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TRANSIENT RESPONSE OF A GUN
SYSTEM UNDER REPEATED FIRING

by

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This report develops an analysis of the transient response of a gun barrel to repeated firing. A finite element approach is used. Initial linear and angular displacements and velocities may be prescribed. A digital computer program based upon the analysis has been developed. The position, velocity, and acceleration of node points are outputs. A plot option is available.

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CONTENTS

Preface:

This report develops an analysis of the transient response of a gun barrel subject to repeated firing. A finite-element approach is used. A piecewise cubic approximation of the deflection ensures continuity of deflections and slopes at the net points. Initially, linear and angular displacements and velocities are prescribed. This arbitrariness in the initial conditions allows for residual motions from the preceding shot. A digital computer program based upon the analysis has been developed. The program may be used for the period during which the projectile leaves the muzzle by setting terms pertaining to the projectile and the gas pressure equal to zero. If desired, the program may be used to determine the static deflection of the gun barrel. Horizontal transverse vibrations of the barrel may be treated by discarding the acceleration due to gravity. The projectile is treated as a point mass. Consequently, projectile balloting is not included in the theory. The effects of a gun-barrel tuning mass, eccentricity of the breech, eccentricity of the recoil mechanism, stiffness and damping of the supports, and prescribed motion of the gun foundation, as well as Bourdon pressure and projectile-barrel friction are included in the model. In the computer program, the time integration routine employs Newmark's beta method. The computer program listing is not included here. However, information on the computer program is available by writing to

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NOTATIONS

t	=	time. Dot denotes time derivative
x	=	axial distance along the barrel from the breech (Fig. 1)
y	=	deflection of the barrel; $\dot{y} = \partial y/\partial t$
θ	=	notation of cross section of barrel. If shear deformation is discarded, $\theta = \partial y/\partial x = y'$
L	=	length of barrel
a	=	radius of the bore
η	=	coordinate of the tuning mass
m_T	=	mass of the tuning mass (Fig. 2)
I_T	=	moment of inertia of the tuning mass about a diametral axis through its center of mass (Fig. 2)
m	=	mass of the projectile
$\bar{M}$	=	mass of the breech block
M	=	mass of the barrel
$\mathcal{m}$	=	mass of the part of the barrel in the range greater than x (Eq. 26)
$\bar{I}$	=	moment of inertia of the breech block about a transverse axis through its center mass
g	=	acceleration of gravity
E	=	Young's modulus
e, d	=	lengths defining the eccentricity of the breech (Fig. 1)
α	=	angle that the barrel forms with the horizontal (Fig. 1)
α_B	=	spring constant for the recoil mechanism (assumed constant
μ_B	=	viscous damping coefficient for the recoil mechanism
v	=	$v(t)$ = speed of the projectile relative to the breech
ξ	=	$\xi(t)$ = displacement of the projective relative to the breech. $v = \dot{\xi}$

p = $p(t)$ = gas pressure behind the projectile

F = $F(t)$ = axial frictional force of the projectile on the barrel

u = $u(t)$ = recoil displacement of the barrel (positive backwards, Fig. 1)

P = $P(t)$ = tension in the barrel at section x due to inertia (Eq. 26)

ρ_0 = mass density of barrel

$x_1, x_2, \ldots, x_{n+1}$ = net points. $x_1 = 0$, $x_{n+1} = L$

Δ_j = $x_{j+1} - x_j$. This is the length of the j-th interval

y_j = $y(x_j, t)$

α_j = constant for translational spring support at point x_j

α'_j = constant for rotational spring support at section x_j

μ_j = constant for translation damper at point x_j

μ'_j = constant for rotational damper at point x_j

P_j = value of P at $x = x_j$. Note that P_j depends on the location ξ of the projectile

θ_j = $\theta_j(t)$ = value of θ at $x = x_j$

r = shear coefficient (Eq. 8)

$B^j_{\alpha\beta}$ = coefficients for the Bourdon effect (Eqs. 24 and 25)

$A^j_{\alpha\beta}$ = coefficients for the axial inertia effect (Eqs. 28 and 31)

x_{T+1} = net point ahead of the tuning mass. $x_T < \eta \leqq x_{T+1}$ (Fig. 2)

e', e'' = eccentricities of breech spring and damper (Fig. 1)

ρ = $\rho(x)$ = mass of barrel per unit length

n = number of segments; $n+1$ = number of net points

1. INTRODUCTION

A finite-element method is used to analyze the transient response of a gun barrel due to repeated firing. The dynamic behavior of the barrel may be computed for the period in which the projectile is in the barrel; of particular importance are the slope and the lateral velocity at the muzzle. By setting to zero terms pertaining to the projectile and the gas pressure, one may follow the gun response also for the period after the projectile leaves the muzzle. The barrel is modeled as a tapered elastic beam with finite degrees of freedom by subdividing it into n intervals (finite elements) with n+1 net points (nodes). The generalized coordinates are the deflections y_j and the rotation θ_j of the barrel cross section at the nodes. Hence, there are 2(n+1) generalized coordinates. Two additional coordinates that enter the equations are the axial recoil displacement u of the barrel and the axial displacement ξ of the projectile relative to the breech. However, these coordinates do not enlarge the system of differential equations if the axial movements of the barrel and the projectile are regarded as known functions of time.

The breech is treated as a rigid mass with its center offset from the axis of the barrel. Viscous dampers (dashpots) may be located at all nodes or they may be provided at only a few points, since in the computer program, a damper can be eliminated by setting its coefficient equal to zero. Also the offset of the breech may be set equal to zero if the center of mass of the breech lies on the axis of the barrel. The structure supporting the barrel and the breech is modeled as a set of independent translational and rotational springs and dashpots at the net points. At nodes where there is no such support, the spring constants and/or dashpot constants are set equal to zero. A tuning mass may be located at an arbitrary point along the barrel.

The foundation upon which the gun barrel and its supports rest is regarded as a rigid body that moves in a prescribed way. In general, it has six degrees of freedom. However, in a linear theory, oscillations of the barrel in a vertical plane are uncoupled from transverse horizontal displacements. Also, the rotation of the barrel about its axis is uncoupled. Consequently, attention may be restricted to a plane motion of the barrel. Oscillations of the barrel in a fixed vertical plane are considered. Horizontal lateral oscillations may be superimposed. The program for horizontal lateral oscillations is the same as for oscillations in a vertical plane, except that effects of gravity are eliminated.

The differential equations of motion may be obtained from Hamilton's principle, as the Lagrangian equations for a nonconservative system. In terms of the virtual work δW and the kinetic energy T, Hamilton's principle states that

$$\int_{t_0}^{t_1} (\delta T + \delta W) dt = 0 \tag{1}$$

where t denotes time. However, because certain contributions to the differential equations of motion can be written down directly in terms of well-known finite element procedures, it is expedient to do so. Hence, in this report Hamilton's principle is used mainly as a guide to verify the form of certain unusual effects such as the Bourdon effect, the axial inertia effect and other pressure effects that are inherent in gun systems.

Contributions to δW come from several sources. One source is the action of gravity on the breech, the barrel, the projectile, and the tuning mass. Another source is the axial frictional force of the projectile on the barrel. It introduces a nonconservative effect. Another contribution comes from the so-called "Bourdon effect", which arises because, in the bent barrel, the area of the bore above the neutral plane, on which gas pressure acts, is slightly greater than the area below the neutral plane. Still another effect contributing to δW is the axial inertia of the recoiling barrel, which initially exerts a straightening influence. The contribution of the strain energy of the barrel due to flexural stresses, etc. may be introduced as a negative contribution to δW. This latter effect essentially results in a finite element model of the gun barrel in the final differential equation form for the gun system. Accordingly, a finite element beam model is used to incorporate this effect in the differential equations, rather than derive this effect from Eq. (1) directly. Finally, to account for the effects of the supporting structure, the support structure is regarded as sets of linear rotational and translational decoupled springs and dashpots located at the nodes of the finite element model of the barrel. These elements are assumed to be attached to the gun foundation which may undergo prescribed motions. The spring constants and the damping constants for the translational and rotational springs and dashpots at node j are respectively (α_j, α'_j) and (μ_j, μ'_j). At a node where there is actually no supporting spring or dashpot, the spring constants and/or dashpot constants are set equal to zero in the computer program.

In general, for a finite element model of the barrel, the net virtual work may be expressed as

$$\delta W = \sum_{j=1}^{n+1} Q_j \delta y_j + \sum_{j=1}^{n+1} Q'_j \delta \theta_j + R\delta u + S\delta\xi \qquad (2)$$

where n = the number of elements (n+1 = number of nodes) and where Q_j, Q'_j, R, and S are the components of generalized force; i.e., the coefficients of the generalized coordinates u, ξ, y_j, θ_j in the expression for δW.

The kinetic energy T is the sum of the kinetic energies of the breech, the projectile, the barrel, and the tuning mass. It is a function of the generalized coordinates u, ξ, $\dot{y}_j$, $\dot{\theta}_j$, where dots denote derivatives with respect to time t. Consequently, the differential equations (the Lagrangian equations) of motion are, by Hamilton's principle,

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{y}_j} - \frac{\partial T}{\partial y_j} = Q_j, \quad \frac{d}{dt}\frac{\partial T}{\partial \dot{\theta}_j} - \frac{T}{\partial \theta_j} = Q'_j;$$

$$j = 1,\ n+1. \qquad (3)$$

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{u}} - \frac{\partial T}{\partial u} = R, \quad \frac{d}{dt}\frac{\partial T}{\partial \dot{\xi}} - \frac{\partial T}{\partial \xi} = S. \qquad \text{(3)cont'd}$$

where Q_j, Q'_j, R, S are the components of generalized force (Eq. 2). The Lagrange equations for u and ξ are irrelevant, if we know u(t) and ξ(t) as functions of time t. It is to be noted that $\partial T/\partial y_j = \partial T/\partial \theta_j = 0$, since $T = T(u, \xi, \dot{y}_j, \dot{\theta}_j)$.

By the usual procedures of finite element methods, Eqs. (3) may be written in the form

$$\underset{\sim}{M}\ddot{\underset{\sim}{x}} + \underset{\sim}{C}\dot{\underset{\sim}{x}} + \underset{\sim}{K}\underset{\sim}{x} = \underset{\sim}{f}(t) \qquad (4)$$

where $\underset{\sim}{M} = \underset{\sim}{M}(t)$, $\underset{\sim}{C} = \underset{\sim}{C}(t)$ and $\underset{\sim}{K} = K(t)$, are the mass, damping and stiffness matrices for the gun system, $\underset{\sim}{f}(t)$ is the driving force vector and $\underset{\sim}{x}$ is the vector of generalized coordinates,

$$\underset{\sim}{x} = \{y_1, \theta_1, y_2, \theta_2, \ldots, y_{n+1}, \theta_{n+1}\}^T . \qquad (5)$$

The Lagrange equations (Eqs. 3 or 4) are linear second order differential equations for the unknown functions $y_j(t)$ and $\theta_j(t)$. They are adaptable to various integration methods, such as the Runge-Kutta method, Newmark's beta method, Houbolt's method, etc. Discontinuities occur at the instantaneous location of the projectile, since, for example, the gas pressure behind the projectile, causing the Bourdon effect, terminates there. However, discontinuities do not obstruct the finite element method. This is an advantage over the direct approach via the differential equations of beams, since with the latter formulation, the action of the projectile is described with a Dirac delta function or a Heaviside step function. These difficulties are avoided by the finite-element treatment. A computer program based on Newmark's beta method has been devised for solving the differential equations for arbitrary initial values $y_j(0)$, $\theta_j(0)$ and $\dot{y}_j(0)$, $\dot{\theta}_j(0)$. Vertical bending and horizontal bending of the gun barrel are decoupled in linear theory. Both of these types of deformation are covered by the program. For horizontal motion, there is no gravitational effect.

The program provides the history of the motion of the barrel from the initial instant at which the projectile is fired, until the projectile leaves the muzzle. In addition, it may be used for the period after the projectile has left the muzzle. To account for this latter period, all terms pertaining to the projectile (e.g., gas pressure, projectile mass, friction) are dropped.

To implement the program, we must know the recoil displacement $u(t)$, the projectile displacement $\xi(t)$ along the barrel, the base pressure $p(t)$ on the projectile, and the frictional force $F(t)$, between the barrel and the projectile. If $\xi(t)$ and $p(t)$ are known, $F(t)$ is determined by Newton's law applied to the axial motion of the projectile. In applying the program to an actual gun system, the spring constants (α_j, α_j') and the damping constants (μ_j, μ_j') of the supporting structure must be estimated. Because of the complicated nature of the structure and the peculiar damping devices in use, this may be difficult.

Of particular significance for the accuracy of shooting are the muzzle displacement, y_{n+1}, muzzle rotation θ_{n+1} and the muzzle velocity $\dot{y}_{n+1}$ at the instant that the projectile leaves the barrel. The program permits a study of these quantities corresponding to various initial motions and deflections of the barrel at the instant of firing.

2. DESCRIPTION OF THE MATHEMATICAL MODEL

Figure 1 represents a gun barrel with mass M and elevation angle α. Axes (x, y) move axially with the barrel as it recoils, but they undergo no lateral displacement. The projectile is regarded as a point mass m. Therefore, balloting of the projectile is not considered. A short time t after the projectile is fired, it lies at the barrel point $x = \xi(t)$, as shown in Fig. 1. The axial velocity of the projectile relative to the barrel is $v(t) = \dot{\xi}$. The axial frictional force of the projectile on the barrel is $F(t)$. The gas pressure driving the barrel is $p(t)$. There is a tuning mass m_T with constant coordinate η. Its center of mass is considered to lie on the axis of the barrel. Its moment of inertia about a transverse axis through its center of mass is I_T. The breech is regarded as a rigid block with mass $\bar{M}$ and moment of inertia $\bar{I}$ about a transverse axis through its center of mass. The location of the center of mass is defined by two lengths, e and d shown in Fig. 1. The axial recoil displacement is $u(t)$. The breech recoil spring and recoil dashpot are assumed to be linear. Their constants are (α_B, μ_B) and their eccentricities are (e', e''), as shown in Fig. 1.

If there is initial bending of the barrel due to weight, unsymmetrical thermal gradients, or manufacturing imperfections, the bending is aggravated by inertial reaction between the barrel and the projectile. Axial friction F(t) also excites motion of the barrel. Another disturbance comes from the so-called "Bourdon" effect, which arises because, in the bent barrel, the area of the bore above the neutral plane, on which the gas pressure acts, is slightly greater than the area below the neutral plane. Still another effect contributing to the bending of the barrel is axial inertia of the recoiling barrel and breech.

3. CONTRIBUTIONS OF VARIOUS PARTS OF THE SYSTEM AND OF VARIOUS EFFECTS TO THE EQUATIONS OF MOTION

3.1. Strain Energy of the Barrel. The barrel may be regarded as a tapered elastic beam of length L. If bending strain energy only is included, the strain energy per unit length is proportional to the square of the curvature. If the effect of transverse shear deformation is included, an appropriate term for the shear energy must be added. In this report these energies are taken, respectively, as (1)

$$U_{Bending} = \frac{1}{2}\int_{0}^{L} EI(y'')^2 dx \tag{6}$$

$$U_{Shear} = \frac{1}{2}\int_{0}^{L} GA(y'-\theta)^2 dx \tag{7}$$

where primes denote derivatives with respect to x, y = the lateral displacement of the barrel, θ = the rotation of the barrel section (Note that if shear deformation is not included the rotation $\theta = y'$, the slope of the axis of the barrel), EI = the bending rigidity and GA = the shear rigidity of the beam. By Hamilton's principle, these energies contribute terms to the components of generalized forces (Eq. 3) or equivalently to the stiffness matrix $\underset{\sim}{K}$ (Eq. 4).

In the usual method of finite elements (2), the barrel is divided into n intervals (elements) of length Δ_1, Δ_2, ..., Δ_j, ..., Δ_n, by points x_1, x_2, ... x_{n+1}, where x_{n+1} is the coordinate of the muzzle (Fig. 1). These points need not be equally spaced. Then with y taken as a cubic polynomial in x in each interval of length Δ_j, and θ taken as a corresponding quadratic polynomial in x, the element stiffness matrix for the jth element is (see Appendix I), for j = 1, 2, ..., n,

$$\underset{\sim}{k}_j = \frac{E_j I_j}{\Delta_j^{\,3}(1+r)_j} \begin{bmatrix} 12 & 6\Delta & -12 & 6\Delta \\ 6\Delta & \Delta^2(4+r) & -6\Delta & \Delta^2(2-r) \\ -12 & -6\Delta & 12 & -6\Delta \\ 6\Delta & \Delta^2(2-r) & -6\Delta & \Delta^2(4+r) \end{bmatrix}_j \tag{8}$$

where $r = 12EI/(AGL^2)$ and $\Delta_j = x_{j+1} - x_j$. Appropriate additions of the matrices $\underset{\sim}{k}_j$ are required to incorporate the element stiffnesses into the structural stiffness matrix $\underset{\sim}{K}$ of Eq. (4) (2). The effects of shear deformation may be discarded by setting r = 0 in Eq. (8).

