R_0 & R_{-1} & R_{-2} & \cdots & R_{-P_1+1} \\ R_1 & R_0 & R_{-1} & \cdots & R_{-P_1+2} \\ R_2 & R_1 & R_0 & \cdots & R_{-P_1+3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ R_{P_1-1} & R_{P_1-2} & R_{P_1-3} & \cdots & R_0 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ \vdots \\ a_{P_1-1} \end{bmatrix} = \begin{bmatrix} S^{(0)} \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \tag{1-10}$$

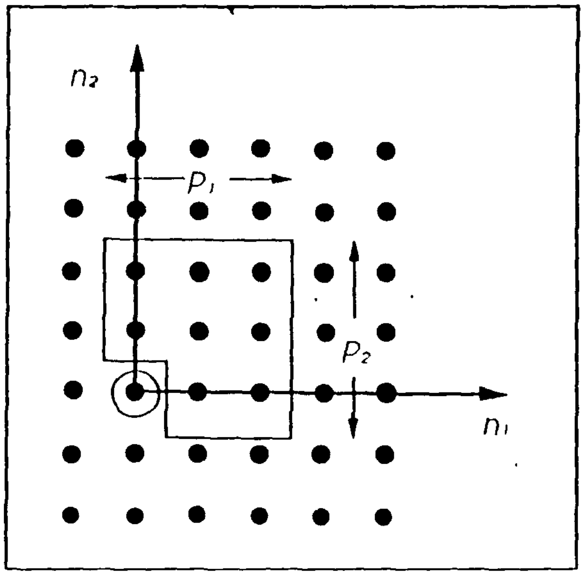

Figure 2. QP support in first quadrant

where each block R_k is given by

$$R_k = R_{-k}^T = \begin{bmatrix} R(k,0) & R(k,-1) & \cdots & R(k,-P_2+1) \\ R(k,1) & R(k,0) & \cdots & R(k,-P_2+2) \\ \vdots & \vdots & \ddots & \vdots \\ R(k,P_2-1) & R(k,P_2-2) & \cdots & R(k,0) \end{bmatrix}, \qquad (1\text{-}11)$$

$$a_k = [\, a(k,0) \quad a(k,1) \quad a(k,2) \quad \cdots \quad a(k,P_2-1) \,]^T, \qquad (1\text{-}12)$$

and

$$S^{(0)} = \left[\sigma^2 \quad 0 \quad \cdots \quad 0\right]^T. \qquad (1\text{-}13)$$

Note that the matrix R has a toeplitz block-toeplitz structure. That is, the blocks along the diagonals are equal, and the elements along the diagonals within the blocks are also equal.

2. Nonsymmetric Half-Plane Support

When $a(i,j)$ has non-zero values in a region of the form shown in Figure 3, the region A is referred to as nonsymmetric half-plane (NSHP) support. In this case the normal equation becomes

$$\sum_{j=1}^{L_2+P_2-1} a(0,j)R_x(l_1,l_2-j) + \sum_{\substack{i=1 \\ (i,j)\neq(0,0)}}^{P_1-1} \sum_{j=L_2}^{P_2-1} a(i,j)R_x(l_1-i,l_2-j) = R_x(l_1,l_2) \qquad \text{(1-14)}$$

where

$$l_1 = 0,1,,2,\dots,P_1-1,$$

$$l_2 = \begin{cases} 0,1,,2,,\dots,L_2+P_2-1, & \text{for } l_1 = 0, \\ L_2,L_2+1,L_2+2,\dots,L_2+P_2-1, & \text{for } l_1 \neq 0, \end{cases}$$

and P_1 and P_2 are the dimensions of A, and L_2 is a negative number as defined in Figure 3. Let $a(0,0) = 1$. Then (1-14) may be written in matrix form as

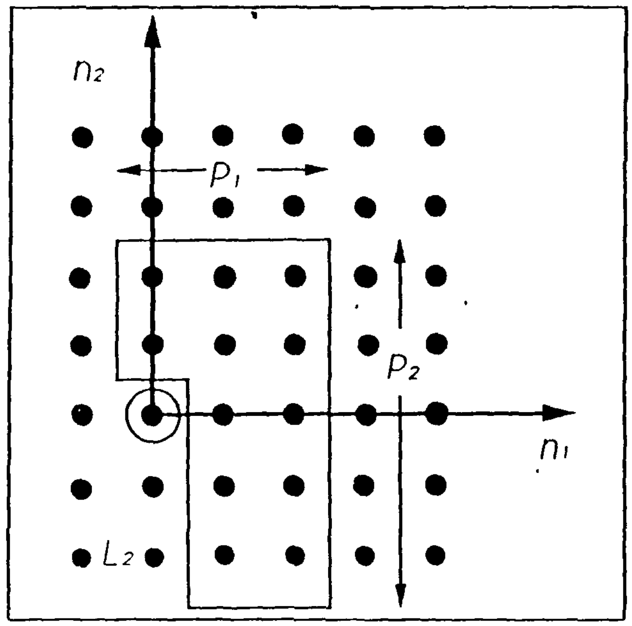

Figure 3. NSHP support

$$\begin{bmatrix} \vec{R}_0 & \vec{R}_{-1} & \vec{R}_{-2} & \cdots & \vec{R}_{-P_1+1} \\ \vec{R}_1 & R_0 & R_{-1} & \cdots & R_{-P_1+2} \\ \vec{R}_2 & R_1 & R_0 & \cdots & R_{-P_1+3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \vec{R}_{P_1-1} & R_{P_1-2} & R_{P_1-3} & \cdots & R_0 \end{bmatrix} \begin{bmatrix} \vec{a}_0 \\ a_1 \\ a_2 \\ \vdots \\ a_{P_1-1} \end{bmatrix} = \begin{bmatrix} \vec{S}^{(0)} \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \qquad (1\text{-}15)$$

where the various blocks $\vec{R}_0$, $\vec{R}_k$, and R_k are given by

$$\vec{R}_0 = \begin{bmatrix} R(0,0) & R(0,-1) & \cdots & R(0,-L_2-P_2+1) \\ R(0,1) & R(0,0) & \cdots & R(0,-L_2-P_2+2) \\ \vdots & \vdots & \ddots & \vdots \\ R(0,L_2+P_2-1) & R(0,L_2+P_2-2) & \cdots & R(0,0) \end{bmatrix}, \qquad (1\text{-}16)$$

$$\tilde{R}_k = \tilde{R}_{-k}^T = \begin{bmatrix} R(k,-L_2) & R(k,-L_2-1) & \cdots & R(k,-L_2-P_2+1) \\ R(k,-L_2+1) & R(k,-L_2) & \cdots & R(k,-L_2-P_2+2) \\ \vdots & \vdots & \ddots & \vdots \\ R(k,-L_2+P_2-1) & R(k,-L_2+P_2-2) & \cdots & R(k,0) \end{bmatrix}, \tag{1-17}$$

$$R_k = R_{-k}^T = \begin{bmatrix} R(k,0) & R(k,-1) & \cdots & R(k,-P_2+1) \\ R(k,1) & R(k,0) & \cdots & R(k,-P_2+2) \\ \vdots & \vdots & \ddots & \vdots \\ R(k,P_2-1) & R(k,P_2-2) & \cdots & R(k,0) \end{bmatrix}, \tag{1-18}$$

the model parameters in vector form are

$$\tilde{a}_0 = [a(0,0)\ a(0,1)\ a(0,2)\ \cdots\ a(0,L_2+P_2-1)]^T, \tag{1-19}$$

$$a_k = [a(k,L_2)\ a(k,L_2+1)\ a(k,L_2+2)\ \cdots\ a(k,L_2+P_2-1)]^T, \tag{1-20}$$

and

$$\tilde{S}^{(0)} = \left[\sigma^2\ \ 0\ \ 0\ \ \cdots\ \ 0\right]^T. \tag{1-21}$$

