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STUDIES ON DAMAGE AGGREGATION FOR WEAPONS SALVOS II

James D. Esary

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This document records three working studies from an ongoing investigation of models and methods for the prediction of the cumulative effect of weapons salvos. It is the second such document, following NPS Technical Report NPS55-90-16, July 1990. Its first two papers are about cellular targeting scenarios which lead to the proportional damage aggregation mechanism which has figured strongly in the investigation so far. The other paper extends an earlier comparison of an empirical rule for damage aggregation to the results of models combining proportional damage aggregation with various weapons hit distributions.

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Foreword

This document records three working studies from an ongoing investigation of models and methods for the prediction of the cumulative effect of weapons salvos. The papers are reproduced in their entirety. It is the second such document, its predecessor being Naval Postgraduate School Technical Report NPS55-90-06, *Studies on damage aggregation for weapons salvos*, July 1990, which recorded three earlier working papers.

The first paper in this sequence, *Proportional damage aggregation for a cellular target with cells of zero value*, describes an important member of an emerging family of target configuration and weapons impact scenarios leading to the proportional damage aggregation mechanism which figures strongly in the first three studies.

The second paper, *A basic lemma on expected damage aggregation for cellular targets and some of its applications*, describes a more abstract approach to expected damage aggregation for cellular targets. This approach is used to demonstrate proportional damage aggregation for an extended class of targeting scenarios which includes the cases previously considered.

The third paper, *Comparisons of an empirical rule for expected damage aggregation from weapons salvos to models assuming a proportional damage aggregation mechanism and dependent weapons hit distributions*, returns to a methodological issue considered in the first working paper in the earlier sequence. Its conclusion is that the empirical rule gives optimistic results in a more general setting than previously shown.

Each working paper was intended to be reasonably self contained. As a result there is some repetition of fundamental material within the sequence.

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Proportional damage aggregation for a cellular target with cells of zero value

1. Introduction

This working paper follows three previous working papers "*A comparison of an empirical rule for aggregating damage from a weapons salvo to a plausible model for the same purpose*" [1], "*Damage aggregation for a weapons salvo by an empirical rule related to the Poisson approximation to the binomial*" [2], and "*A stochastic model for hit overlap in a weapons salvo directed against an area target that leads to a proportional mechanism for damage aggregation*" [3]. The material in the next three paragraphs of this introduction is summarized from [1] and [3] is discussed in greater detail there.

The scenario that has been considered is that a salvo of n weapons is launched against a target. The number of weapons that hit the target is a random variable N with possible values $0, 1, \ldots, n$. Possible damage to the target is measured as a percentage (or proportion) of the whole ranging from 0% to 100%. The damage to a pristine target resulting from a single hit is a deterministic proportion d of the whole. The aggregate proportion of damage to the target from the salvo is a random variable D, the randomness in D resulting from the randomness in the number of hits N.

The premise of the **proportional effects** mechanism for damage aggregation has been that *if the proportion of a pristine target that is damaged by a single hit is d, then each additional hit damages the same proportion d of that part of the target not previously damaged.* Thus if $D(k)$ is the aggregate proportion of damage to a pristine target from exactly k hits, then

$$D(k) = 1 - (1-d)^k, \quad k = 0, \ldots, n$$

as is shown in Section 3 of [1].

The objective has been to predict the expected proportion of damage $E(D)$ to the target resulting from the salvo. Since

$$E(D) = \sum_{k=0}^{n} D(k)\, P[N=k]$$

where $P[N=k]$ is the probability of exactly k hits from the salvo, this prediction involves modeling the probability distribution of the number of hits N. So far it has been assumed that each weapon in the salvo hits independently with the same probability p, so that the number of hits N has a binomial probability distribution. The impact of this assumption on $E(D)$ is discussed early in Section 4 of [1].

Effects mechanisms can be derived from assumptions about the geometry of the target, the coverage of the weapon, and the probabilities of hitting locations within the target area. Then the proportion of damage from k hits becomes a random variable $\Delta(k)$, and with $D(k) = E\{\Delta(k)\}$

$$E(D) = \sum_{k=0}^{n} E\{\Delta(k)\}P[N{=}k] = \sum_{k=0}^{n} D(k)P[N{=}k]$$

Viewing the model at this deeper level of detail can provide a better picture of its applicability.