3.2. Certain Effects Related to Pressure, Rigid Parts and Supports. The generalized coordinates are taken to be the displacements y_j, the rotations θ_j, the displacement ξ of the projectile with respect to the breech, and the axial displacement

u of the barrel and the breech (as noted previously; see Fig. 1). The breech is regarded as a rigid block of mass $\bar{M}$ and moment of inertia $\bar{I}$ about its center of mass. The center of mass of the breech may be off center from the axis of the barrel. The eccentricity of the breech is specified by two lengths, d and e (Fig. 1). There is an axial spring with constant α_B and an axial dashpot with constant μ_B at the breech. These elements are offset distances e' and e'' from the axis of the barrel (Fig. 1) Movement of the breech transverse to the barrel is restrained by a spring with constant α_1 and a dashpot with constant μ_1. These are included among springs and dashpots with constants α_j and μ_j that act at the nodal points on the barrel. Their effects will be discussed later.

3.2.1. Action of Gravity on the Breech. By Fig. 1, the descent of the center of mass of the breech due to displacement at $x = 0$ is

$$u \sin\alpha - y_1 \cos\alpha + (d \cos\alpha - e \sin\alpha)\theta_1$$

where α is the angle of inclination of the barrel with respect to the horizontal. Therefore, the contribution of the action of gravity on the breech to the variation of the virtual work is

$$\bar{M}g[\delta u \sin\alpha - \delta y_1 \cos\alpha + (d \cos\alpha - e \sin\alpha)\delta\theta_1] \tag{9}$$

By Eqs. (2) and (3), it is seen that the coefficients of δu, δy_1, $\delta\theta_1$ in Eq. (9) contribute to the right-hand side of Eqs. (3). With Eqs. (3) written in the form of Eqs. (4), the coefficients of δu, δy_1, $\delta\theta_1$ contribute to the right-hand side of Eqs. (4).

3.2.2. Contribution to δW of Axial Force on Projectile. The projectile is subjected to the axial force $\pi a^2 p-F$, where p is the gas pressure behind the projectile, a is the radius of the bore, and F is the axial frictional force of the barrel on the projectile. The effect of the shock wave ahead of the projectile can be included in F. The contribution to δW of the axial force on the projectile is, therefore,

$$(\pi a^2 p-F)(\delta\xi - \delta u) \tag{10}$$

3.2.3. Contribution to δW of Axial Movement of the Barrel. Since pressure p also acts on the breech, the contribution of the axial movement of the barrel to δW is

$$Mg(\sin\alpha)\delta u + (\pi a^2 p-F)\delta u \tag{11}$$

where M is the mass of the barrel, and where the term $Mg(\sin\alpha)$ is the contribution of the axial component of gravity.

Accordingly, the net contribution to δW from the breech, the axial force acting on the projectile, the axial movement of the barrel, the breech spring, and the breech damper is, by Eqs. (9), (10), (11),

$$\begin{aligned}\delta W = [&-\alpha_B(u-e'\theta_1)-\mu_B(\dot{u}-e''\dot{\theta}_1)\\ &+(\bar{M}+M)g \sin\alpha]\delta u +(\pi a^2 p-F)\delta\xi\\ &-\bar{M}g(\cos\alpha)\delta y_1 + [\bar{M}g(d \cos\alpha - e \sin\alpha)\\ &+\alpha_B e'(u-e'\theta_1)+ \mu_B e''(\dot{u}-e''\dot{\theta}_1)]\delta\theta_1\end{aligned} \tag{12}$$

The terms involving α_B and μ_B in Eq. (15) come from the fact that the compression of the breech spring is $u-e'\theta_1$. Consequently, the resisting forces of these elements are $\alpha_B(u-e'\theta_1)$ and $\mu_B(\dot{u}-e''\dot{\theta}_1)$. The coefficients of δu, $\delta\xi$, δy_1, $\delta\theta_1$ contribute to the corresponding terms in Eqs. (3) or (4).

3.2.4. Contribution to δW of Dampers and Supporting Structure. The contribution to δW from rotational and translational dashpots at point x_j is

$$-\mu_j\dot{y}_j\delta y_j - \mu_j'\dot{\theta}_j\delta\theta_j \tag{13}$$

where (μ_j, μ_j') are constants for the dashpots, and the dot denotes time derivative. At a node where there is no dashpot, $\mu_j = \mu_j' = 0$.

If the supporting structure can be regarded as a set of decoupled springs located at the nodes, the contribution of the springs at node j to δW is

$$-\alpha_j y_j\delta y_j - \alpha_j'\theta_j\delta\theta_j \tag{14}$$

Accordingly, the contributions of the supports and dampers to δW is

$$\delta W = -\sum_{j=1}^{n+1}(\mu_j\dot{y}_j+\alpha_j y_j)\delta y_j - \sum_{j=1}^{n+1}(\mu_j'\dot{\theta}_j+\alpha_j'\theta_j)\delta\theta_j \tag{15}$$

Again, the coefficients of δy_j and $\delta\theta_j$ contribute in the appropriate manner to Eqs. (3) and (4). In particular, the coefficients of y_j, θ_j contribute to the matrix $\underset{\sim}{K}$ and the coefficients of $\dot{y}$, $\dot{\theta}_j$ contribute to the matrix $\underset{\sim}{C}$ in Eq. (4).

3.2.5. Action of Gravity on the Barrel. In the finite element method employed, the external loads (forces and couples) are applied at the nodes only. If a distributed load or a concentrated load is applied between nodes, they must be replaced by equivalent systems of concentrated loads at the nodes. We require these equivalent loads to be derived in such a manner as to be consistent with the method used in deriving the stiffness matrix (Appendix A). Since the lateral component of the gravity load acting on an element of the barrel is uniformly distributed between nodes, the equivalent nodal force matrix for the jth element of the barrel is

$$\underset{\sim}{f}_j^B = \frac{-m_j \Delta_j \; g \cos\alpha}{12} \begin{bmatrix} 6 \\ 1 \\ 6 \\ -1 \end{bmatrix} , \quad j = 1,2,\ldots,n \qquad (16)$$

where m_j = the mass of the jth element and Δ_j = the length of the jth element. By the method of finite elements, appropriate additions of the matrices $\underset{\sim}{f}_j^B$ are required to incorporate the element nodal forces into the driving force vector $\underset{\sim}{f}(t)$, Eq. (4).

The effect of the axial component of gravity on the barrel has been accounted for previously (see Eq. 11).

3.2.6. Action of Gravity on the Tuning Mass. In general, the tuning mass is located at some point between nodes, Figs. 1 and 2. Since the lateral component of gravity force on the tuning mass is m_T g cosα, this force must be transformed into equivalent nodal forces for the element in which the tuning mass is located. By the method of finite elements (Appendix A), the equivalent nodal force matrix for the element in which the tuning element lies is

$$\underset{\sim}{f}_T = -m_T\, g \cos\alpha \begin{bmatrix} 1 - 3s^2 + 2s^3 \\ (1 - 2s + s^2)s\Delta \\ 3s^2 - 2s^3 \\ (-s + s^2)s\Delta \end{bmatrix}_T \tag{17}$$

where Δ_T = the length of the tuning mass element and $s_T\Delta_T$ = the distance from the left node of the element to the tuning mass. Equation (17) accounts for the lateral component of gravity acting on the tuning mass. In addition, the axial component of gravity contributes to δW in the form

$$\delta W = m_T\, g\, (\sin\alpha)\delta u \tag{18}$$

3.2.7. Action of Gravity on the Projectile. The absolute axial displacement of the projectile at time t is $(\xi - u)$, Figs. 1 and 3. Then, the contribution to δW of the axial component of gravity on the projectile is

$$\delta W = mg\, (\sin\alpha)\, (\delta u - \delta\xi) \tag{19}$$

The effect of the lateral component of gravity on the projectile is included in the same manner as for the tuning mass (Eq. 17) where Δ_T, s_T are replaced by Δ_p, s_p, respectively. In addition, it is to be noted that s_p is a function of time t, since $\xi = \xi(t)$. In studying the effect of gravity on the projectile, it has been found by actual computation that this effect is extremely small for projectiles of the 20-40 mm class.

3.2.8. Axial Friction Effect of Projectile on Barrel. The axial frictional force $F(t)$ exerted by the projectile on the barrel is assumed to be a known function of time t. Alternatively, it may be computed if the pressure acting on the projectile and the projectile acceleration are known as functions of time. Also, it may be expressed as $F(\xi)$, where $x = \xi(t)$ locates the projectile in the barrel. This is a distance s along the curve of the axis of the barrel (Fig. 4).

There is a subtlety in the calculation of the work that force F performs on the barrel, because the projectile is sliding in the barrel. For example, if a grinding wheel acts on a fixed plate, the frictional force of the grinder performs no work on the plate because the plate does not move. However, the frictional force of the plate performs negative work on the grinder.

The axial movement of the projectile during an infinitesimal time dt does not affect the work that the projectile performs on the barrel. Consequently, in computing δW, we set $dt = 0$. The corresponding point on the curve $y + \delta y$ consequently also lies a distance s along the curve. The component of F along the virtual displacement determines δW (Fig. 4). We have

$$y = y(x), \quad s = \int_0^{x_1} \sqrt{1 + (y')^2}\, dx$$

Also, along the curve $y + \delta y$,

$$s = \int_0^{x_2} \sqrt{1 + (y' + \delta y')^2}\, dx$$

Since $y \ll 1$,

$$s \approx x_1 + \frac{1}{2}\int_0^{x_1} (y')^2 \, dx$$

and

$$s \approx x_2 + \frac{1}{2}\int_0^{x_2} (y'^2 + 2y'\delta y') \, dx$$

Therefore,

$$x_2 - x_1 = \varepsilon = \frac{1}{2}\int_0^{x_1} (y')^2 dx - \frac{1}{2}\int_0^{x_2} (y'^2 + 2y'\delta y') \, dx$$

and since $x_1 = \xi(t)$,

$$\varepsilon = \int_0^{\xi} y'^2 \, dx - \frac{1}{2}\int_0^{\xi+\varepsilon} (y'^2 + 2y'\delta y') \, dx$$

and we have

$$\varepsilon = \frac{1}{2}\int_0^{\xi} y'^2 \, dx - \frac{1}{2}\int_0^{\xi} (y'^2 + 2y'\delta y') \, dx$$

$$- \int_{\xi}^{\xi+\varepsilon} y'^2 \, dx$$

or

$$\varepsilon = -\int_0^{\xi} y'\delta y'dx - \frac{1}{2}\int_{\xi}^{\xi+\varepsilon} y'^2\ dx$$

However, since ε is small,

$$\int_{\xi}^{\xi+\varepsilon} y'^2\ dx = \varepsilon[y'(\xi)]^2$$

Hence,

$$\varepsilon\{1 + \tfrac{1}{2}[y'(\xi)]^2\} = -\int_0^{\xi} y'\delta y'dx$$

and since $y' << 1$

$$\varepsilon \approx -\int_0^{\xi} y'\delta y'dx$$

Using the trapezoid rule to evaluate this integral, and setting $\xi = x_r$, we get

$$\begin{aligned}\varepsilon = -\tfrac{1}{2}[&(\theta_1\delta\theta_1+\theta_2\delta\theta_2)\Delta_1+(\theta_2\delta\theta_2+\theta_3\delta\theta_3)\Delta_2 \\ &+ (\theta_3\delta\theta_3+\theta_4\delta\theta_4)\Delta_3 + \dots \\ &+ (\theta_{r-1}\delta\theta_{r-1} + \theta_r\delta\theta_r)\Delta_{r-1}]\end{aligned}$$

The contribution to δW from axial friction is the component of F in the x direction times ε plus the component of F in the y direction times δy_r. Thus, we have approximately

$$\delta W = F\varepsilon + F\theta_r\delta y_r$$

Accordingly, for the j-th interval,

$$\delta W_j = -\frac{F}{2}(\theta_j \delta\theta_j + \theta_{j+1}\delta\theta_{j+1})\Delta_j, \quad 1 \leq j \leq r-1$$

$$\delta W_j = 0 \ , \ j \geq r$$

In addition, there is the contribution $F\theta_r \delta y_r$ at point x_r.

Consequently, the contribution of axial friction of the projectile on the barrel to δW is

$$\begin{aligned} \delta W = F\theta_r \theta y_r - \frac{1}{2} F\Delta_1\theta_1\delta\theta_1 - \frac{1}{2} F(\Delta_1+\Delta_2)\theta_2\delta\theta_2 \\ - \frac{1}{2} F(\Delta_2+\Delta_3)\theta_3\delta\theta_3 - \ldots\ldots \\ - \frac{1}{2} F(\Delta_{r-2}+\Delta_{r-1})\theta_{r-1}\delta\theta_{r-1} \\ - \frac{1}{2} F \Delta_{r-1}\theta_r\delta\theta_r \ , \end{aligned} \tag{20}$$

As with previous contributions to δW, the terms in Eq. (20) must be incorporated appropriately into Eqs. (3) or (4), with the understanding that $F = F(t)$, or alternatively, $F = F(\xi)$. In particular, terms in Eq. (20) contribute to the $\underset{\sim}{K}$ matrix of Eq. (4).

3.2.9. The Bourdon Effect. The tendency of the pressure p in a gun barrel to straighten the barrel is called the "Bourdon effect". A line element of a generator of the bore above the neutral plane has length $(1 - ay_{xx} \sin\theta)dx$ (Fig. 5). The corresponding element below the neutral plane has length $(1 + ay_{xx} \sin\theta)dx$. The difference of these lengths is $2ay_{xx} \sin\theta \, dx$, and the difference between the areas of infinitesimal strips is accordingly $2a^2y_{xx} \sin\theta \, d\theta \, dx$. The projection of this area on the neutral plane is $2a^2y_{xx} \sin^2\theta \, d\theta \, dx$.

Consequently, the effective load per unit length of the barrel, caused by the gas pressure is

$$- 2a^2 p y_{xx} \int_0^{\pi} \sin^2\theta \, d\theta = - \pi a^2 p y_{xx} \tag{21}$$

Accordingly, the contribution of the gas pressure to δW is

$$\delta W = - \pi a^2 p \int_0^{\xi} y_{xx} \, \delta y \, dx$$

where ξ is the location of the projectile. We suppose that the projectile lies at a net point x_r; i.e., $\xi = x_r$. This is not unrealistic, since the terminal point of the pressure is rather indefinite because of the shock wave ahead of the projectile and the finite length of the projectile. By Eq. (21),

$$\delta W = - \pi a^2 p \left\{ \int_{x_1}^{x_2} y_{xx} \, \delta y dx + \int_{x_2}^{x_3} y_{xx} \, \delta y dx + \ldots\ldots \right. \tag{22}$$

$$\left. + \int_{x_{r-1}}^{x_r} y_{xx} \, \delta y dx \right\}$$

Hence,

$$\delta W = \sum_{j=1}^{r-1} \delta W_j$$

where δW_j is the contribution of the "Bourdon effect" to δW in the j-th interval ($\Delta_j = x_{j+1} - x_j$). We let $\delta W_j = 0$ if $j > r-1$, since, for $x > x_r$, $p = 0$.

In the j-th interval, Eq. (1) of Appendix B yields

$$y_{xx} = 2a_2^j + 6a_3^j\, x$$

and $$\delta y = \delta a_0^j + x\delta a_1^j + x^2\, \delta a_2^j + x^3\, \delta a_3^j$$

Hence,

$$\delta W_j = -\pi a^2 p \int_{x_j}^{x_{j+1}} (2a_2^j + 6a_3^j x)(\delta a_0^j + x\delta a_1^j + x^2 \delta a_2^j + x^3 \delta a_3^j)\, dx$$

Integration yields

$$\begin{aligned}
\delta W_j = -\pi a^2 p\{ & 2(x_{j+1} - x_j)\, a_2^j \delta a_0^j \\
& + (x_{j+1}^2 - x_j^2)(a_2^j \delta a_1^j + 3a_3^j \delta a_0^j) \\
& + (x_{j+1}^3 - x_j^3)(\tfrac{2}{3}\, a_2^j \delta a_2^j + 2a_3^j \delta a_1^j) \\
& + (x_{j+1}^4 - x_j^4)(\tfrac{1}{2}\, a_2^j \delta a_3^j + \tfrac{3}{2}\, a_3^j \delta a_2^j) \\
& + \tfrac{6}{5}(x_{j+1}^5 - x_j^5)\, a_3^j \delta a_3^j\}
\end{aligned} \qquad (23)$$

Introducing Eq. (2) of Appendix B into Eq. (23), we get an expression of the following form:

$$
\begin{aligned}
\delta W_j &= B^j_{11} y_j \delta y_j + B^j_{12} y_{j+1} \delta y_j + B^j_{13} \theta_j \delta y_j + B^j_{14} \theta_{j+1} \delta y_j \\
&+ B^j_{21} y_j \delta y_{j+1} + B^j_{22} y_{j+1} \delta y_{j+1} + B^j_{23} \theta_j \delta y_{j+1} + B^j_{24} \theta_{j+1} \delta y_{j+1} \\
&+ B^j_{31} y_j \delta \theta_j + B^j_{32} y_{j+1} \delta \theta_j + B^j_{33} \theta_j \delta \theta_j + B^j_{34} \theta_{j+1} \delta \theta_j \qquad (24) \\
&+ B^j_{41} y_j \delta \theta_{j+1} + B^j_{42} y_{j+1} \delta \theta_{j+1} + B^j_{43} \theta_j \delta \theta_{j+1} + B^j_{44} \theta_{j+1} \delta \theta_{j+1}
\end{aligned}
$$

If we introduce Eq. (2) of Appendix B into Eq. (23), and compare the resulting lengthy expression with Eq. (24), we get expressions for the coefficients $B^j_{\alpha\beta}$. With Eq. (3) of Appendix B, they simplify greatly. Finally, they reduce to the following:

$$
\begin{aligned}
&B^j_{11} = B^j_{22} = \frac{6\pi a^2 p}{5\Delta_j}, && B^j_{12} = B^j_{21} = -\frac{6\pi a^2 p}{5\Delta_j} \\
&B^j_{13} = \frac{11\pi a^2 p}{10}, && B^j_{31} = \frac{\pi a^2 p}{10} \\
&B^j_{14} = B^j_{41} = \frac{\pi a^2 p}{10}, && B^j_{24} = -\frac{11\pi a^2 p}{10}, \quad B^j_{42} = -\frac{\pi a^2 p}{10} \\
&B^j_{23} = B^j_{32} = -\frac{\pi a^2 p}{10}, && B^j_{33} = B^j_{44} = \frac{2\pi a^2 p \Delta_j}{15} \qquad (25) \\
&B^j_{34} = B^j_{43} = -\frac{\pi a^2 p}{30} \Delta_j
\end{aligned}
$$

These equations hold for $j > r-1$, with $p = 0$. Hence, $B^j_{\alpha\beta} = 0$ if $j > r-1$. Also, the equations hold for $r = 1$; $j = r-1 = 0$ with the understanding that $B^0_{\alpha\beta} = 0$. The virtual displacements δu and $\delta\xi$ contribute nothing to the Bourdon effect. It is to be noted that the coefficients $B^j_{\alpha\beta}$ are not entirely symmetrical in α and β. The terms in Eq. (24) contribute in an appropriate manner to the stiffness matrix of Eq. (4).