The blocks of the autocorrelation matrix have three structures given by $\tilde{R}_0$, $\tilde{R}_k$, and R_k. Except for the upper and left borders, the matrix is block-toeplitz with toeplitz blocks. Quarter-plane support is a special case of NSHP support with $L_2 = 0$.

II. SOLUTION OF NORMAL EQUATION USING ITERATION

The solution of the normal equation requires inversion of the autocorrelation matrix. As previously stated, direct inversion becomes increasingly computationally intensive for larger matrices. However, the toeplitz block-toeplitz structure of the autocorrelation matrix enables the inversion of the matrix via an iterative method which employs successive partitioning of the normal equation [Ref. 8].

A. QUARTER-PLANE SUPPORT

To develop the iterative algorithm we first divide both sides of (1-10) by σ^2. This results in the modified normal equation

$$\begin{bmatrix} R_0 & R_{-1} & R_{-2} & \cdots & R_{-P_1+1} \\ R_1 & R_0 & R_{-1} & \cdots & R_{-P_1+2} \\ R_2 & R_1 & R_0 & \cdots & R_{-P_1+3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ R_{P_1-1} & R_{P_1-2} & R_{P_1-3} & \cdots & R_0 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ \vdots \\ a_{P_1-1} \end{bmatrix} = \begin{bmatrix} S^{(0)} \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \qquad (2\text{-}1)$$

where $S^{(0)} = [1 \; 0 \; 0 \cdots 0]^T$. The normal equation is then partitioned as follows.

$$
\begin{bmatrix}
R_0 & R_{-1} & R_{-2} & \cdots & \vdots & R_{-P_1+1} \\
R_1 & R_0 & R_{-1} & \cdots & \vdots & R_{-P_1+2} \\
R_2 & R_1 & R_0 & \cdots & \vdots & R_{-P_1+3} \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
R_{P_1-1} & R_{P_1-2} & R_{P_1-3} & \cdots & \vdots & R_0
\end{bmatrix}
\begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ \vdots \\ \cdots \\ a_{P_1-1} \end{bmatrix}
=
\begin{bmatrix} S^{(0)} \\ 0 \\ 0 \\ \vdots \\ \cdots \\ 0 \end{bmatrix}. \tag{2-2}
$$

Equation (2-2) can be expressed as

$$
\begin{bmatrix} G_1 & H_1 \\ H_1^T & R_0 \end{bmatrix}
\begin{bmatrix} \gamma_1 \\ a_{P_1-1} \end{bmatrix}
=
\begin{bmatrix} \varphi_1 \\ 0 \end{bmatrix} \tag{2-3}
$$

where

$$
G_1 = \begin{bmatrix}
R_0 & R_{-1} & R_{-2} & \cdots & R_{-P_1+2} \\
R_1 & R_0 & R_{-1} & \cdots & R_{-P_1+3} \\
R_2 & R_1 & R_0 & \cdots & R_{-P_1+4} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
R_{P_1-2} & R_{P_1-3} & R_{P_1-4} & \cdots & R_0
\end{bmatrix}, \tag{2-4}
$$

$$
H_1^T = [R_{P_1-1}\ R_{P_1-2}\ R_{P_1-3}\ \cdots\ R_1], \tag{2-5}
$$

$$
\gamma_1 = [a_0^T\ a_1^T\ a_2^T\ \cdots\ a_{P_1-2}^T]^T, \tag{2-6}
$$

and

$$\varphi_1 = [1\ 0\ 0\ \cdots\ 0]^T. \tag{2-7}$$

Both sides of (2-3) are then premultiplied by a matrix F_1 which results in

$$\begin{aligned} \gamma_1 &= G_1^{-1}[\varphi_1 - H_1\ a_{P_1-1}] \\ a_{P_1-1} &= -R_0^{-1} H_1^T \gamma_1 \end{aligned} \tag{2-8}$$

where

$$F_1 = \begin{bmatrix} G_1^{-1} & H_1 \\ H_1^T & R_0^{-1} \end{bmatrix}. \tag{2-9}$$

These last equations suggest an iterative solution. In particular, (2-8) may be expressed as

$$\begin{aligned} \gamma_1^{(k)} &= G_1^{-1}[\varphi_1 - H_1\ a_{P_1-1}^{(k-1)}] \\ a_{P_1-1}^{(k)} &= -R_0^{-1} H_1^T \gamma_1^{(k-1)}\ . \end{aligned} \tag{2-10}$$

Equation (2-10) requires the inversion of the submatrix G_1, which is not much smaller than the autocorrelation matrix, and hence requires nearly as many multiplications to invert. Therefore, G_1, γ_1, and φ_1 may be further partitioned to yield

$$\begin{bmatrix} G_2 & H_2 \\ H_2^T & R_0 \end{bmatrix} \begin{bmatrix} \gamma_2 \\ a_{P_1-2} \end{bmatrix} = \begin{bmatrix} \varphi_2 \\ 0 \end{bmatrix} \tag{2-11}$$

where

$$G_2 = \begin{bmatrix} R_0 & R_{-1} & R_{-2} & \cdots & R_{-P_1+3} \\ R_1 & R_0 & R_{-1} & \cdots & R_{-P_1+4} \\ R_2 & R_1 & R_0 & \cdots & R_{-P_1+5} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ R_{P_1-3} & R_{P_1-4} & R_{P_1-5} & \cdots & R_2 \end{bmatrix}, \tag{2-12}$$

$$H_2^T = [R_{P_1-1}\ R_{P_1-2}\ R_{P_1-3} \cdots R_2], \tag{2-13}$$

$$\gamma_2 = [a_0^T\ a_1^T\ a_2^T \cdots a_{P_1-3}^T]^T, \tag{2-14}$$

and

$$\varphi_2 = [1\ 0\ 0 \cdots 0]^T. \tag{2-15}$$

Premultiplication of both sides of (2-11) by the matrix F_2 results in

$$\begin{aligned} \gamma_2 &= G_2^{-1}[\varphi_2 - H_2\ a_{P_1-2}] \\ a_{P_1-2} &= -R_0^{-1} H_2^T \gamma_2 \end{aligned} \tag{2-16}$$

where

$$F_2 = \begin{bmatrix} G_2^{-1} & H_2 \\ H_2^T & R_0^{-1} \end{bmatrix}. \tag{2-17}$$