3.2.10. Bending Caused by Axial Inertia. The tension $P(x, t)$ in the barrel due to recoil acceleration may be calculated by Newton's law with the assumption that the barrel is rigid. Accordingly,

$$P = m\ddot{u} \tag{26}$$

where m is the mass of the part of the barrel in the range greater than x.

If we suppose that the barrel is inextensional, the shortening of the chord due to bending is

$$\frac{1}{2} \int_0^L y_x^2 \, dx$$

where $L = x_{n+1}$ is the length of the barrel. Effectively, then, an element of length dx is shortened by the amount $\frac{1}{2} y_x^2 dx$, and the potential energy of that element due to shortening is $\frac{1}{2} P y_x^2 dx$. Accordingly, the strain energy is augmented by the amount

$$\frac{1}{2} \int_0^L P y_x^2 \, dx$$

The integral is a function of t, since y and P are functions of x and t. Section properties of the barrel do not enter here, except insofar as they affect $P(x, t)$. Equation (1) of Appendix B now yields

$$V_j = \frac{1}{2} \int_{x_{j-1}}^{x_j} P(a_1^j + 2a_2^j x + 3a_3^j x^2)^2 \, dx \tag{27}$$

in which V_j is the potential energy in the j-th interval due to shortening of the chord.

We may write V_j in the following form, by virtue of Eq. (2) of Appendix B,

$$\begin{aligned} 2V_j = {} & A^j_{11}y_jy_j + A^j_{12}y_{j+1}y_j + A^j_{13}\theta_jy_j + A^j_{14}\theta_{j+1}y_j \\ & + A^j_{21}y_jy_{j+1} + A^j_{22}y_{j+1}y_{j+1} + A^j_{23}\theta_jy_{j+1} + A^j_{24}\theta_{j+1}y_{j+1} \\ & + A^j_{31}y_j\theta_j + A^j_{32}y_{j+1}\theta_j + A^j_{33}\theta_j\theta_j + A^j_{34}\theta_{j+1}\theta_j \\ & + A^j_{41}y_j\theta_{j+1} + A^j_{42}y_{j+1}\theta_{j+1} + A^j_{43}\theta_j\theta_{j+1} + A^j_{44}\theta_{j+1}\theta_{j+1} \end{aligned} \tag{28}$$

Without loss of generality, we may set $A^j_{\alpha\beta} = A^j_{\beta\alpha}$.

It is reasonable to assume that P varies linearly in an interval. Then, in the jth interval $x_j < x < x_{j+1}$)

$$P = P_j\Delta_j^{-1}(x_{j+1} - x) + P_{j+1}\Delta_j^{-1}(x - x_j) \tag{29}$$

Consequently,

$$\begin{aligned} \int_{x_j}^{x_{j+1}} x^n P dx = {} & \frac{P_j}{(n+1)(n+2)\Delta_j}\{x_{j+1}^{n+2} - (n+2)x_j^{n+1}x_{j+1} + (n+1)x_j^{n+2}\} \\ & + \frac{P_{j+1}}{(n+1)(n+2)\Delta_j}\{x_j^{n+2} - (n+2)x_jx_{j+1}^{n+1} + (n+1)x_{j+1}^{n+2}\} \end{aligned} \tag{30}$$

With Eq. (29), the integrals in Eq. (27) can be evaluated. Then, with Eq. (2) of Appendix B, we get V_j expressed as a quadratic form in y_j, y_{j+1}, θ_j, θ_{j+1}. Using Eq. (3) of Appendix B, and comparing with Eq. (28), we get

$$A_{11}^{j} = A_{22}^{j} = \frac{3}{5}\,\Delta_j^{-1}(P_j+P_{j+1})$$

$$A_{33}^{j} = \frac{1}{30}\,\Delta_j^{-1}(3P_j+P_{j+1}), \qquad A_{44}^{j} = \frac{1}{30}\,\Delta_j^{-1}(P_j+3P_{j+1}) \tag{31}$$

$$A_{12}^{j} = A_{21}^{j} = -\frac{3}{5}\,\Delta_j^{-1}(P_j+P_{j+1}), \qquad A_{13}^{j} = A_{31}^{j} = \frac{1}{10}\,P_{j+1}$$

$$A_{14}^{j} = A_{41}^{j} = \frac{1}{10}\,P_j, \qquad A_{23}^{j} = A_{32}^{j} = -\frac{1}{10}\,P_{j+1}$$

$$A_{24}^{j} = A_{42}^{j} = -\frac{1}{10}\,P_j, \qquad A_{34}^{j} = A_{43}^{j} = -\frac{1}{60}\,\Delta_j(P_j+P_{j+1})$$

$$A_{\alpha\beta}^{0} = 0, \qquad A_{\alpha\beta}^{n+1} = 0$$

The contribution to δW due to bending caused by axial inertia is

$$\begin{aligned}
\delta W_j = & -A_{11}^{j} y_j \delta y_j - A_{12}^{j} y_{j+1} \delta y_j - A_{13}^{j} \theta_j \delta y_j - A_{14}^{j} \theta_{j+1} \delta y_j \\
& -A_{21}^{j} y_j \delta y_{j+1} - A_{22}^{j} y_{j+1} \delta y_{j+1} - A_{23}^{j} \theta_j \delta y_{j+1} - A_{24}^{j} \theta_{j+1} \delta y_{j+1} \\
& -A_{31}^{j} y_j \delta \theta_j - A_{32}^{j} y_{j+1} \delta \theta_j - A_{33}^{j} \theta_j \delta \theta_j - A_{34}^{j} \theta_{j+1} \delta \theta_j \qquad (32) \\
& -A_{41}^{j} y_j \delta \theta_{j+1} - A_{42}^{j} y_{j+1} \delta \theta_{j+1} - A_{43}^{j} \theta_j \delta \theta_{j+1} - A_{44}^{j} \theta_{j+1} \delta \theta_{j+1}
\end{aligned}$$

The terms in Eq. (32) contribute in an appropriate manner to Eq. (4).

4. KINETIC ENERGY EFFECTS

The kinetic energy is the sum of the kinetic energies of the breech, the tuning mass, the projectile, and the barrel. These energies contribute to the mass matrix of Eq. (4) or the total kinetic energy T of Eq. (3).

4.1. Kinetic Energy Contribution of the Breech. The velocity components of the center of mass of the breech are $e\dot{\theta}_1 - \dot{u}$, $\dot{y}_1 - d\dot{\theta}_1$ (see Fig. 1) and the angular velocity of the breech is $\dot{\theta}_1$. Consequently, the kinetic energy of the breech is

$$T_B = \frac{1}{2}\bar{M}(e\dot{\theta}_1 - \dot{u})^2 + \frac{1}{2}\bar{M}(\dot{y}_1 - d\dot{\theta}_1)^2 + \frac{1}{2}\bar{I}\dot{\theta}_1{}^2$$

where $\bar{M}$ is the mass of the breech and $\bar{I}$ is the moment of inertia of the breech about a transverse axis through its center of mass.

Accordingly by Eq. (3), the breech contributes the term $\bar{M}(\ddot{y}_1 - \ddot{\theta}_1 d)$ to the first row of $\underset{\sim}{M}\underset{\sim}{\ddot{x}}$ and the term $[\bar{I}+\bar{M}(e^2+d^2)]\cdot\ddot{\theta}_1 - \bar{M}d\ddot{y}_1$ to the second row of $\underset{\sim}{M}\underset{\sim}{\ddot{x}}$, and in addition, the term $\bar{M}e\ddot{u}$ must be added to the right hand side of Eq. (4) in the second row.

4.2. Kinetic Energy Contribution of the Tuning Mass. The contribution of the tuning mass to the kinetic energy may be treated by means of D'Alamberts Principle. Thus, we represent the kinetic effects of the tuning mass by an inertial force $m_T\ddot{y}$ and an inertial couple $I_T\ddot{\theta}$ directed as shown in Fig. 2. Hence representation of y ($\theta = y'$) by a cubic for the element in which the tuning mass lies (see Eq. 1, Appendix A or Eq. 1, Appendix B) yields the concentrated force $m_T\ddot{y}$ and the concentrated couple $I_T\ddot{\theta}$ as functions of the nodal displacements and nodal rotations of the tuning mass element, $x_T \leq x \leq x_{T+1}$. Then by the results for a concentrated force and a concentrated couple (see Appendix A) $m_T\ddot{y}$ and $I_T\ddot{\theta}$ may be transformed into equivalent nodal forces and couples.

4.3. Kinetic Energy Contribution of the Barrel. The kinetic energy of the barrel contributes to the term $\underset{\sim}{M}\ddot{\underset{\sim}{x}}$ in Eq. (4). By the method of finite elements, the mass matrix m for a beam element may be computed and the contributions to the total mass matrix $\underset{\sim}{M}$ may be obtained. (See Appendix A for the mass matrix m.)

4.4. Kinetic Energy Contribution of the Projectile. As for the barrel element, the projectile kinetic energy contributes to the term $\underset{\sim}{M}\ddot{\underset{\sim}{x}}$. However, in addition the projectile kinetic energy contributes to the terms $\underset{\sim}{C}\dot{\underset{\sim}{x}}$ and $\underset{\sim}{K}\underset{\sim}{x}$, because of the nature of its motion (Coriolis acceleration, centripetal acceleration, etc.). The analysis of these contributions are best undertaken by a direct consideration of the acceleration effects of the projectile upon the equations of motion of the system (see Appendix C).

5. THE COMPUTER PROGRAM AND RESULTS

The computer program uses a finite element method to analyze the dynamic motion of a gun barrel. The problem is illustrated in Fig. 6. The projectile tuning mass and breech are considered to be rigid bodies. The barrel is treated as a beam and is divided into a number of finite elements. The division points for these elements are called nodes or node points, and there are two degrees of freedom at each node; lateral deflection y, positive upward and rotation y', positive counter clockwise. The nodes are numbered from the left end (the breech end) towards the muzzle (see Fig. 7). If there are N elements, there are N+1 nodes and 2N+2 degrees of freedom.

Each of the elements may have a different length. The barrel may be subdivided into a number of segments. Within each segment, the elements are the same length, but they may differ in diameter if the barrel is tapered (Fig. 7). The tapered portions are modeled in a "stairstep" manner.

The individual elements are treated as beams with four degrees of freedom (deflection and rotation at both ends). Usually, the Bernoulli-Euler beam theory is sufficient for most problems. However, the program provides the option of including shearing deformation and rotary inertia (Timoshenko beam theory). In order to include shearing deformation, a cross-section shear constant must be input. This may be obtained from the literature [3]. For a hollow tube cross section, the value of the shear constant is

$$\text{SHEAR} = \frac{6(1+\nu)(1+\bar{m}^2)^2}{(7+6\nu)(1+\bar{m}^2)^2 + (20+12\nu)\ \bar{m}^2}$$

where $\bar{m} = \frac{\text{inside radius}}{\text{outside radius}}$, ν = Poisson's ratio.

The program permits translational and rotational springs and dashpot supports at any number of node points in addition to a recoil spring and dashpot at the breech. A Bourdon pressure and a projectile-barrel friction force are incorporated in the program, and may be included at the option of the user.

The program is intended primarily to be used for a numerical time integration study of a gun barrel motion from the time of first firing onward including the option of more than one round. The numerical integration routine utilizes the Newmark "Beta Method" [4]. If desired, the program can be used for determination of the static deflected shape with no time variance involved. In addition, provision for a shaking of the supports of the barrel (due to vehicle motion or other cause) is available as an option.

The program output is printed as follows: (1) the problem title, (2) the input data, (3) the initial values of the position, velocity and acceleration of the node points, (4) the position, velocity, and acceleration of the node points at time intervals dictated by the input requests, (5) a plot of the muzzle deflection as a function of time.

Typical plots of muzzle deflection as a function of time are shown in Figs. 8-16. The data is for the British Rarden system for a range of spring stiffnesses K, damping coefficients C and various locations of barrel lateral stopper pads. Figure 8 is for a single shot, whereas Figs. 9-16 are for bursts of three shots. Figures 8-9 are for shots without the lateral motion stopper pads, and Figs. 10-16 are with pads at various locations of the pads. Study of these figures reveals the great importance of the stopper pads. In general, the use of a stopper pad reduces the tip (muzzle) deflections by a factor of ten or more. Study of Figs. 11-14, for the case of the pad located 54 inches from the breech end of the barrel, shows that increasing the damping coefficient C also tends to reduce the muzzle deflection, particularly in the later firings. However, a predominant role is played by the stopper pad in the control of muzzle displacement.

With appropriate input data, other gun systems may be readily studied.

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APPENDIX A: FINITE ELEMENT MODELS

A1. Beam Element

A typical element of beam has 4 degrees of freedom as shown in Figure A1. The displacement of a typical point in the element is assumed to be

$$y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 \tag{1}$$

With the four boundary conditions $y(0)=u_1, y'(0)=u_2, y(\Delta)=u_3, y'(\Delta)=u_4$, the displacement at a typical point can be expressed in terms of the nodal displacements (u_1, u_2, u_3, u_4) as follows:

$$[y] = \left[1 - \frac{3x^2}{\Delta^2} + \frac{2x^3}{\Delta^3},\ x - \frac{2x^2}{\Delta} + \frac{x^3}{\Delta^2},\ \frac{3x^2}{\Delta^2} - \frac{2x^3}{\Delta^3},\ -\frac{x^2}{\Delta} + \frac{x^3}{\Delta^2}\right] \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} \tag{2}$$

or

$$\underset{\sim}{y} = \underset{\sim}{N}\underset{\sim}{u} \tag{2a}$$

The slope y' may thus be expressed in terms of the nodal displacement as

$$[y'] = \left[-\frac{6x}{\Delta^2} + \frac{6x^2}{\Delta^3},\ 1 - \frac{4x}{\Delta} + \frac{3x^2}{\Delta^2},\ \frac{6x}{\Delta^2} - \frac{6x^2}{\Delta^3},\ -\frac{2x}{\Delta} + \frac{3x^2}{\Delta^2}\right] \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} \tag{3}$$

The curvature y" may likewise be expressed in terms of the nodal displacements as

$$[y''] = \left[-\frac{6}{\Delta^2} + \frac{12x}{\Delta^3},\ -\frac{4}{\Delta} + \frac{6x}{\Delta^2},\ \frac{6}{\Delta^2} - \frac{12x}{\Delta^3},\ -\frac{2}{\Delta} + \frac{6x}{\Delta^2}\right] \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} \tag{4}$$

$$\underset{\sim}{\varepsilon} = \underset{\sim}{B}\underset{\sim}{u}$$

The relation between moment and curvature is (ignoring shear deformation; see Art. A2.)

$$[M] = [EI]\ [y''] \tag{5}$$

$$\underset{\sim}{\sigma} = \underset{\sim}{E}\underset{\sim}{\varepsilon}$$

The general expression for stiffness matrix is (derived by the principle of virtual work)(2)

$$k = \int_0^{\Delta} \underset{\sim}{B}^T\ \underset{\sim}{E}\underset{\sim}{B}\ dx = EI \begin{bmatrix} \frac{12}{\Delta^3} & \frac{6}{\Delta^2} & -\frac{12}{\Delta^3} & \frac{6}{\Delta^2} \\ \frac{6}{\Delta^2} & \frac{4}{\Delta} & -\frac{6}{\Delta^2} & \frac{2}{\Delta} \\ -\frac{12}{\Delta^3} & -\frac{6}{\Delta^2} & \frac{12}{\Delta^3} & -\frac{6}{\Delta^2} \\ \frac{6}{\Delta^2} & \frac{2}{\Delta} & -\frac{6}{\Delta^2} & \frac{4}{\Delta} \end{bmatrix} \tag{6}$$

This stiffness matrix for the beam element relates nodal loads to nodal displacements. The nodal loads are shown in Fig. A2.

$$\begin{bmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \end{bmatrix} = \begin{bmatrix} k_{11} & k_{12} & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} \tag{7}$$

$$\underset{\sim}{f} = \underset{\sim}{k}\underset{\sim}{u}$$

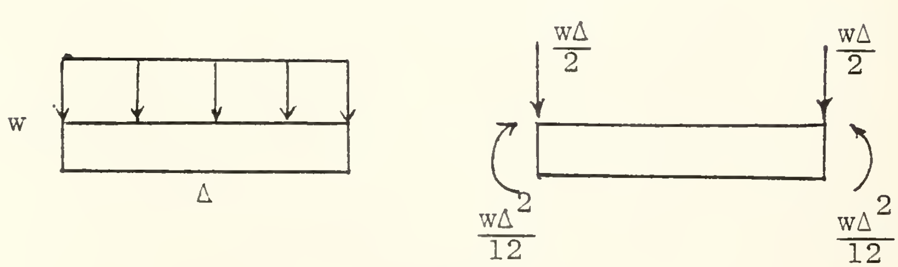
The equivalent load matrix for a distributed load is obtained from

$$\underset{\sim}{f}_{eq} = \int_0^{\Delta} \underset{\sim}{N}^T \underset{\sim}{p} \, dx \tag{8}$$

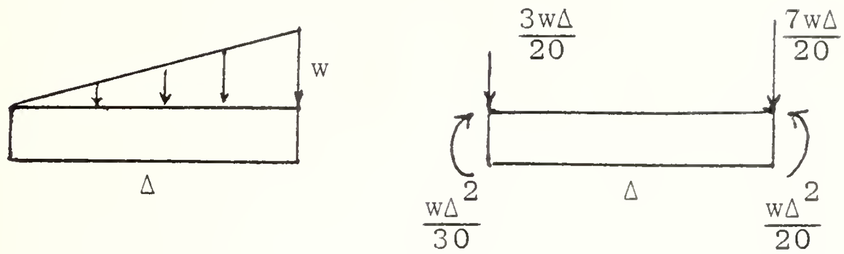
where $\underset{\sim}{p} = [p]$ is the distributed load, e.g., if $p = p_o$ (a constant), then

$$\underset{\sim}{f}_{eq} = \int_0^{\Delta} \underset{\sim}{N}^T [p_o] \, dx = p_o \begin{bmatrix} 1/2 \\ \Delta/12 \\ 1/2 \\ -\Delta/12 \end{bmatrix} \tag{9}$$

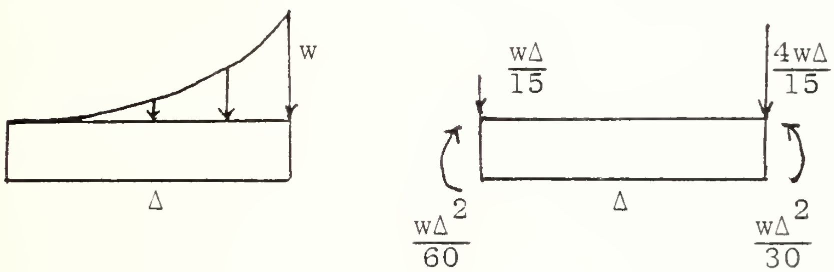
(1) Uniform distribution load equivalent



(2) Linear distributed load equivalent

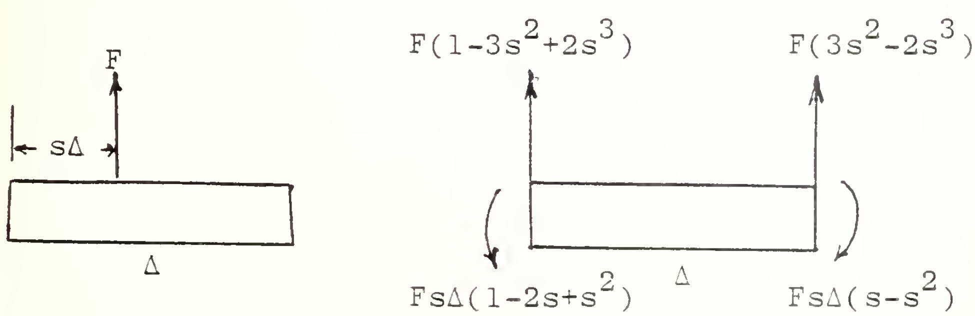


(3) Quadratic distributed load equivalent

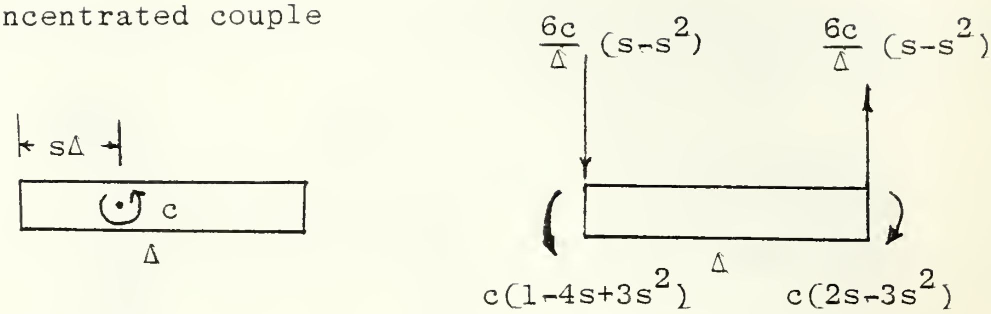


If a concentrated load is to be applied at some point other than a node, it must be replaced by an equivalent system of nodal loads also. Examples of equivalent systems are given below:

(4) Concentrated force



(5) Concentrated couple



A2. Beam Element Including Shear Deformation

Consider a beam element of length ΔZ (Fig. A3). Under deformation cross section plane TOB of the beam, which is initially perpendicular to the beam axis, rotates into the position T*O*B*. Because of shear effects, the plane T*O*B* is no longer perpendicular to the displaced axis of the beam, Fig. A3, but rotates through an angle $\phi \neq y'$. For equilibrium of an infinitesimal length dZ of the beam element, Fig. A4, we have

$$\frac{dM}{dZ} = -F, \quad \frac{dF}{dZ} = -p \tag{10}$$

where M denotes the bending moment, F denotes the shear force and p denotes the lateral pressure (+ as shown) that act on the element dZ. The stress-strain relations for the beam element are

$$M = EI\phi', \quad F = GA(y'-\phi) \tag{11}$$

where E = the modulus of elasticity, G = the shear modulus, I = the moment of inertia of the area, A = the beam cross sectional area, and prime denotes derivative with respect to axial length Z.

Substitution of Eq. (11) into Eq. (10) yields the equilibrium equations in terms of ϕ and y as

$$\begin{aligned} EI\phi'' &= GA(\phi - y') \\ GA(y'' - \phi') &= -p \end{aligned} \tag{12}$$

In the simplest finite element shear model, the displacements are assumed in the form (2)

$$\begin{aligned} \phi &= A_1 + A_2 Z + A_3 Z^2 \\ y &= A_4 + A_5 Z + A_6 Z^2 + A_7 Z^3 \end{aligned} \tag{13}$$

Assuming that the pressure p (or any other load between node points of the beam element) is transformed into nodal forces (see Eq. 8, Art. A1), we find that substitution of Eqs. (13) into Eqs. (12) yields the result

$$\begin{aligned} \phi &= A_1 + A_2 Z + A_3 Z^2 \\ y &= A_4 + (A_1 - \frac{\Delta^2 r}{6} A_3) Z + \frac{A_2}{2} Z^2 + \frac{A_3}{3} Z^3 \end{aligned} \tag{14}$$

where $$r = \frac{12EI}{AG\Delta^2} \tag{15}$$

is the shear deformation ratio. With Eqs. (14) the method of finite elements (2) yields the beam element stiffness matrix

$$k = \frac{EI}{\Delta^3(1+r)} \begin{vmatrix} 12 & 6\Delta & -12 & 6\Delta \\ 6\Delta & \Delta^2(4+r) & -6\Delta & \Delta^2(2-r) \\ -12 & -6\Delta & 12 & -6\Delta \\ 6\Delta & \Delta^2(2-r) & -6\Delta & \Delta^2(4+r) \end{vmatrix} \tag{16}$$

We observe that if $r = 0$, Eq. (16) reduces to Eq. (6).

The mass matrix $\underset{\sim}{m}$ for the beam element is given by

$$\underset{\sim}{m} = \int_0^\Delta \underset{\sim}{N}^T \, \underset{\sim}{\rho} \, \underset{\sim}{N} \, dZ \tag{17}$$

where the $\underset{\sim}{N}$ matrix relates the end (nodal) displacements x_1, x_2, x_3, x_4 of the beam element to the displacements y, ϕ (see Eq. 2) and the $\underset{\sim}{\rho}$ matrix is the mass density matrix (2)

$$\underset{\sim}{\rho} = \begin{bmatrix} \mu & 0 \\ & \\ 0 & \mu R^2 \end{bmatrix} \tag{18}$$

where μ = mass per unit length and R = the radius of gyration of the beam cross section.

Explicitly, the 4x4 matrix $\underset{\sim}{m}$ may be written as

$$\underset{\sim}{m} = \frac{\mu\Delta}{840(1+r)^2} \begin{bmatrix} m_{11} & m_{12} & m_{13} & m_{14} \\ m_{12} & m_{22} & m_{23} & m_{24} \\ m_{13} & m_{23} & m_{33} & m_{34} \\ m_{14} & m_{24} & m_{34} & m_{44} \end{bmatrix} \tag{19}$$

where

$$
\begin{aligned}
m_{11} &= 312 + 588r + 280r^2 + 1008 \frac{R^2}{\Delta^2} \\
m_{12} &= \Delta[44 + 77r + 35r^2 + \frac{R^2}{\Delta^2} (84 - 420r)] \\
m_{13} &= 108 + 252r + 140r^2 - 1008 \frac{R^2}{\Delta^2} \\
m_{14} &= \Delta[-26 - 63r - 35r^2 + \frac{R^2}{\Delta^2} (84 - 420r)] \\
m_{22} &= \Delta^2[8 + 14r + 7r^2 + \frac{R^2}{\Delta^2} (112 + 140r + 280r^2)] \\
m_{23} &= \Delta[26 + 63r + 35r^2 + \frac{R^2}{\Delta^2} (-84 + 420r)] \\
m_{24} &= \Delta^2[-6 - 14r - 7r^2 + \frac{R^2}{\Delta^2} (-28 - 140r + 140r^2)] \\
m_{33} &= 312 + 588r + 280r^2 + 1008 \frac{R^2}{\Delta^2} \\
m_{34} &= \Delta[-44 - 77r - 35r^2 + \frac{R^2}{\Delta^2} (-84 + 420r)] \\
m_{44} &= \Delta^2[8 + 14r + 7r^2 + \frac{R^2}{\Delta^2} (112 + 140r + 280r^2)]
\end{aligned}
\tag{20}
$$

We note that if shear deformation is discarded r = 0, and if rotary inertia effects are discarded R = 0 in Eq. (20), and $\underset{\sim}{m}$ is greatly simplified.

A3. Initial Stress Stiffness Matrix Due to Axial Tension Load P

The following stiffness matrix due to the axial tension load P is employed (see 2, p. 262).

$$\underset{\sim}{k}_p = \frac{P}{30\Delta}\begin{bmatrix} 36 & 3\Delta & -36 & 3\Delta \\ 3\Delta & 4\Delta^2 & -3\Delta & -\Delta^2 \\ -36 & -3\Delta & 36 & -3\Delta \\ 3\Delta & -\Delta^2 & -3\Delta & 4\Delta^2 \end{bmatrix} \tag{21}$$

APPENDIX B: PIECEWISE CUBIC APPROXIMATION

Rectangular coordinates (x, y) lie in a vertical plane, with the x-axis coinciding with the axis of the undeformed barrel (Fig. 1). The axes (x, y) are attached to the barrel, so that they recoil with it, but they do not rotate or move in the y-direction. The origin, x = y = 0, lies at the breech. The barrel is divided into intervals by points $x_1 = 0$, x_2, x_3, ..., x_{n+1}, where x_{n+1} is the coordinate of the muzzle. These points need not be equally spaced. The j-th interval is (x_j, x_{j+1}). In the j-th interval, we use the cubic approximation,

$$y = a_0^j + a_1^j x + a_2^j x^2 + a_3^j x^3;\quad x_j \leqq x \leqq x_{j+1} \tag{1}$$

$$a_\alpha^j = A_\alpha^j\, y_j + B_\alpha^j\, y_{j+1} + C_\alpha^j\, \theta_j + D_\alpha^j\, \theta_{j+1} \tag{2}$$

where $\theta_j = y'_j$, in which the prime denotes the derivative with respect to x. We set $\Delta_j = x_{j+1} - x_j$. By algebra

$$\begin{aligned}
A_0^j &= \Delta_j^{-3} x_{j+1}^2 (x_{j+1} - 3x_j), & B_0^j &= \Delta_j^{-3} x_j^2 (3x_{j+1} - x_j) \\
C_0^j &= -\Delta_j^{-2} x_j\, x_{j+1}^2, & D_0^j &= -\Delta_j^{-2} x_j^2\, x_{j+1} \\
A_1^j &= 6\Delta_j^{-3} x_j\, x_{j+1}, & B_1^j &= -6\Delta_j^{-3} x_j\, x_{j+1} \\
C_1^j &= \Delta_j^{-2} x_{j+1} (2x_j + x_{j+1}), & D_1^j &= \Delta_j^{-2} x_j (2x_{j+1} + x_j) \\
A_2^j &= -3\Delta_j^{-3} (x_j + x_{j+1}), & B_2^j &= 3\Delta_j^{-3} (x_j + x_{j+1})
\end{aligned} \tag{3}$$

$$C_2^j = -\Delta_j^{-2}(x_j + 2x_{j+1}), \qquad D_2^j = -\Delta_j^{-2}(x_{j+1} + 2x_j)$$

$$A_3^j = 2\Delta_j^{-3}, \quad B_3^j = -2\Delta_j^{-3}, \qquad C_3^j = D_3^j = \Delta_j^{-2}$$

A_α^j, B_α^j, C_α^j, D_α^j are constants for the j-th interval. It is noted that we get B_α^j from A_α^j and D_α^j from C_α^j by interchanging x_{j+1} and x_j. This property simplifies programming. Equations (1), (2), and (3) define a piecewise cubic polynomial that is continuous with its first derivative.

APPENDIX C:

ACCELERATION EFFECTS OF PROJECTILE

The following kinematic analysis of the projectile acceleration entails no approximations. Let coordinate axes (x, y) be fixed (Fig. C1). The equation of the tube at time t is $x = x(s,t)$, $y = y(s,t)$, where s is arc length along the tube. Hence,

$$
\begin{aligned}
x_s^2 + y_s^2 &= 1 \\
x_s = \cos\theta, \quad y_s &= \sin\theta
\end{aligned}
\tag{1}
$$

The location of the particle (projectile) mass m at time t is

$$x = x[\xi(t),t], \quad y = y[\xi(t),t] \tag{2}$$

where $\xi(t)$ is the arc length s to the projectile. Hence, by Eqs. (1) and (2),

$$\sin\theta = y_s[\xi(t),t], \quad \cos\theta = x_s[\xi(t),t] \tag{3}$$

where θ is the angle of the tangent at point m (Fig. C1).

The velocity components of the particle m are

$$
\begin{aligned}
v_x &= \frac{dx}{dt} = \dot{\xi}x_s(\xi,t)+x_t(\xi,t) = \dot{\xi}\cos\theta + x_t(\xi,t) \\
v_y &= \frac{dy}{dt} = \dot{\xi}y_s(\xi,t)+y_t(\xi,t) = \dot{\xi}\sin\theta + y_t(\xi,t)
\end{aligned}
\tag{4}
$$

The terms $x_t(\xi,t)$ and $y_t(\xi,t)$ are the velocity components of the contiguous point of the tube. The normal velocity component of m is

$$v_n = -v_x \sin\theta + v_y \cos\theta \tag{5}$$

Hence, by Eqs. (4) and (5),

$$v_n = -x_t(\xi,t)\sin\theta + y_t(\xi,t)\cos\theta \tag{6}$$

This result shows that v_n for mass m is the same as the normal velocity of the contiguous point of the tube. The tangential velocity of mass m is

$$\begin{aligned} v_s &= v_x \cos\theta + v_y \sin\theta \\ &= \dot{\xi} + x_t(\xi,t)\cos\theta + y_t(\xi,t)\sin\theta \end{aligned} \tag{7}$$

or $$v_s = \dot{\xi} + x_t x_s + y_t y_s \tag{7a}$$

where $x_t x_s + y_t y_s$ denotes the tangential velocity of the contiguous point of the tube.