An iterative solution to (2-16) can then be formulated as

$$\begin{aligned} \gamma_2^{(k)} &= G_2^{-1}[\varphi_2 - H_2\, a_{P_1-2}^{(k-1)}] \\ a_{P_1-2}^{(k)} &= -R_0^{-1} H_2^T \gamma_2^{(k-1)} . \end{aligned} \tag{2-18}$$

Since (2-18) requires the inversion of submatrix G_2, which may be large, the matrices G_2, γ_2, and φ_2 are further partitioned. Repetitive partitioning of the normal equation finally results in

$$\begin{bmatrix} G_{P_1-1} & H_{P_1-1} \\ H_{P_1-1}^T & R_0 \end{bmatrix} \begin{bmatrix} \gamma_{P_1-1} \\ a_1 \end{bmatrix} = \begin{bmatrix} \varphi_{P_1-1} \\ 0 \end{bmatrix} \tag{2-19}$$

where $G_{P_1-1} = R_0$, $H_{P_1-1}^T = R_1$, $\gamma_{P_1-1} = a_0$, and $\varphi_{P_1-1} = S^{(0)}$. Premultiplication of (2-19) by the matrix F_{P_1-1} yields an iterative solution

$$\begin{aligned} a_0^{(k)} &= R_0^{-1}[S^{(0)} - R_{-1}\, a_1^{(k-1)}] \\ a_1^{(k)} &= -R_0^{-1} R_1 a_0^{(k-1)} \end{aligned} \tag{2-20}$$

where

$$F_{P_1-1} = \begin{bmatrix} R_0^{-1} & R_{-1} \\ R_1 & R_0^{-1} \end{bmatrix}. \qquad (2\text{-}21)$$

Combining all of the iterative solutions from the successive levels of partitioning results in the final algorithm

$$\begin{aligned} a_0^{(k)} &= a_0^{(0)} - R_0^{-1} \sum_{i=1}^{P_1-1} R_{-i}\, a_i^{(k-1)} \\ a_j^{(k)} &= -R_0^{-1} \sum_{\substack{i=0 \\ i \neq j}}^{P_1-1} R_{j-i} a_i^{(k-1)} \end{aligned} \qquad (2\text{-}22)$$

where a_i are P_2x1 vectors of AR parameters, $a_0^{(0)} = R_0^{-1} S^{(0)}$, $a_j^{(0)} = R_0^{-1} R_j a_0^{(0)}$, j = 1,2,...,P_1-1, and k is the index of iteration. Solution of (2-22) requires $O(P_1^2 P_2^2 K)$ multiplications where K is the number of iterations required for convergence to the true parameters.

B. NONSYMMETRIC HALF-PLANE SUPPORT

As noted previously, the autocorrelation matrix which results from NSHP support consists of three different types of blocks. The derivation of the iteration follows the same procedure of succesive partitioning of the normal equation as that for the QP support. However, the assymmetry of the autocorrelation matrix results in a slightly different final recursion given by

$$\tilde{a}_0^{(k)} = \tilde{a}_0^{(0)} - \tilde{R}_0^{-1} \sum_{i=1}^{P_1-1} \tilde{R}_{-i} \, a_i^{(k-1)}$$
$$a_j^{(k)} = -R_0^{-1} \tilde{R}_j a_0^{(k-1)} - R_0^{-1} \sum_{\substack{i=1 \\ i \neq j}}^{P_1-1} R_{j-i} a_i^{(k-1)} \tag{2-23}$$

where a_i are P_2x1 vectors of AR parameters, $\tilde{a}_0^{(0)} = \tilde{R}_0^{-1}\tilde{S}^{(0)}$, $a_j^{(0)} = \tilde{R}_0^{-1}\tilde{R}_j\tilde{a}_0^{(0)}$, $j = 1,2,\ldots, P_1-1$, and k is the index of iteration. As with (2-22), the solution of (2-23) requires $O(P_1^2 P_2^2 K)$ mutiplications to converge to the true values of the parameters.

III. COVARIANCE METHOD

The iterative solutions to the normal equations given in (2-22) and (2-23) assume that the autocorrelation matrix is known and that it is toeplitz block-toeplitz. In general, however, the autocorrelation matrix must be estimated from the available data. Assuming a 2-D AR model, the covariance method provides a means to estimate the autocorrelatiom matrix and may be formulated in terms of linear prediction.

In the 2-D linear prediction problem, the error between the true value of a random process $x(n_1,n_2)$ and the estimated value $\hat{x}(n_1,n_2)$ is given by

$$e(n_1,n_2) = x(n_1,n_2) - \hat{x}(n_1,n_2) \tag{3-1}$$

$$e(n_1,n_2) = x(n_1,n_2) + \sum_{\substack{(i,j) \in A \\ (i,j) \neq (0,0)}} \sum a(i,j)x(n_1-i,n_2-j) \ . \tag{3-2}$$

The objective of the covariance method is to minimize the sum of the actual squared errors from a particular set of these terms [Ref. 1]. Let $a(0,0) = 1$. Then (3-2) can be written as

$$e(n_1,n_2) = \sum_{(i,j) \in A} \sum a(i,j)x(n_1-i,n_2-j) \ . \tag{3-3}$$

This last equation can be represented in vector form as

$$e = Xa \tag{3-4}$$

where the rows of the matrix X are the points of the random process within the region A as the filter is moved over the data. In the covariance method the filter is positioned so that it never falls even partially outside of the available data. The elements of the vector e are the error between the observed and the predicted values of the random process for each position of the filter. Premultiplication of both sides of (3-4) by X^T and application of the the orthogonality principle results in the following normal equation

$$Ra = S, \tag{3-5}$$

where $R = X^T X$, $S = X^T e = [S^{(0)T}\ 0 \cdots 0]^T$, and $S^{(0)} = [\sigma^2\ 0 \cdots 0]^T$. The structure of R depends upon the shape of the region A. Quarter-plane and NSHP support are discussed below.

A. QUARTER-PLANE SUPPORT

If the region A is a quarter-plane in the first quadrant, the matrix X becomes

$$X = \begin{bmatrix} x_{n_1,n_2} \\ \vdots \\ x_{n_1+N_1-P_1,n_2} \\ x_{n_1,n_2+1} \\ \vdots \\ x_{n_1+N_1-P_1,n_2+1} \\ \vdots \\ x_{n_1,n_2+N_2-P_2} \\ \vdots \\ x_{n_1+N_1-P_1,n_2+N_2-P_2} \end{bmatrix} \tag{3-6}$$

where each row is given by

$$x_{p,q}^{T} = \begin{bmatrix} x(n_1+p,n_2+q) \\ \vdots \\ x(n_1+p,n_2+q-P_2+1) \\ x(n_1+p-1,n_2+q) \\ \vdots \\ x(n_1+p-1,n_2+q-P_2+1) \\ \vdots \\ x(n_1+p-P_1+1,n_2+q) \\ \vdots \\ x(n_1+p-P_1+1,n_2+q-P_2+1) \end{bmatrix}, \tag{3-7}$$

and N_1 and N_2 are the number of columns and rows in the 2-D random process, respectively, and P_1, P_2 are the dimensions of the region A as shown in Figure 4.

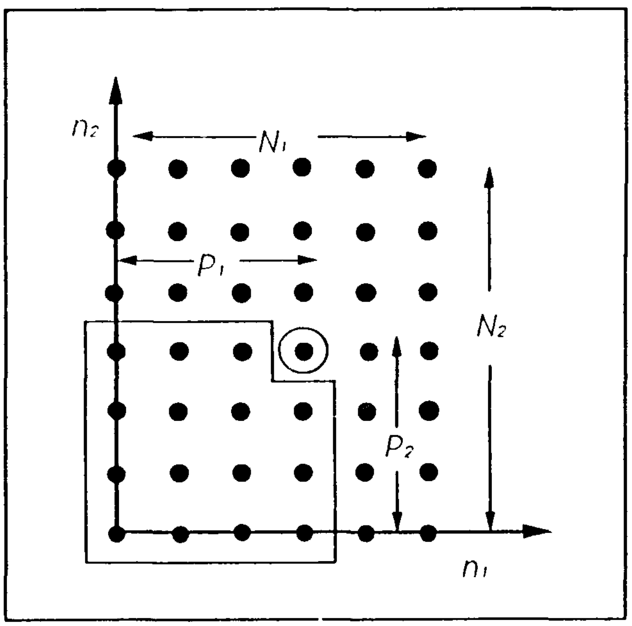

Figure 4. QP support used to estimate autocorrelation matrix

The sample autocorrelation matrix then becomes

$$R = \begin{bmatrix} R_{0,0} & R_{0,1} & \cdots & R_{0,P_1-1} \\ R_{1,0} & R_{1,1} & \cdots & R_{1,P_1-1} \\ \vdots & \vdots & \ddots & \vdots \\ R_{P_1-1,0} & R_{P_1-1,1} & \cdots & R_{P_1-1,P_1-1} \end{bmatrix} \tag{3-8}$$

where $R_{m,n} = R^T_{n,m}$. The matrix R is symmetric; however, it is not toeplitz block-toeplitz. Nevertheless, an iterative solution to estimate the AR parameters may still be obtained as follows [Ref. 8].

The first step is to average the blocks along the main diagonal of the autocorrelation matrix. Refer to this block as R_{avg}. From R_{avg} a toeplitz approximation T of the diagonal blocks is formed. One approximation is given in Ref. 9, where the diagonal elements of T, $t(i)$, are given by

$$t(i) = \frac{1}{P_2 - i} \sum_{j=0}^{P_2-i-1} R_{avg}(i+j,j) \tag{3-9}$$

where $i = 0,1,\ldots P_2$-1. The normal equation is then successively partitioned as shown previously.

The first partitioning of the normal equation, similiar to that in (2-3), leads to

$$\begin{aligned} \gamma_1^{(k)} &= G_1^{-1}[\varphi_1 - H_1\, a_{P_1-1}^{(k-1)}] \\ a_{P_1-1}^{(k)} &= -R_{P_1-1,P_1-1}^{-1} H_1^T \gamma_1^{(k-1)} \,. \end{aligned} \tag{3-10}$$

The block R_{P_i-1,P_i-1} may be expressed as

$$R_{P_i-1,P_i-1} = T + D_{P_i-1,P_i-1} \tag{3-11}$$

where D_{P_i-1,P_i-1} is the difference between the toeplitz approximation T and R_{P_i-1,P_i-1}.

Substituting (3-11) into (3-10) leads to

$$\begin{aligned} \gamma_1^{(k)} &= G_1^{-1}[\varphi_1 - H_1\ a_{P_i-1}^{(k-1)}] \\ a_{P_i-1}^{(k)} &= -T^{-1} D_{P_i-1,P_i-1} a_{P_i-1}^{(k-1)} - T^{-1} H_1^T \gamma_1^{(k-1)}\ . \end{aligned} \tag{3-12}$$

Subsequent partitioning of the normal equation then results in

$$\begin{aligned} \gamma_{P_i-1}^{(k)} &= G_{P_i-1}^{-1}[\varphi_{P_i-1} - H_{P_i-1}\ a_1^{(k-1)}] \\ a_1^{(k)} &= -R_{1,1}^{-1} H_{P_i-1}^T \gamma_{P_i-1}^{(k-1)} \end{aligned} \tag{3-13}$$

where $\gamma_{P_i-1} = a_0, G_{P_i-1} = R_{0,0}$, and $H_{P_i-1} = R_{0,1}$. The submatrix $R_{1,1}$ can be expressed

as

$$R_{1,1} = T + D_{1,1}\ , \tag{3-14}$$

and $R_{0,0}$ can be expressed as

$$R_{0,0} = T + D_{0,0}\ . \tag{3-15}$$

Substituting (3-14) and (3-15) into (3-13) gives

$$\begin{aligned} \gamma_{P_1-1}^{(k)} &= G_{P_1-1}^{-1}[\varphi_{P_1-1} - H_{P_1-1}\, a_1^{(k-1)}] \\ a_1^{(k)} &= -T^{-1}D_{1,1}a_1^{(k-1)} - T^{-1}H_1^T\gamma_{P_1-1}^{(k-1)} \, . \end{aligned} \tag{3-16}$$

Finally, combining all of the succesive iterations leads to

$$\begin{aligned} a_0^{(k)} &= a_0^{(0)} - T^{-1}D_{0,0}a_0^{(k-1)} - T^{-1}\sum_{i=1}^{P_1-1} R_{0,i}\, a_i^{(k-1)} \\ a_j^{(k)} &= -T^{-1}D_{j,j}a_j^{(k-1)} - T^{-1}\sum_{\substack{i=0 \\ i\neq j}}^{P_1-1} R_{j,i}a_i^{(k-1)} \end{aligned} \tag{3-17}$$

where $a_0^{(0)} = T^{-1}S^{(0)}$, $a_j^{(0)} = T^{-1}R_{j,0}a_0^{(0)}$, and $j = 1,2,\ldots,P_1\text{-}1$. Equation (3-17) is then iterated to solve for the model parameters in (3-5).