Now the acceleration components of the mass m are

$$a_x = \frac{d^2x}{dt^2}, \quad a_y = \frac{d^2y}{dt^2}$$

where $x = f[\xi(t),t]$. To derive expressions for a_x, a_y, we must more generally consider the form $x = f[p(t), q(t)]$. Then,

$$\frac{dx}{dt} = \frac{\partial f}{\partial p}\dot{p} + \frac{\partial f}{\partial q}\dot{q}$$

$$\frac{d^2x}{dt^2} = \frac{\partial f}{\partial p}\ddot{p} + \frac{\partial f}{\partial q}\ddot{q} + \dot{p}^2\frac{\partial^2 f}{\partial p^2} + 2\dot{p}\dot{q}\frac{\partial^2 f}{\partial p \partial q} + \dot{q}^2\frac{\partial^2 f}{\partial q^2}$$

and since $p(t) = \xi(t)$, and $q(t) = t$, we have

$$a_x = \ddot{\xi}x_s(\xi,t)+\dot{\xi}^2x_{ss}(\xi,t)+2\dot{\xi}x_{st}(\xi,t)+x_{tt}(\xi,t) \tag{8}$$

Similarly

$$a_y = \ddot{\xi}y_s(\xi,t)+\dot{\xi}^2y_{ss}(\xi,t)+2\dot{\xi}y_{st}(\xi,t)+y_{tt}(\xi,t) \tag{9}$$

The normal component of acceleration of m is

$$a_n = -a_x\sin\theta + a_y\cos\theta = -a_xy_s(\xi,t) + a_yx_s(\xi,t)$$

For brevity, we may write $x_s(\xi,t) = x_s$, etc. Hence, by Eqs. (8) and (9),

$$a_n = \dot{\xi}^2(x_sy_{ss} - x_{ss}y_s) + 2\dot{\xi}(x_sy_{st} - y_sx_{st})$$

$$+ (x_sy_{tt} - y_sx_{tt}) \tag{10}$$

Note that the factor $\ddot{\xi}$ cancels out of Eq. (10).

Recalling that the curvature of the tube is

$$\frac{1}{r} = (x_{ss}^2 + y_{ss}^2)^{1/2}$$

since $x_{ss} << y_{ss}$, $1/r \approx y_{ss}$. Also, $x_s \approx 1$. Hence the term $x_sy_{ss} - x_{ss}y_s$ in Eq. (10) is approximately 1/r.

The tangential component of acceleration is

$$a_s = a_x\cos\theta + a_y\sin\theta = a_xx_s + a_yy_s$$

Consequently with Eqs. (8) and (9), we find

$$a_s = \ddot{\xi}(x_s^2 + y_s^2) + \dot{\xi}^2(x_sx_{ss} + y_sy_{ss})$$

$$+2\dot{\xi}(x_sx_{st} + y_sy_{st}) + x_sx_{tt} + y_sy_{tt}$$

Now, by geometry, $x_s^2 + y_s^2 = 1$. Hence, $x_s x_{st} + y_s y_{st} = 0$ and $x_s x_{ss} + y_s y_{ss} = 0$. Therefore,

$$a_s = \ddot{\xi} + x_s x_{tt} + y_s y_{tt} = \ddot{\xi} + x_{tt}\cos\theta + y_{tt}\sin\theta \tag{11}$$

If we employ linearizing approximations $\cos\theta \approx 1$, $\sin\theta = \theta = y_s$. Also, then, $x = s$, $x_s = 1$, $x_{ss} = 0$, $x_t = 0$. Then, by Eqs. (6) and (7),

$$v_n = y_t(\xi,t),\ v_s = \dot{\xi} + y_x(\xi,t)\ y_t(\xi,t) \tag{12}$$

and by Eqs. (9) and (10)

$$a_y = \ddot{\xi} y_x(\xi,t) + \dot{\xi}^2 y_{xx}(\xi,t) + 2\dot{\xi} y_{xt}(\xi,t) + y_{tt}(\xi,t) \tag{13}$$

$$a_x = \ddot{\xi}$$

and

$$a_n = \dot{\xi}^2 y_{xx}(\xi,t) + 2\dot{\xi} y_{xt}(\xi,t) + y_{tt}(\xi,t) \tag{14}$$

In the differential equations of motion of the barrel, the question arises as to whether we should use a_y or a_n for the force exerted on the barrel by the projectile. This question may be resolved by considering the tube. When the tube is bent, it is effectively a curved beam. For a curved beam, it is natural to deal with tractions or normal cross sections of the beam. For a curved beam, we have the familiar equation

$$\frac{dM}{ds} + Q = 0 \tag{15}$$

where M is the bending moment on a cross section and Q is the shearing force on the cross section. If we deal with the large deflections of a straight elastic beam, we have

$$M = \frac{EI}{r} = EI\kappa \tag{16}$$

where $\kappa = 1/r$ is the curvature of the neutral axis. Hence,

$$Q = -EI \frac{d\kappa}{ds}$$

The curvature κ is continuous at the point $s = \xi$, but the derivative $d\kappa/ds$ is discontinuous. The discontinuity is equal to ma_n. If we write y as a function of x

$$\kappa = \frac{y_{xx}}{(1 + y_x^2)^{3/2}} = y_{xx}(1 - \frac{3}{2} y_x^2 + \ldots)$$

Hence, with linear approximations, $\kappa = y_{xx}$. Also, $ds = (1 + y_x^2)^{1/2}\, dx = (1 + \frac{1}{2} y_x^2)\, dx \approx dx$, and therefore,

$$Q = -EI\, y_{xx}$$

Thus, if we treat the problem by means of the differential equations, we must use

$$\Delta Q = ma_n \quad \text{at } x = \xi$$

or

$$-EI\Delta y_{xx} = ma_n \quad \text{at } x = \xi$$

Hence, it follows from this argument that we should use a_n, as given by Eq. (14), to determine the force exerted on the barrel by the projectile. Effectively the terms $\dot{\xi}^2\, y_{xx}$, $2\dot{\xi}\, y_{xt}$, y_{tt}, contribute to the terms $\underset{\sim}{K}\underset{\sim}{x}$, $\underset{\sim}{C}\dot{\underset{\sim}{x}}$, and $\underset{\sim}{M}\ddot{\underset{\sim}{x}}$, respectively.

LIST OF FIGURES

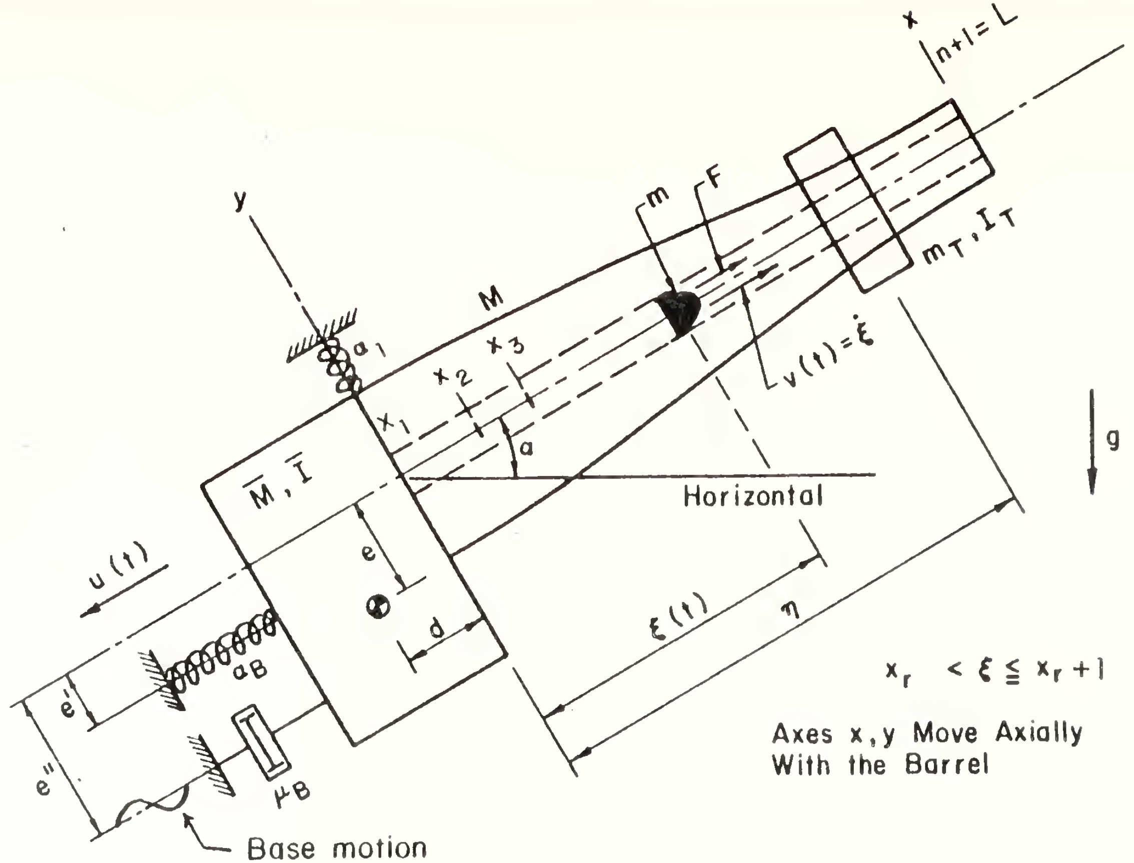
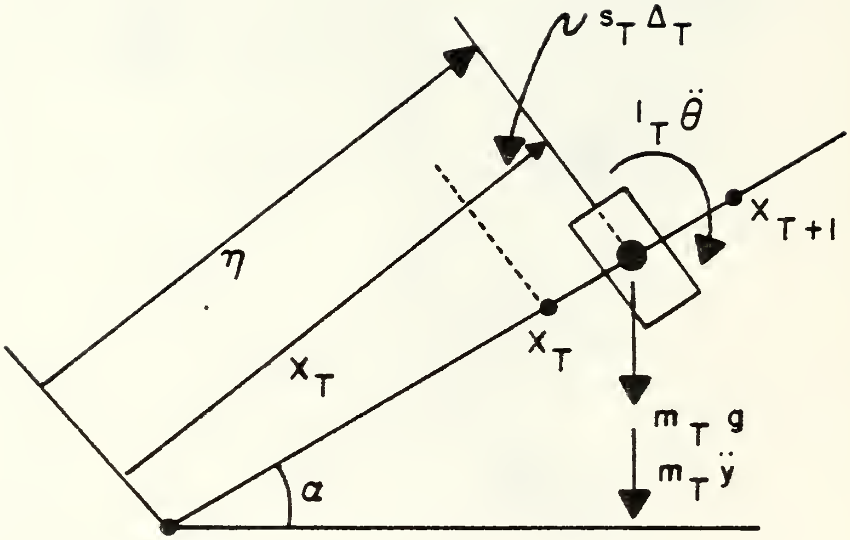
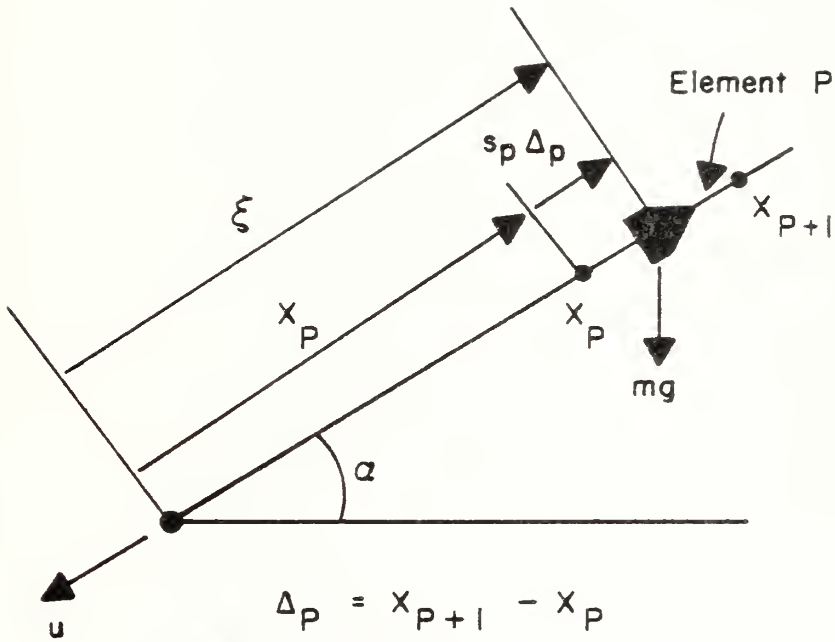