B. NSHP SUPPORT

An iteration similiar to that used for QP support may be used for NSHP support. Use of NSHP suppoι to estimate the autocorrelation matrix results in a matrix with an asymmetric structure similiar to that of (1-15). The sample autocorrelation matrix can be expressed as

$$R = \begin{bmatrix} \tilde{R}_{0,0} & \tilde{R}_{0,1} & \cdots & \tilde{R}_{0,P_1-1} \\ \tilde{R}_{1,0} & R_{1,1} & \cdots & R_{1,P_1-1} \\ \vdots & \vdots & \ddots & \vdots \\ \tilde{R}_{P_1-1,0} & R_{P_1-1,1} & \cdots & R_{P_1-1,P_1-1} \end{bmatrix} . \tag{3-18}$$

The dimensions of the blocks along the top and left edges are reduced by a number of rows and columns, respectively, equal to the number $-L_2$ associated with region A.

The development of the iteration begins by forming a toeplitz approximation T of the average of the blocks along the main diagonal of R. The upper-left block is not included in the average, however, because it is not of the same dimension as the other blocks along the diagonal. A separate toeplitz approximation $\tilde{T}$ is determined for this block.

The normal equation is succesively partitioned as discussed previously in sections II.A and II.B. At each stage of the partitioning, the blocks along the main diagonal are expressed as the sum of the toeplitz approximation and a difference matrix. The final form of the algorithm is

$$\begin{aligned} \tilde{a}_0^{(k)} &= \tilde{a}_0^{(0)} - \tilde{T}^{-1} \tilde{D}_{0,0} \tilde{a}_0^{(k-1)} - T^{-1} \sum_{i=1}^{P_1-1} R_{0,i} \, a_i^{(k-1)} \\ a_j^{(k)} &= -T^{-1} D_{j,j} a_j^{(k-1)} - T^{-1} \tilde{R}_{j,0} \tilde{a}_0^{(k-1)} - T^{-1} \sum_{\substack{i=1 \\ i \neq j}}^{P_1-1} R_{j,i} a_i^{(k-1)} \end{aligned} \tag{3-19}$$

where $\tilde{a}_0^{(0)} = \tilde{T}^{-1} \tilde{S}^{(0)}$, $a_j^{(0)} = T^{-1} \tilde{R}_{j,0} \tilde{a}_0^{(0)}$, and $j = 1,2,...P_1-1$.

C. COMBINED-QUADRANT METHOD

In deriving the iteration for the quarter-plane case, support in the first quadrant was assumed. However, support in any of the quadrants may have been used without any loss in generality. Due to the hermitian symmetry and toeplitz block-toeplitz property of the autocorrelation matrix, support in the third quadrant results in an identical spectral estimate to that obtained from support in the first quadrant. In general, however, spectral estimates derived from support in the first and second quadrants are different.

Spectral estimates obtained from quarter-plane support often result in contours of constant power spectral density that are elliptical. This means that the frequency estimate in one direction is more accurate than the estimate for the other direction. To mitigate this problem, the following combined-quadrant (CQ) spectral estimate has been suggested [Ref. 10]

$$\hat{P}_x = \frac{\sigma^2}{\frac{1}{2}\left[\frac{1}{\hat{P}_{x_1}} + \frac{1}{\hat{P}_{x_2}}\right]} \tag{3-20}$$

where $\hat{P}_{x_1}$ and $\hat{P}_{x_2}$ are the spectral estimates obtained from QP support in the first and second quadrants, respectively.

It can easily be shown that QP support in the second quadrant results in the following normal equation

$$\begin{bmatrix} R_{0,0} & R_{0,1} & \cdots & R_{0,P_1-1} \\ R_{1,0} & R_{1,1} & \cdots & R_{1,P_1-1} \\ \vdots & \vdots & \ddots & \vdots \\ R_{P_1-1,0} & R_{P_1-1,1} & \cdots & R_{P_1-1,P_1-1} \end{bmatrix} \begin{bmatrix} b_{P_1-1} \\ b_{P_1-2} \\ \vdots \\ b_0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ S^{(0)} \end{bmatrix} \tag{3-21}$$

where $R_{m,n} = R^T_{n,m}$, $b_k^T = [b(k,0)\ b(k,1)\ b(k,2) \cdots b(k,P_2-1)]$, $b(0,0) = 1$, and $S^{(0)} = [\sigma^2\ 0 \cdots 0]^T$. The autocorrelation matrix is identical to that obtained using QP support in the first quadrant. However, the AR parameters b_i are not the same in general.

The estimation of the AR parameters is obtained by an iteration similiar to (3-17). The difference is the manner in which the normal equation is partitioned. Rather than beginning the partitioning at the lower-right corner and continuing toward the opposite corner, as previously shown with support in the first quadrant, the partitioning begins with the upper-left corner and continues to the opposite corner. The toeplitz approximation T is the same as that used for support in the first quadrant. The iteration is therefore

$$
\begin{aligned}
b_0^{(k)} &= b_0^{(0)} - T^{-1} D_{P_1-1,P_1-1} b_0^{(k-1)} - T^{-1} \sum_{i=1}^{P_1-1} R_{P_1-1,P_1-1-i}\, b_{P_1-1-i}^{(k-1)} \\
b_j^{(k)} &= -T^{-1} D_{P_1-1-j,P_1-1-j} b_j^{(k-1)} - T^{-1} \sum_{\substack{i=1 \\ i \neq j}}^{P_1-1} R_{P_1-1-j,P_1-1-i}\, b_{P_1-1-i}^{(k-1)}
\end{aligned}
\tag{3-22}
$$

where $b_0^{(0)} = T^{-1} S^{(0)}$, $b_j^{(0)} = T^{-1} R_{P_1-1,j} b_0^{(0)}$, and $j = 1, 2, \ldots, P_1-1$. The CQ method requires $O(2P_1^2 P_2^2 K)$ multiplications to converge to the true parameter values.

IV. RESULTS OF ITERATIVE SPECTRAL ESTIMATION

The performance of the iterative spectral etimation method was investigated by testing its ability to detect multiple sinusoids in white noise. The data was given by

$$x(n_1,n_2) = B_1\cos(0.125n_1+0.125n_2)+B_2\cos(0.33n_1+0.33n_2)+w(n_1,n_2) \qquad (4\text{-}1)$$

where B_i are the amplitudes of the sinusoids, and $w(n_1,n_2)$ is a sample function of zero mean white noise with unit variance. The signal-to-noise ratio (SNR) of the random process is defined by [Ref. 1]

$$SNR = \sum_{i=1}^{m} \frac{B_i^2}{\sigma^2} \qquad (4\text{-}2)$$

where m is the number of sinusoids present.

Two sizes of data sets were used to evaluate the performance of the iterative spectral estimation method: 16x16 and 8x8. For each size a SNR of 10 dB and 0 dB were used. The SNR was altered by varying the values of B_i while holding σ^2 constant. Comparisons were made between QP and NSHP support and the CQ method.

Spectral estimates of the 16x16 data set with a 10 dB SNR are shown in Figure 5. Best results were obtained for QP support after 12 iterations for $P_1 = P_2 = 4$. The estimated frequencies do not match the true values, and there are many spurious peaks. Other values of P_1 and P_2 failed to produce more accurate results.

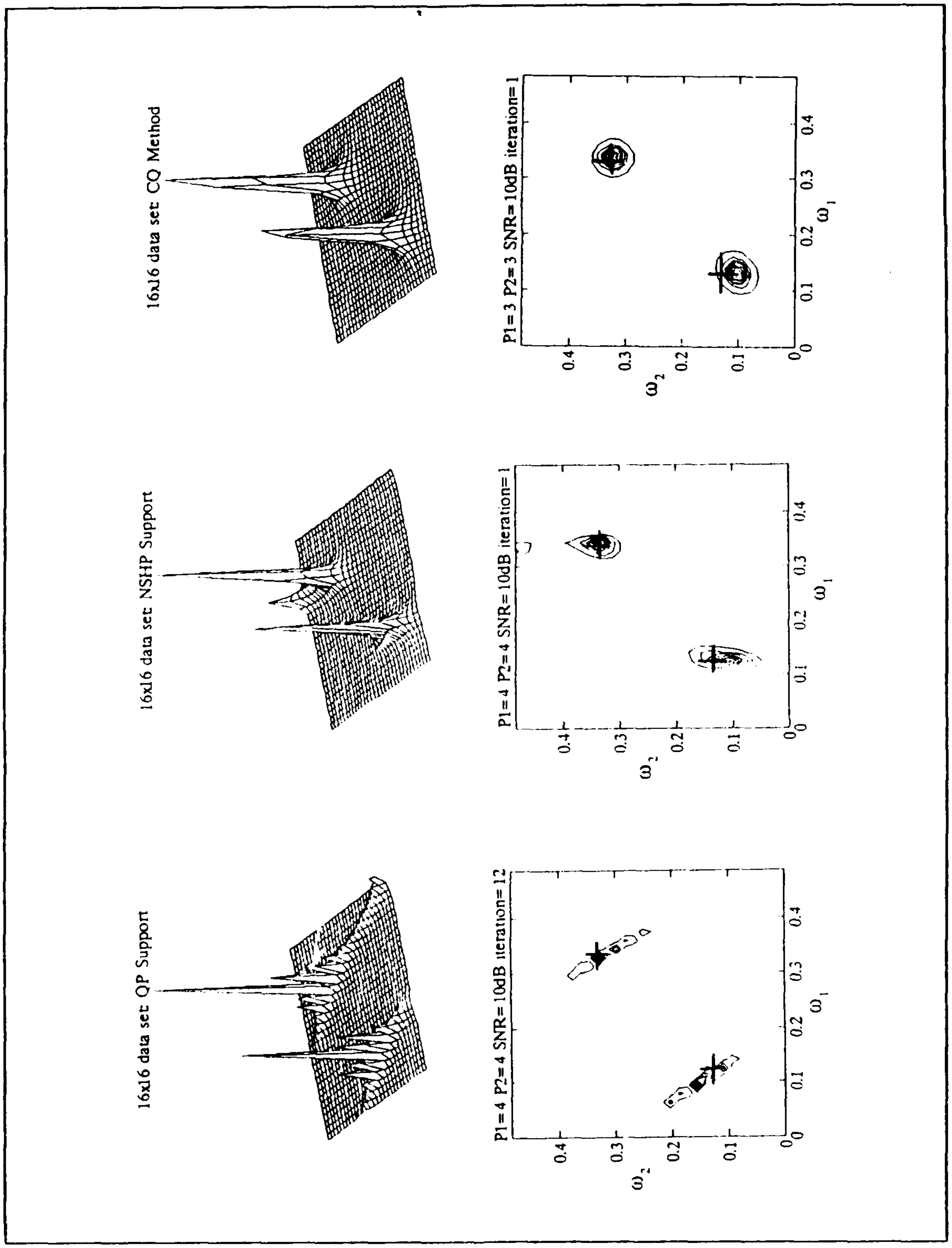

Figure 5. Spectral estimates of 16x16 data set with 10 dB SNR

Better results were obtained using NSHP support with $P_1=P_2=4$, and $L_2=-2$. After one iteration an acceptable spectral estimate was obtained. The estimated frequencies are closer to the true values, and the spectral peaks are sharp.

The CQ method produced even better results. An acceptable spectral estimate was obtained after a single iteration with $P_1 = P_2 = 3$. The spectral peaks are sharp, and the estimated frequencies are closer to the true values than with NSHP support.

Results of spectral estimation applied to a 0 dB SNR data set of size 16x16 are shown in Figure 6. Meaningful spectral estimates could not be obtained using QP support for any values of P_1 and P_2. Employment of NSHP support led to a useful spectral estimate after three iterations for $P_1=P_2=4$, and $L_2=-2$. However, the spectral peaks are broad.

The CQ method yielded a useful spectral estimate after one iteration for P_1 and P_2 equal to three. The estimated frequencies are very close to the true values, but the spectral peaks are very broad.

Spectral estimates of a 10 dB SNR data set of size 8x8 are seen in Figure 7. After five iterations QP support produced a mediocre spectral estimate for both P_1 and P_2 equal to four. The estimated frequencies are very close to the true values. However, the presence of spurious peaks makes the spectral estimate very difficult to interpret accurately, in spite of the sharp spectral peaks.

A more nearly-accurate spectral estimate was obtained from NSHP support after a single iteration for $P_1=P_2=4$, and $L_2=-2$. The accuracy of the higher-frequency estimate