Figure 1. GUN SCHEMATIC



$$s_T \Delta_T = \eta - x_T$$
$$\Delta_T = x_{T+1} - x_T$$

Figure 2. TUNING MASS FORCES



$$\Delta_p = x_{p+1} - x_p$$

$$s_p\,\Delta_p = \xi - x_p$$

Figure 3. PROJECTILE GRAVITY EFFECT

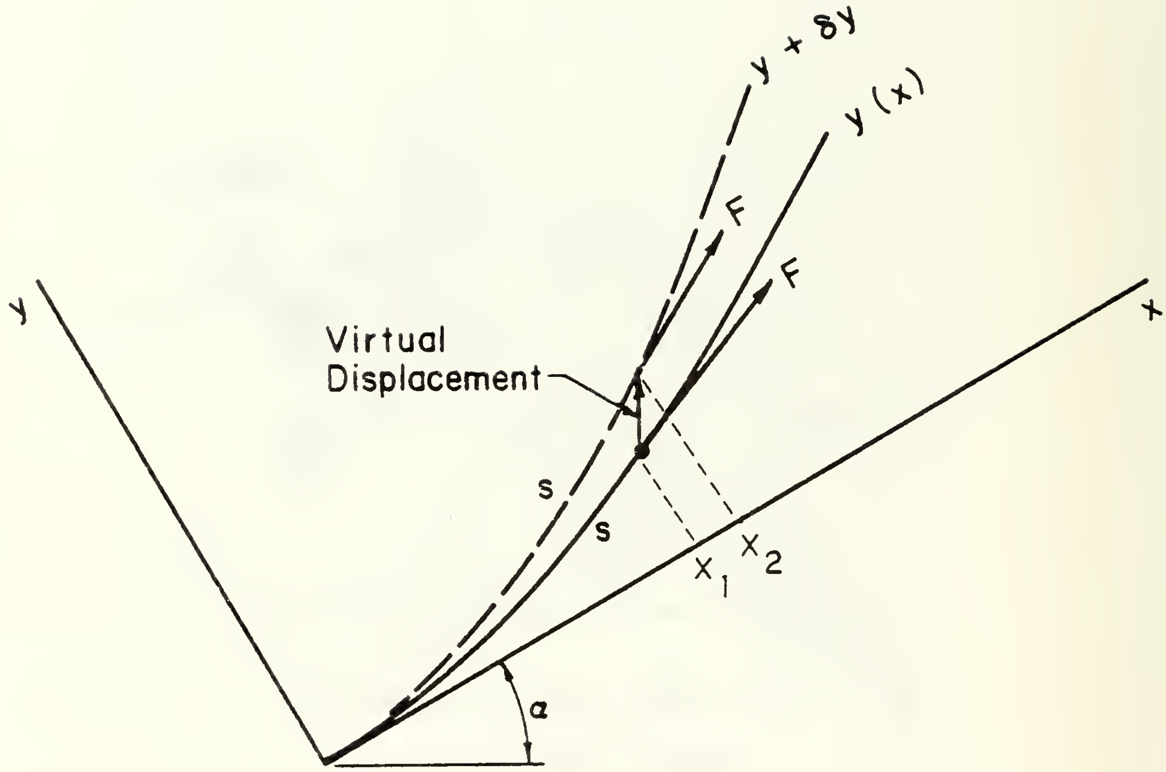


Figure 4. VIRTUAL DISPLACEMENT

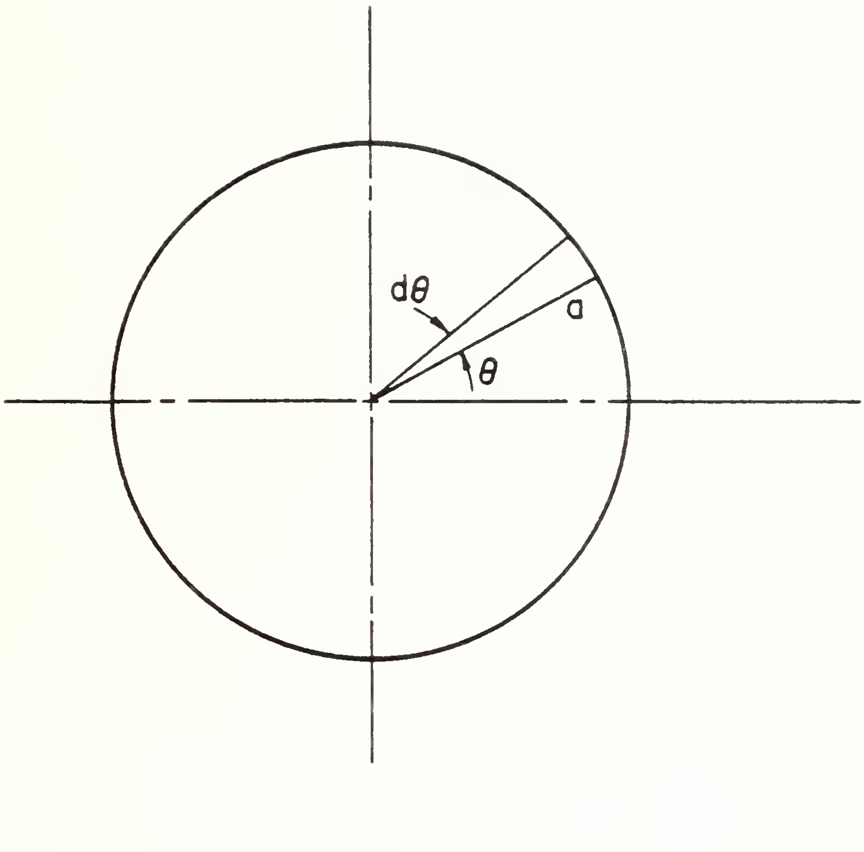


Figure 5. BOURDON EFFECT

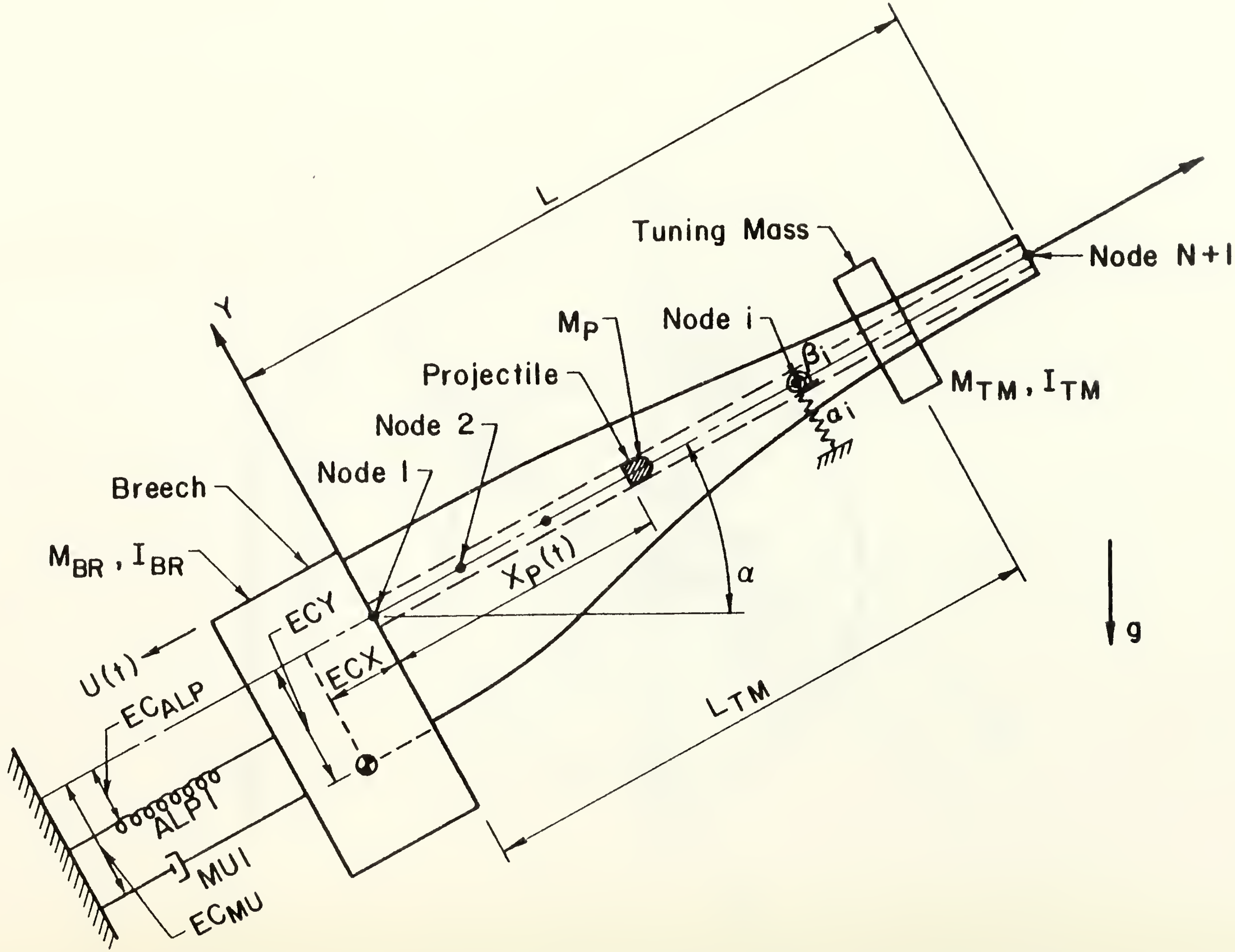


Figure 6. SCHEMATIC OF GUN TUBE

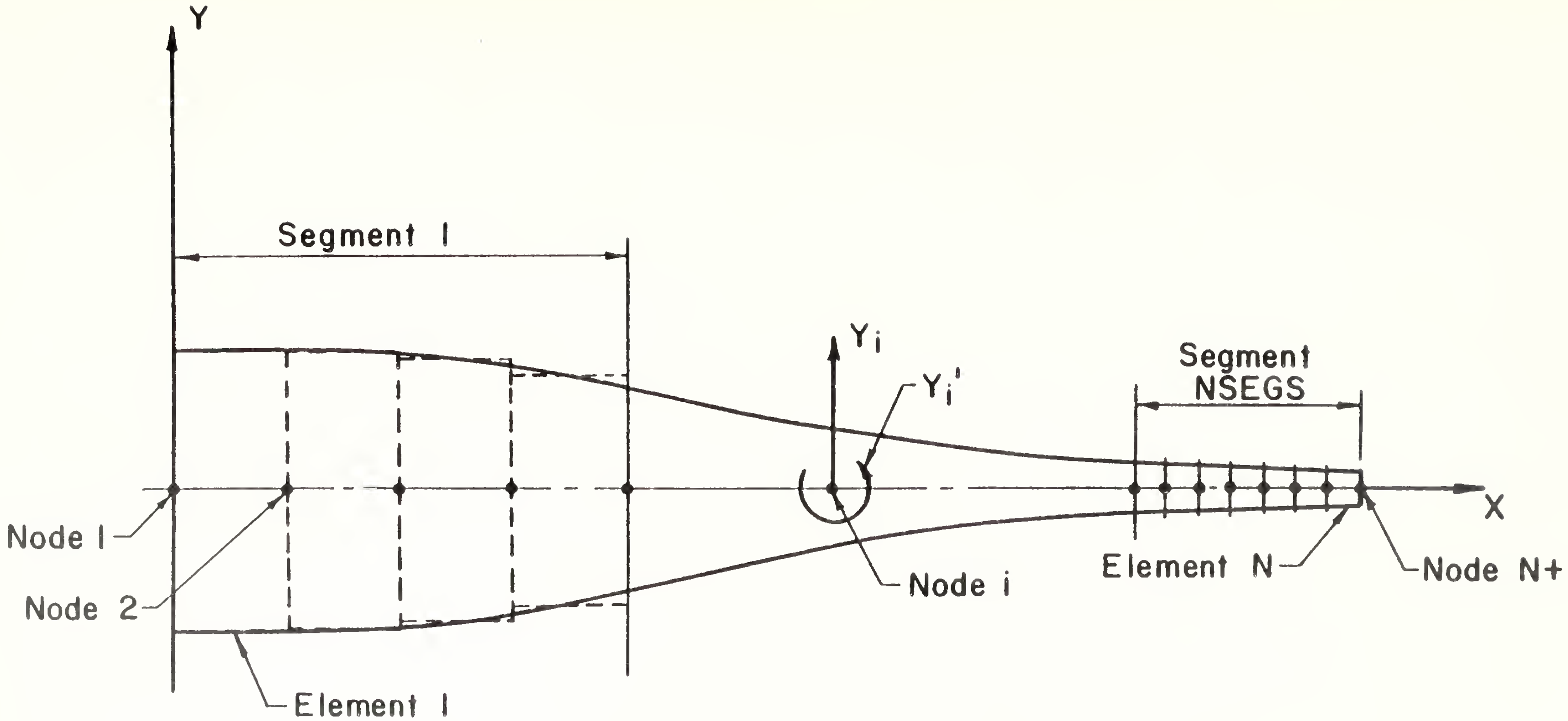


Figure 7. FINITE ELEMENT IDEALIZATION OF GUN TUBE

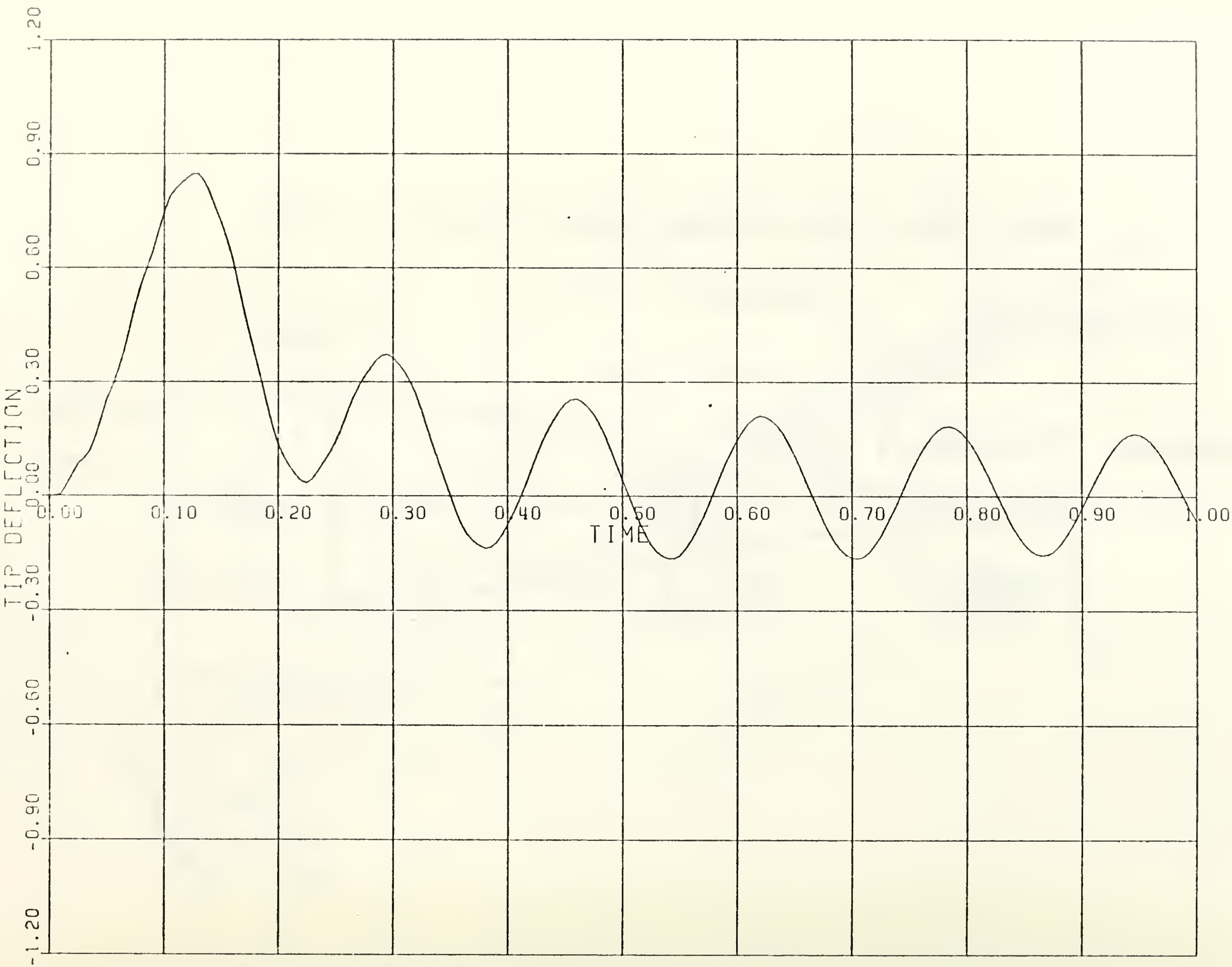


FIGURE 8. RARDEN GUN WITHOUT PADS

K = 5000, C = 1.5

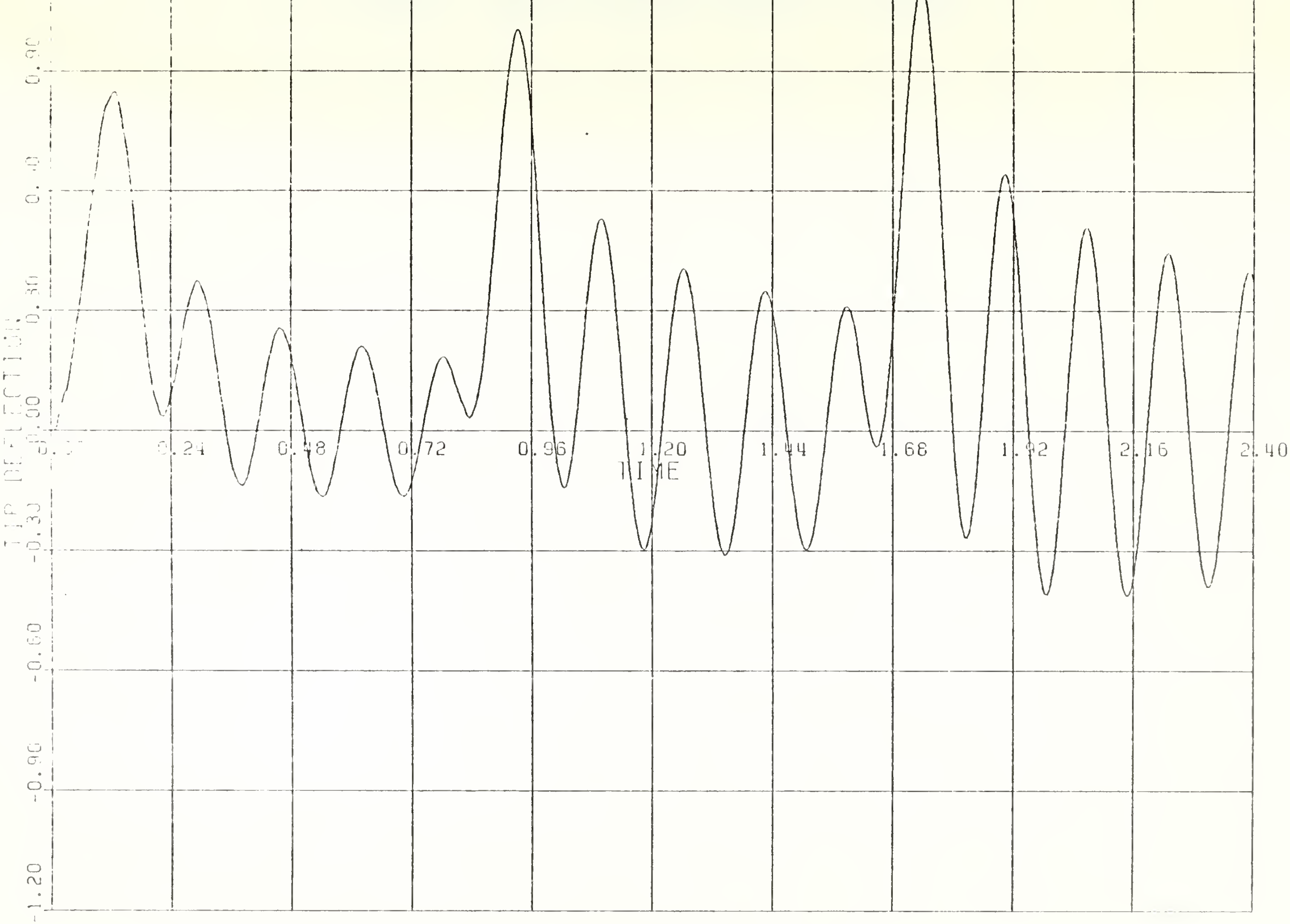


FIGURE 9. RARDEN GUN WITHOUT PADS

K = 5000, C = 1.5

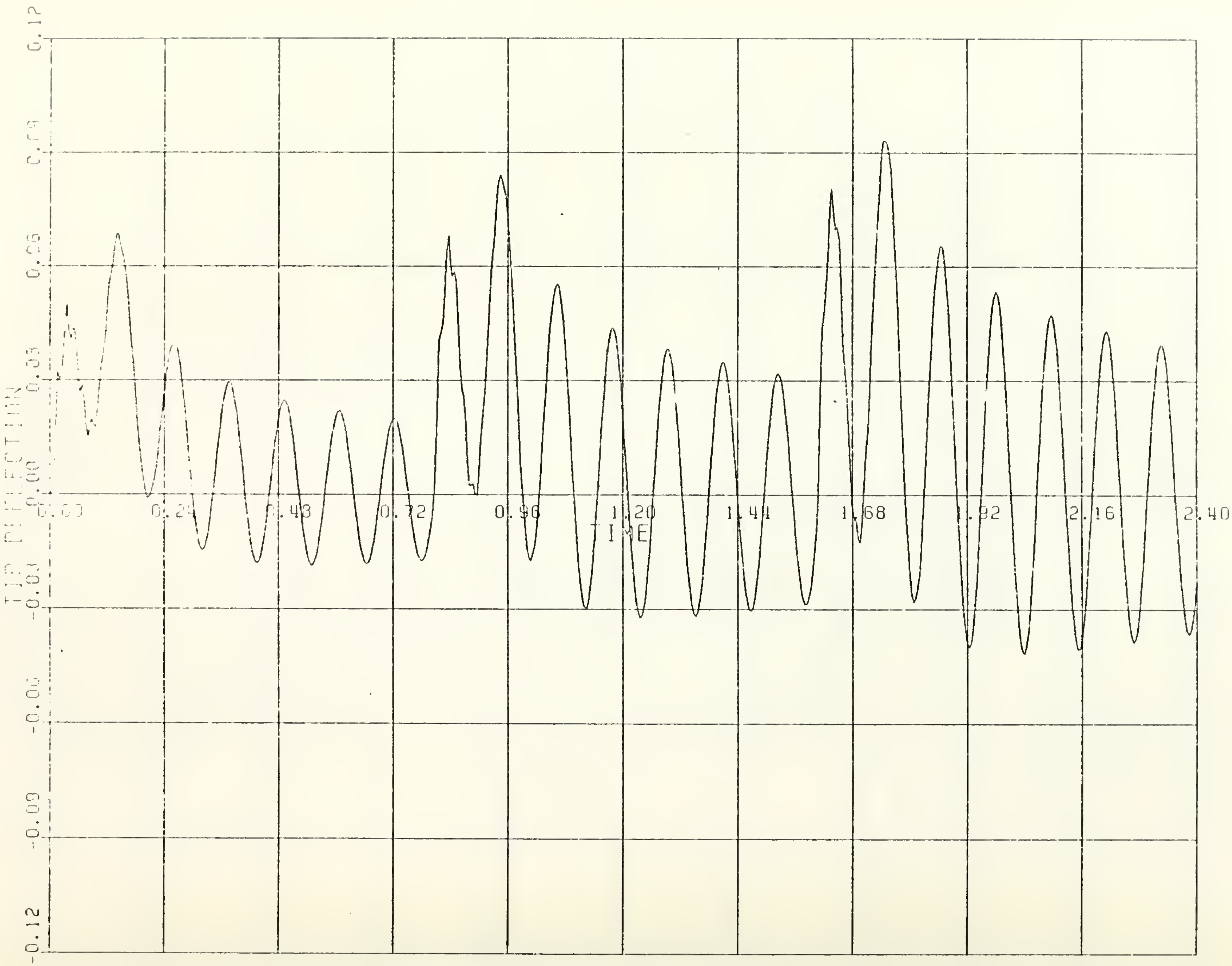


FIGURE 10. RARDEN GUN WITH PADS AT 37.5 INCHES

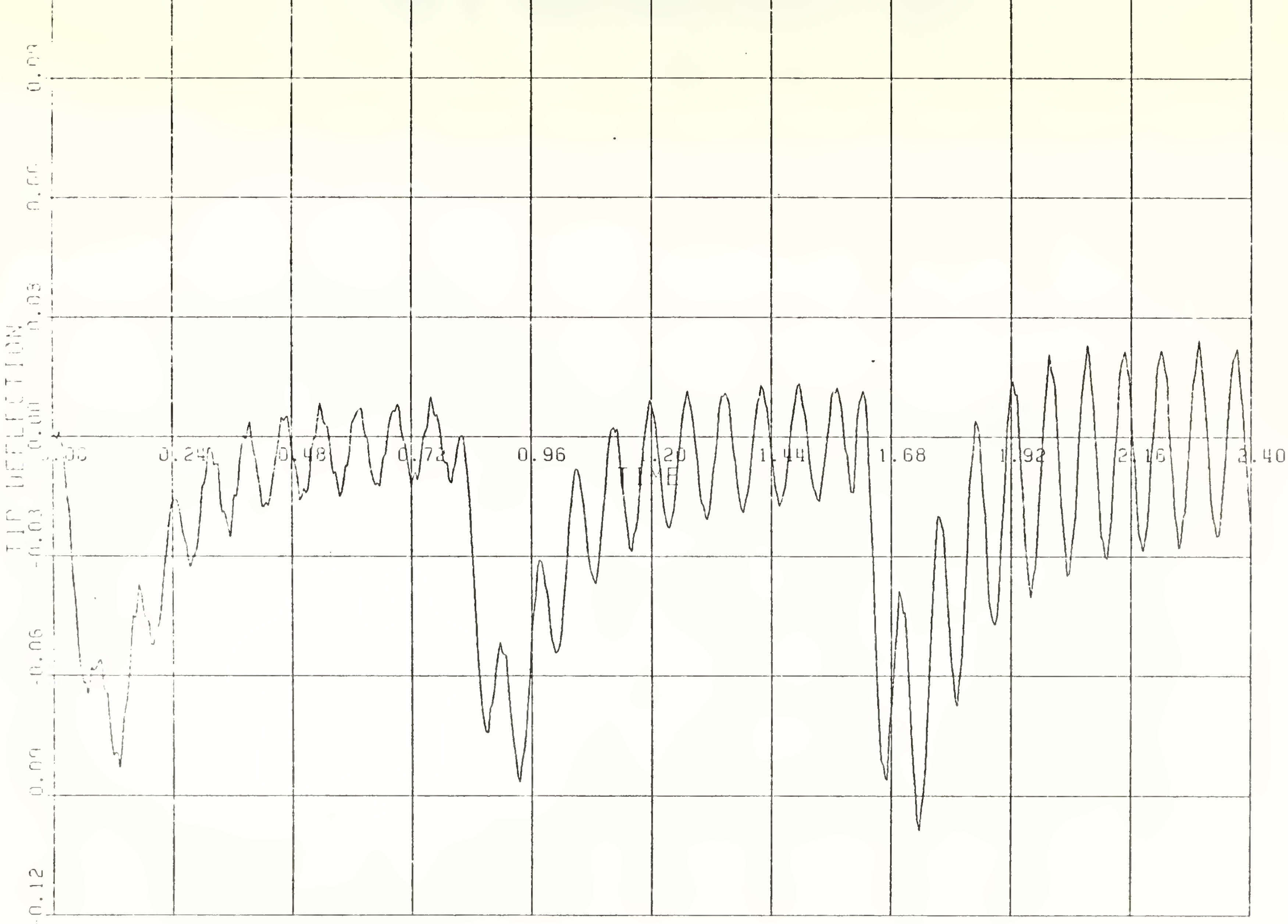


FIGURE 11. RARDEN GUN WITH PADS AT 54 INCHES

K = 5000, C = 0.0

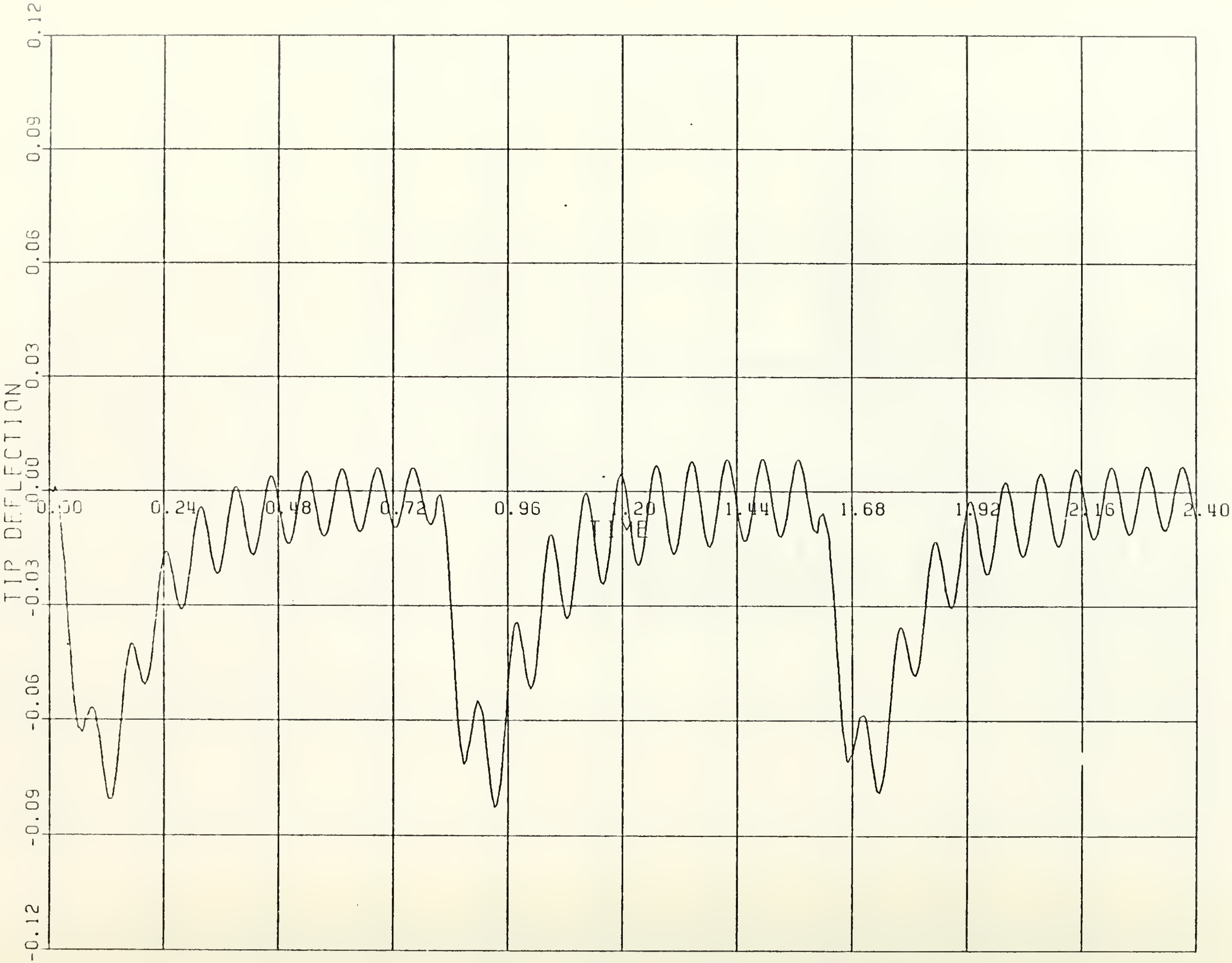


FIGURE 12. RARDEN GUN WITH PADS AT 54 INCHES

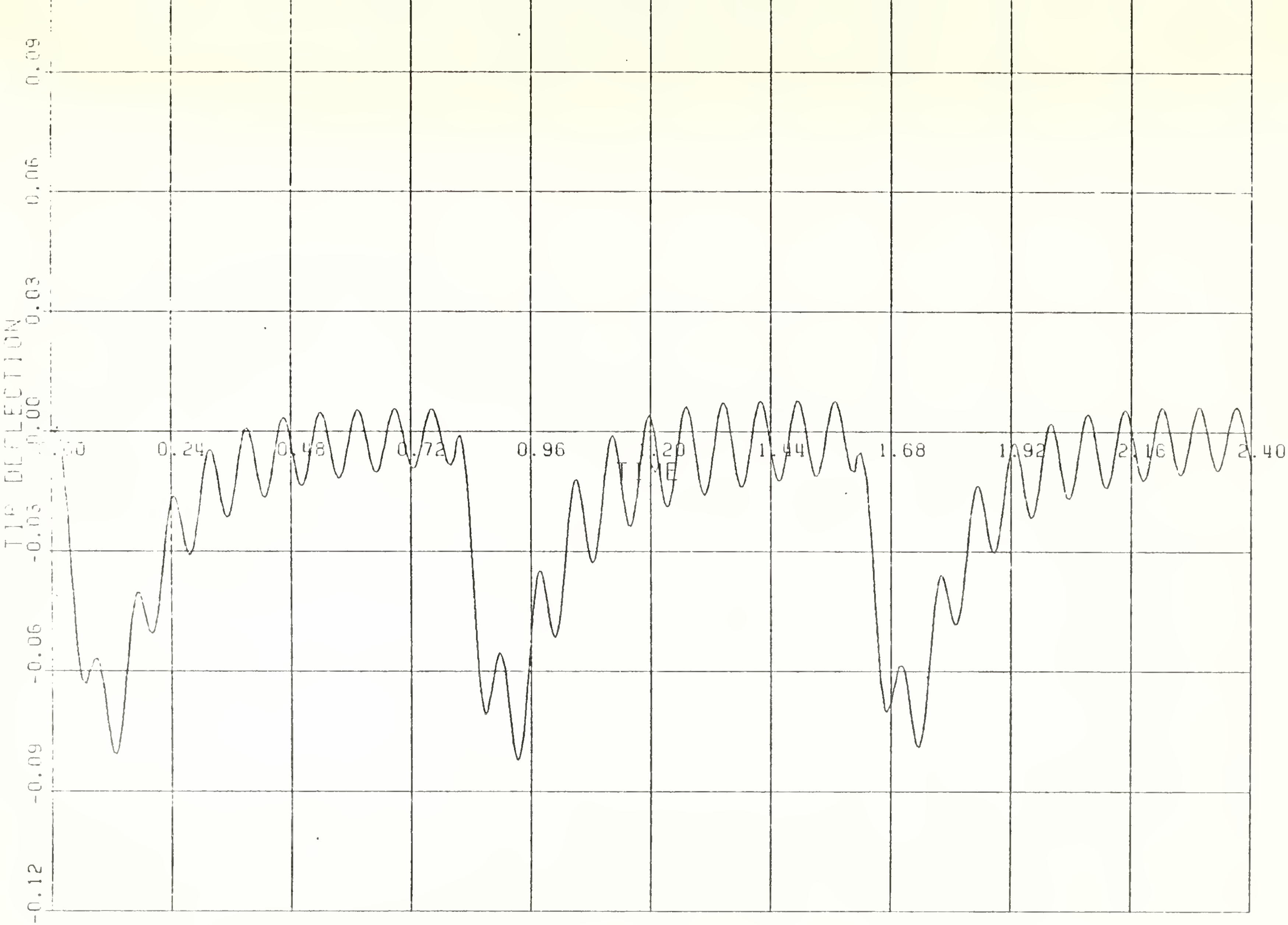


FIGURE 13. RARDEN GUN WITH PADS AT 54 INCHES

K = 5000, C = 2.0

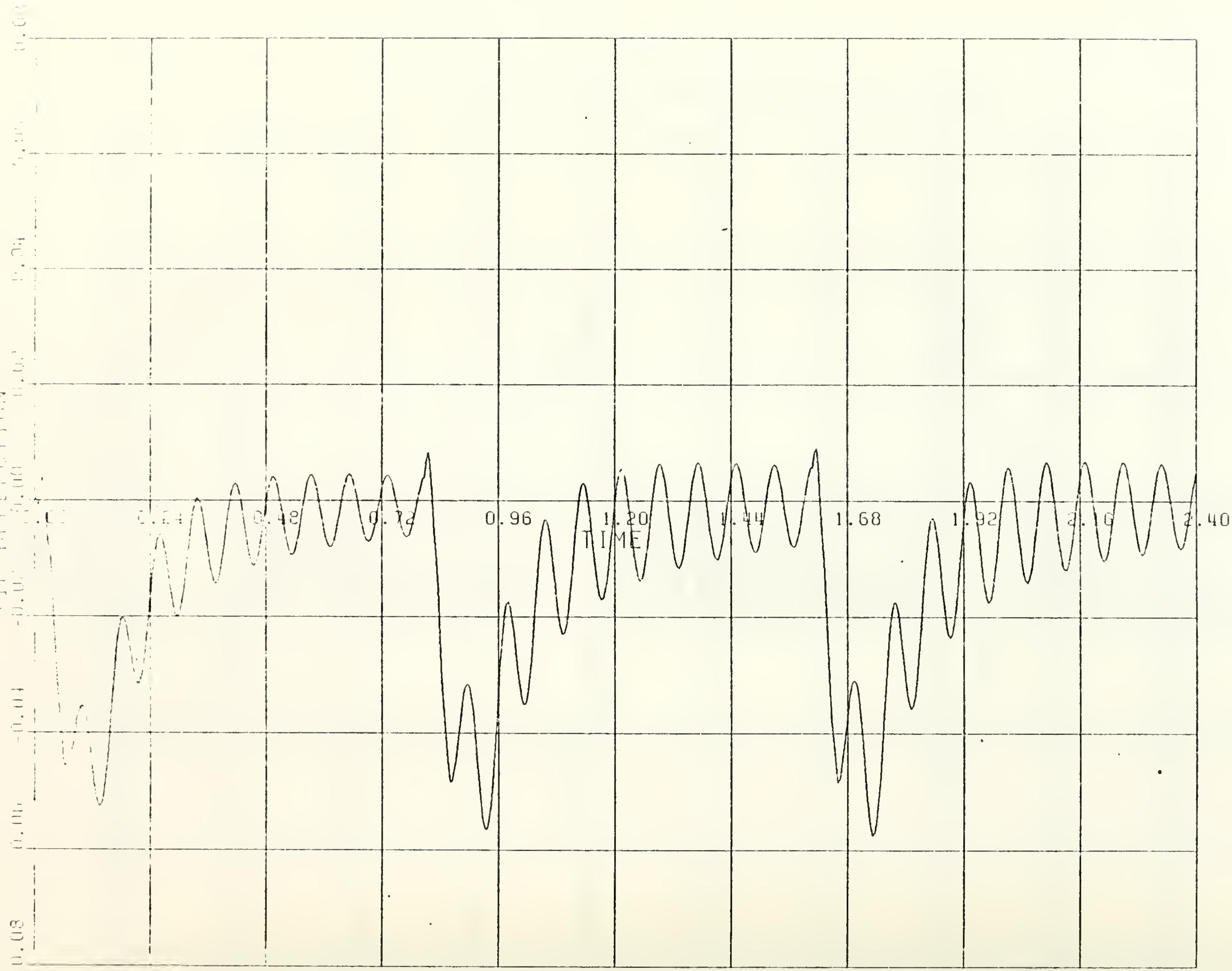


FIGURE 14. RARDEN GUN WITH PADS AT 54 INCHES

K = 3000, C = 2.0

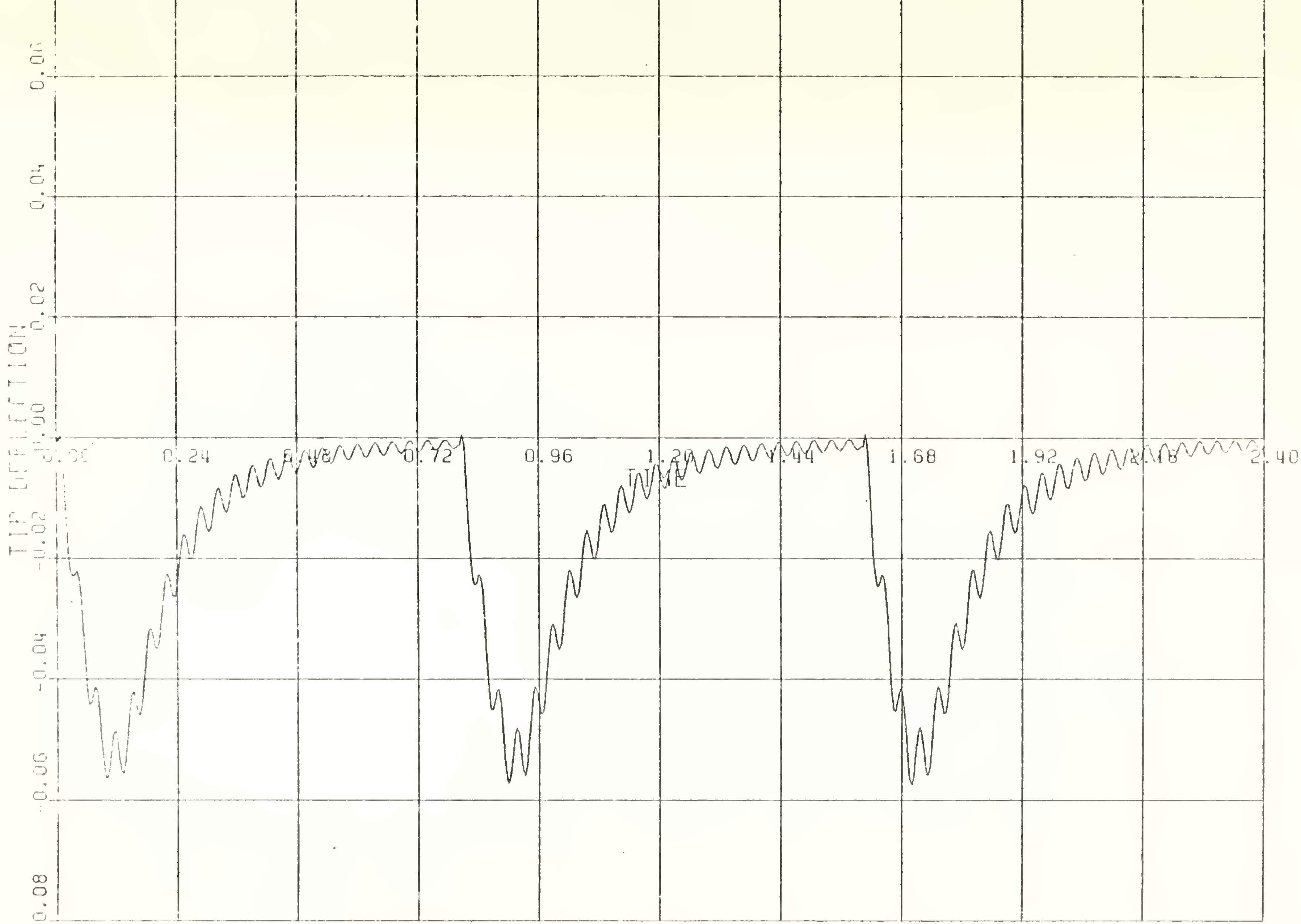


FIGURE 15. RARDEN GUN WITH PADS AT 75 INCHES