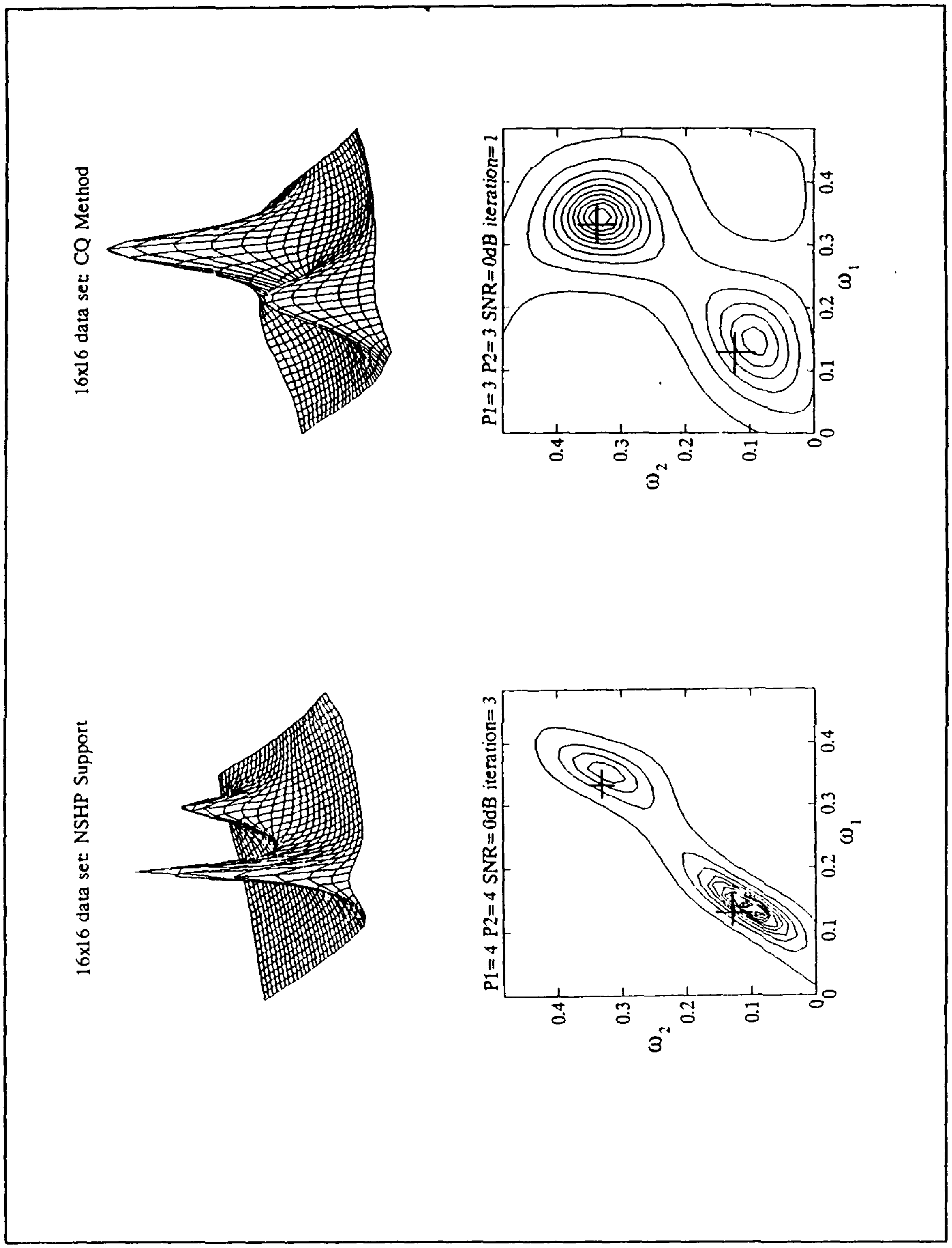

Figure 6. Spectral estimates of 16x16 data set with 0 dB SNR

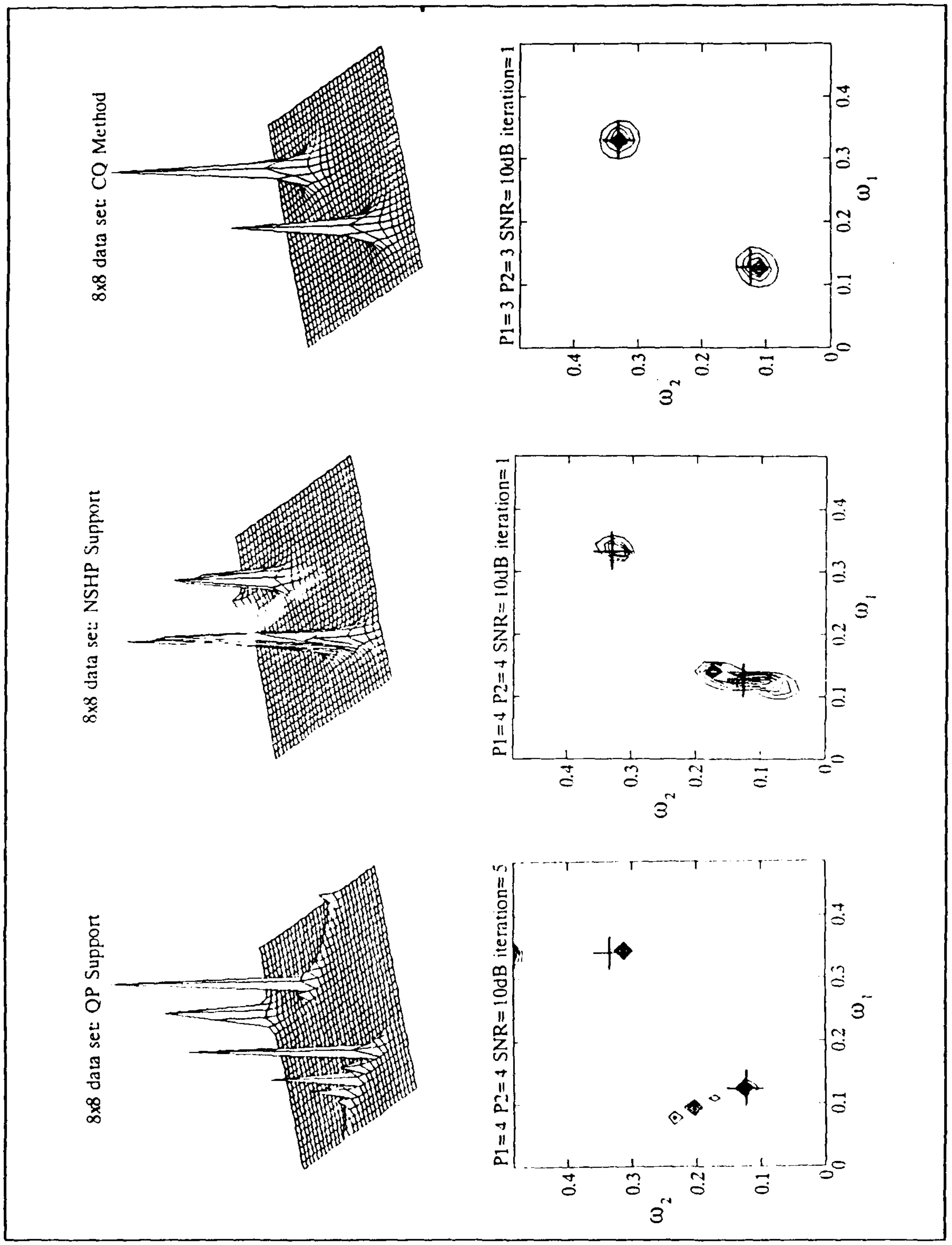

Figure 7. Spectral estimates of 8x8 data set with 10 dB SNR

is very good, whereas the resolution of the lower-frequency estimate is only fair.

The CQ method produced a very accurate spectral estimate after one iteration for P_1 and P_2 equal to three. As seen in the figure, the spectral peaks are very sharp, and the estimated frequencies are very close to the true values.

Only the CQ method produced an acceptable spectral estimate of the 0 dB SNR data set of size 8x8. A meaningful spectral estimate was obtained after a single iteration for P_1 and P_2 equal to three, as seen in Figure 8. The frequency estimates are close to the true values, but the spectral peaks are broad.

Several other cases were tested for different sinusoidal frequencies in white noise. Additionally, the values for the SNR were varied. In all of these cases, the results were consistent with those presented above.