K = 5000, C = 2.0

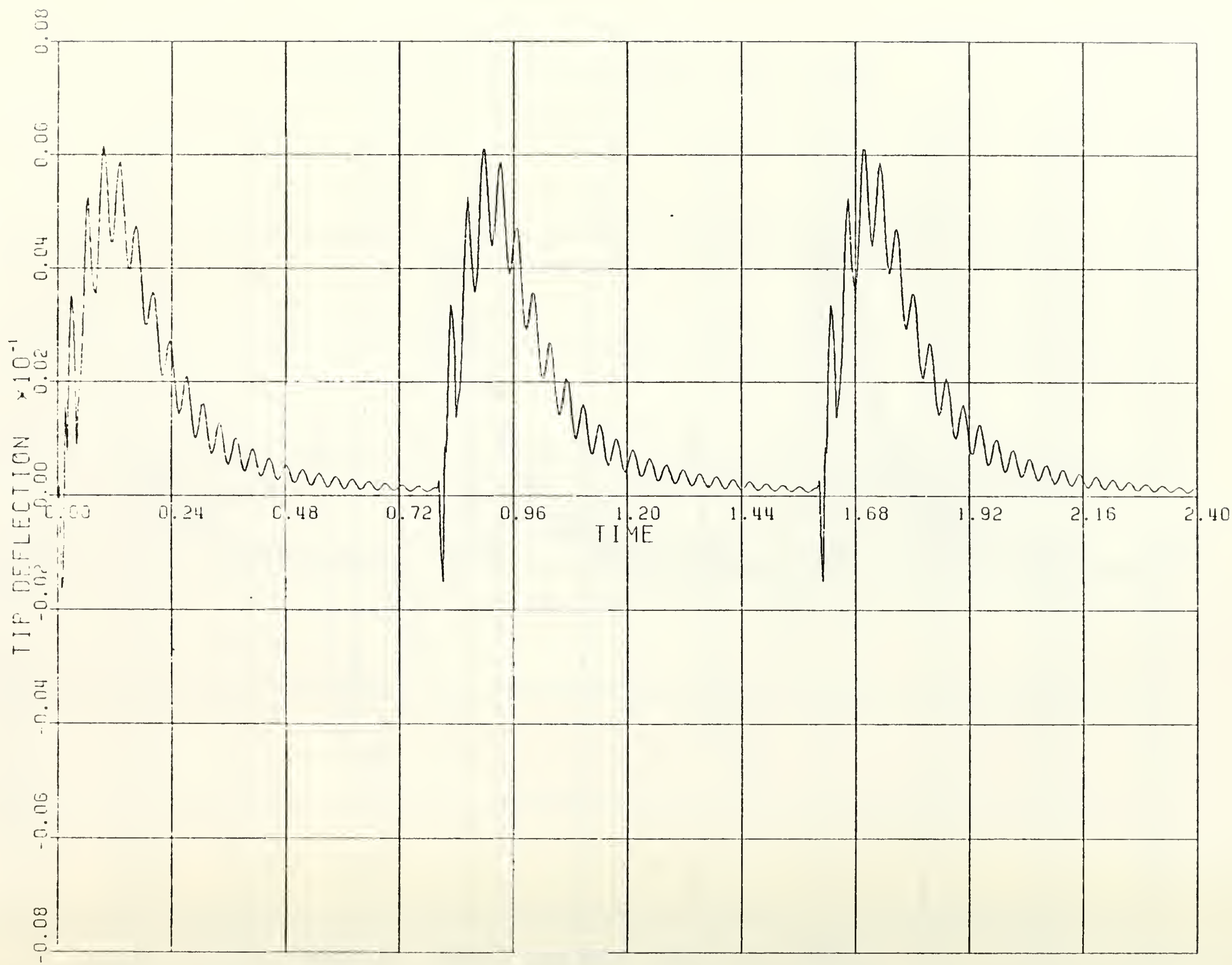


FIGURE 16. RARDEN GUN WITH PADS AT 96 INCHES

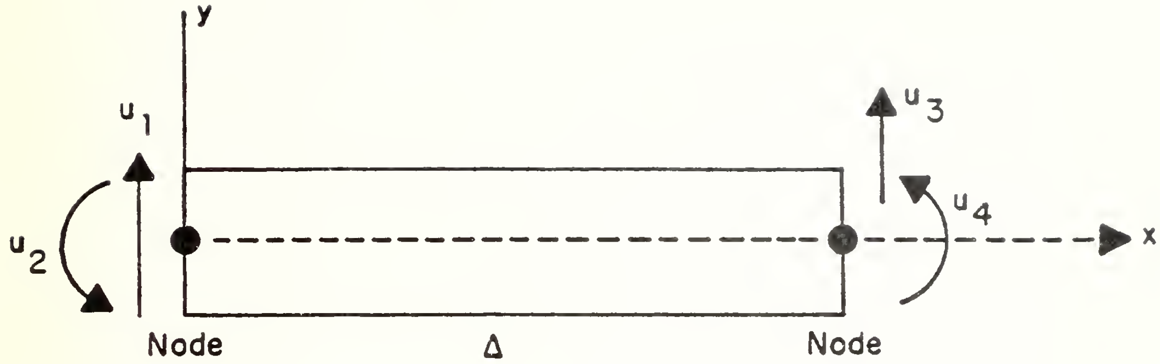


Figure A1. Beam Element

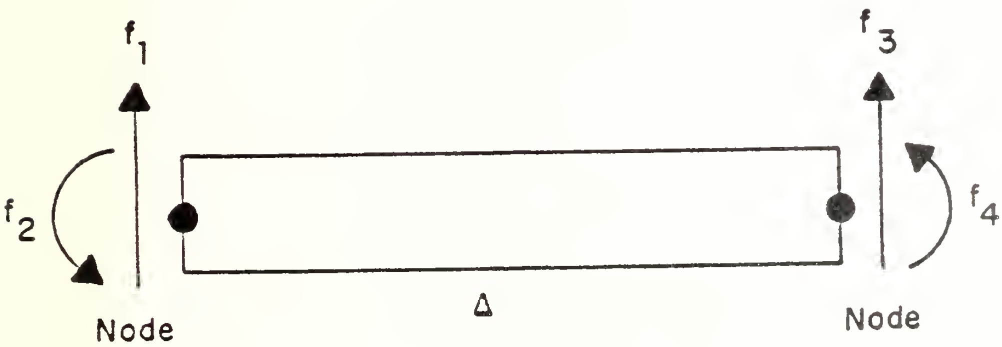


Figure A2. Beam Element Including Shear Deformation

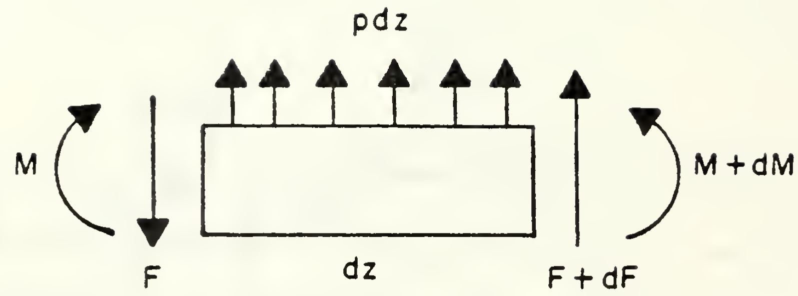


Figure A3. Initial Stress Stiffness Matrix Due to Axial Tension Load P

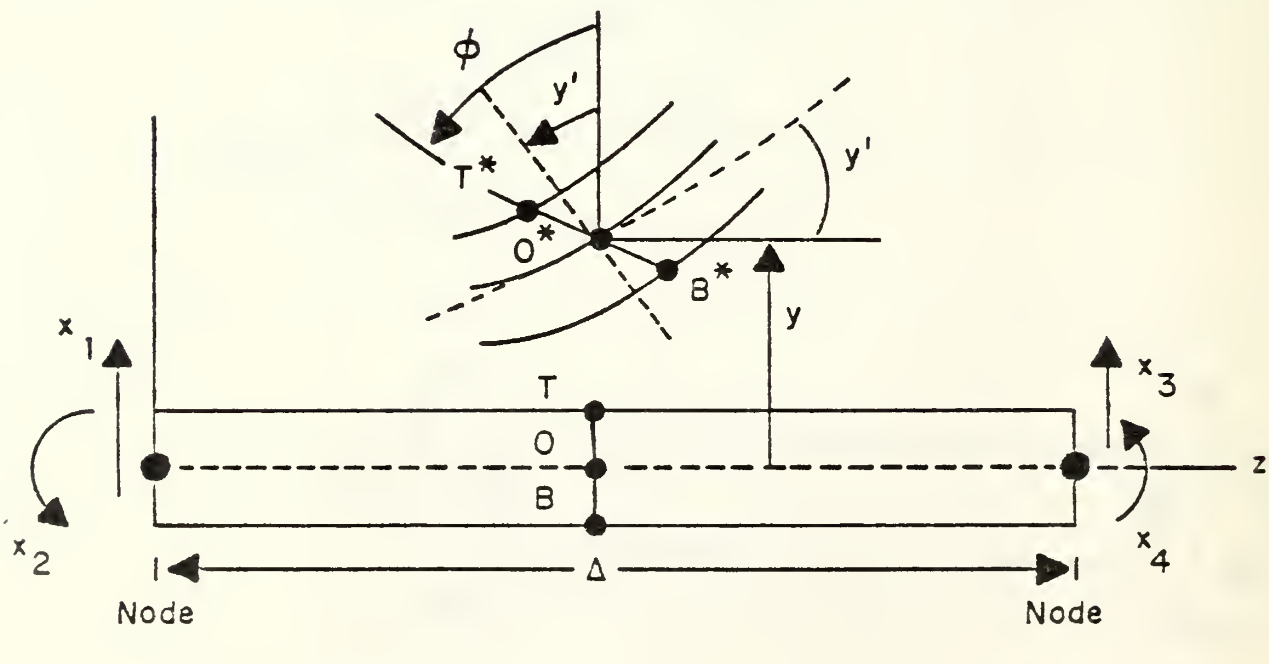


Figure A4. Equilibrium of an Infinitesimal Length dZ of the Beam Element

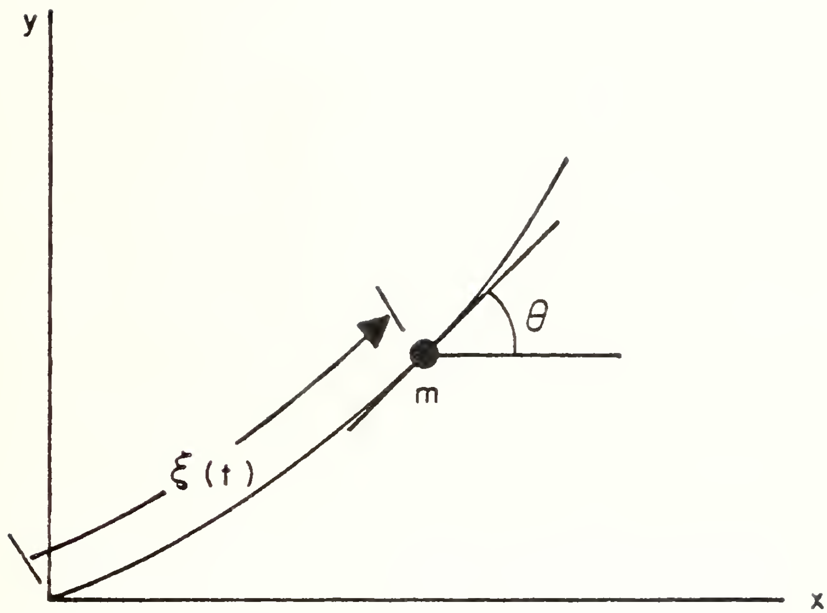


Figure C1. Projectile Coordinates

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