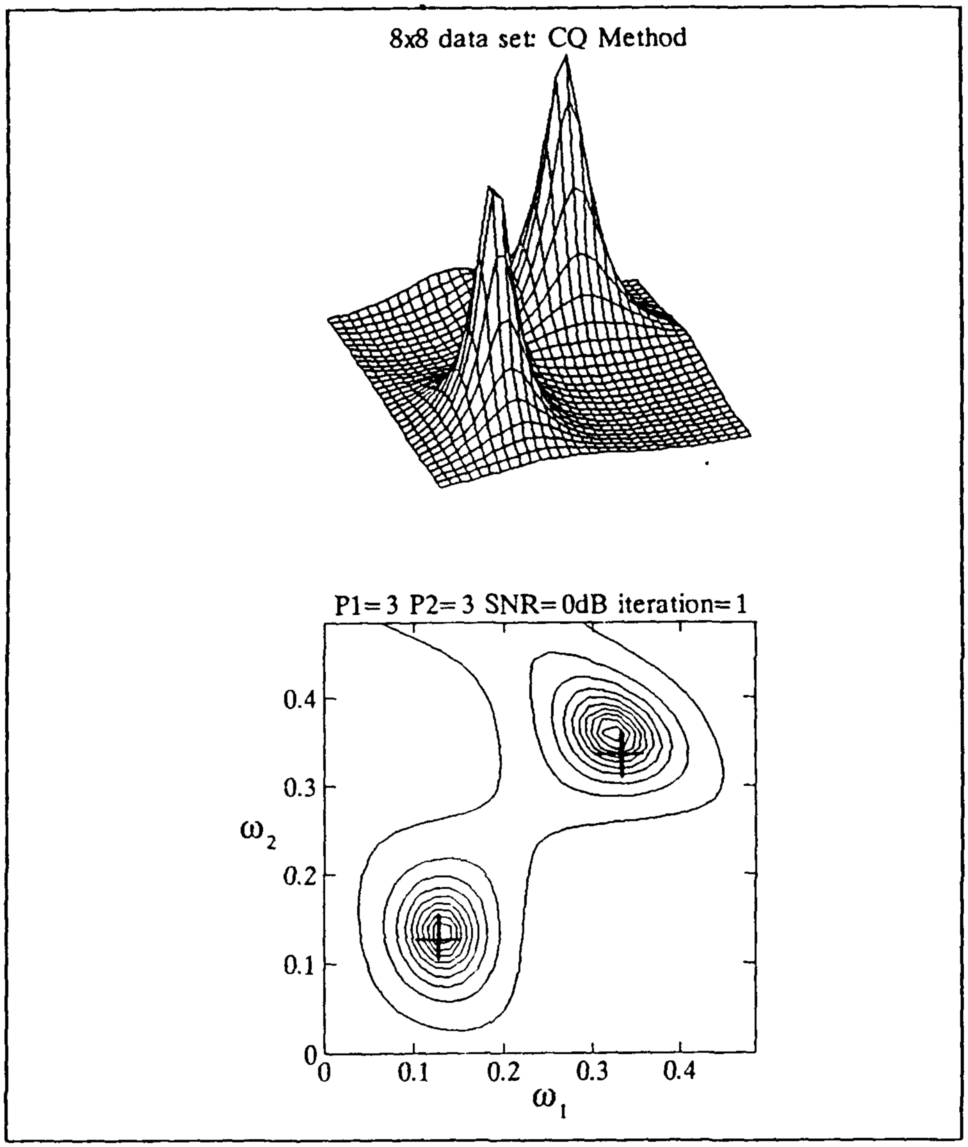

Figure 8. Spectral estimate of 8x8 data set with 0 dB SNR

V. CONCLUSIONS

The results presented in Chapter IV show that high-resolution spectral estimates can be obtained using iterative methods. In several cases, a single iteration was sufficient to produce meaningful results. The net gain is a reduction in the number of computations required to estimate the spectra.

Comparisons of spectral estimates obtained from QP and NSHP support and the CQ method show that the latter produced superior results. Although the CQ method requires $O(2P_1^2P_2^2)$ multiplications to arrive at a solution as compared to $O(P_1^2P_2^2)$ for QP and NSHP support, it requires a smaller region of support. Therefore, the CQ method not only provides spectral estimates of superior resolution to QP and NSHP support, it also accomplishes these results in fewer iterations.

It was observed that the resolution of the spectral estimates did not necessarily improve with more iterations. In the case of the CQ method, the resolution did not improve significantly after the first iteration. Results for QP and NSHP support indicate that the estimates of the AR parameters converge to the true values, and may then diverge. This phenomenom is not understood and provides a basis for continued study.

Additional areas of research arise from the findings of this thesis. The fast solution of the normal equation via iteration leads to the possibility of following spectral lines of slowly time-varying signals. Because the values of the AR parameters are dependent upon previous values, the previous values could possibly be used to re-initialize the estimates when the random process changes in time. Exploration of this idea would serve as good follow-on study. Another follow-on study is the posible application of iterative methods to the multichannel case, where the AR parameters are in matrix form.

LIST OF REFERENCES

1. Jae S. Lim, *Two-Dimensional Signal and Image Processing*, pp. 346-396, Prentice Hall, Englewood Cliffs, New Jersey, 1990.

2. Steven M. Kay, *Modern Spectral Estimation: Theory and Application*, pp. 479-512, Prentice Hall, Englewood, New Jersey, 1988.

3. Dan E. Dudgeon and Russell M. Mersereau, *Multidimensional Digital Signal Processing*, pp. 315-338, Prentice Hall, Englewood, New Jersey, 1984.

4. D.H. Johnson, "The application of spectral estimation methods to bearing estimation problems," *Proc. IEEE*, Vol. 70, No. 9, pp. 1018-1028, 1982.

5. J.S. Lim and F.U. Dowla, "A new algorithm for high resolution two-dimensional spectral estimation," *IEEE Trans. Acoust., Speech, Signal Processing*, Vol. ASSP-29, No. 6, pp. 284-285, 1981.

6. C.W. Therrien and H.T. El-Shaer, "A direct algorithm for computing 2-D AR power spectrum estimates," *IEEE Trans. Acoust., Speech, Signal Processing*, Vol. ASSP-27, No. 11, pp. 1795-1798, 1989.

7. J.W. Woods, "Two-dimensional Markov spectral estimation," *IEEE Trans. Inf. Theory*, Vol. IT-22, No. 5, pp. 552-559, 1976.

8. Murali Tummala, "New algorithm for solving block matrix equations with applications in 2-D AR spectral estimation," *IEEE Trans. Acoust., Speech, Signal Processing*, Vol. ASSP-39, No. 3, 1991.(To appear)

9. Murali Tummala, "Efficient iterative methods for FIR least squares identification," *IEEE Trans. Acoust., Speech, Signal Processing*, Vol. ASSP-38, No. 5, pp. 887-890, 1990.

10. L.B. Jackson and H.C. Chien, "Frequency and bearing estimation by two-dimensional linear prediction," *Proc. IEEE Int. Conf. Acoust., Speech, Signal Processing*, pp. 665-668, Washington, D.C., 1979.

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