One scenario for weapons overlap on a cellular target which leads to a proportional effects mechanism, in the sense that

$$D(k) = E\{\Delta(k)\} = 1 - (1-d)^k, \quad k = 0, \ldots, n$$

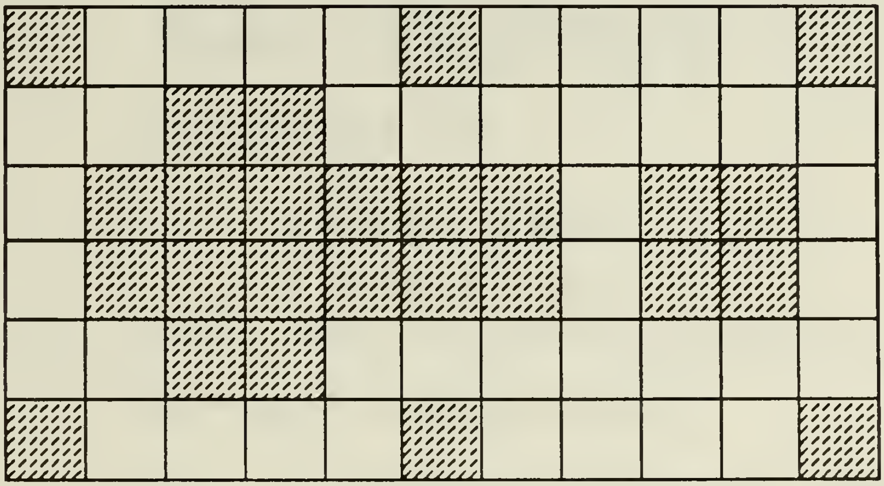
for a pertinent value of d, was presented in [3], and the probability distributions for $\Delta(k)$ that it implied were examined. *Probability distributions for $\Delta(k)$ can be combined with probability distributions for N to obtain deeper rooted probability distributions for D, the proportion of damage to the target.*

The purpose of this working paper is to explore a generalization of the cellular target model of [3] which again leads to a proportional effects mechanism. In this generalized model hits that impact some target cells do not increase the proportion of damage to the target.

2. A cellular target with zero valued cells

Suppose that a cellular target is divided into m disjoint cells. Each cell represents a portion of the target which would be affected by a single weapon which impacts on that cell. The cells have different **target values**. There are v cells with value $1/v$ and $m-v$ cells with value 0. Thus the total target value of all m cells is 1. The first weapon to impact a cell raises the aggregate proportion of damage to the target by the value of the cell.

Subsequent impacts on the cell do not raise the aggregate proportion of damage to the target. If $v = m$, this target configuration reduces to the target configuration considered in [3].

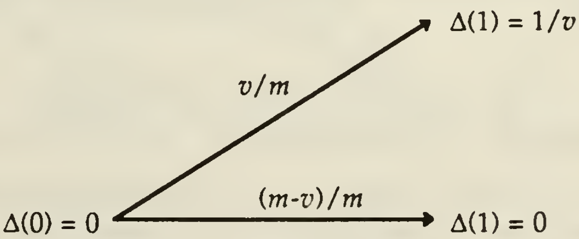


A cellular target with $m = 66$ cells. *The* $v = 26$ *shaded cells have target value* $1/26$ *and the* $m\text{-}v = 40$ *unshaded cells have target value* 0.

Now suppose, as in [3], that each hit from the salvo impacts a randomly chosen cell within the target independently of the cells impacted by the other hits. Then if k hits impact j different cells with value $1/v$, the proportion of the target which is damaged is

$$\Delta(k) = \frac{j}{v}$$

If there are no hits ($k = 0$), then $\Delta(0) = 0$ and $D(0) = E\{\Delta(0)\} = 0$. If there is only one hit ($k = 1$), then the possibilities are illustrated by the transition diagram below.



The arrows indicate transitions to the possible proportions of target damage after one hit, with the probabilities that those transitions occur shown as labels on the arrows. Then

$$D(1) = E\{\Delta(1)\} = 0 \cdot \frac{m-v}{m} + \frac{1}{v} \cdot \frac{v}{m} = \frac{1}{m}$$

Thus

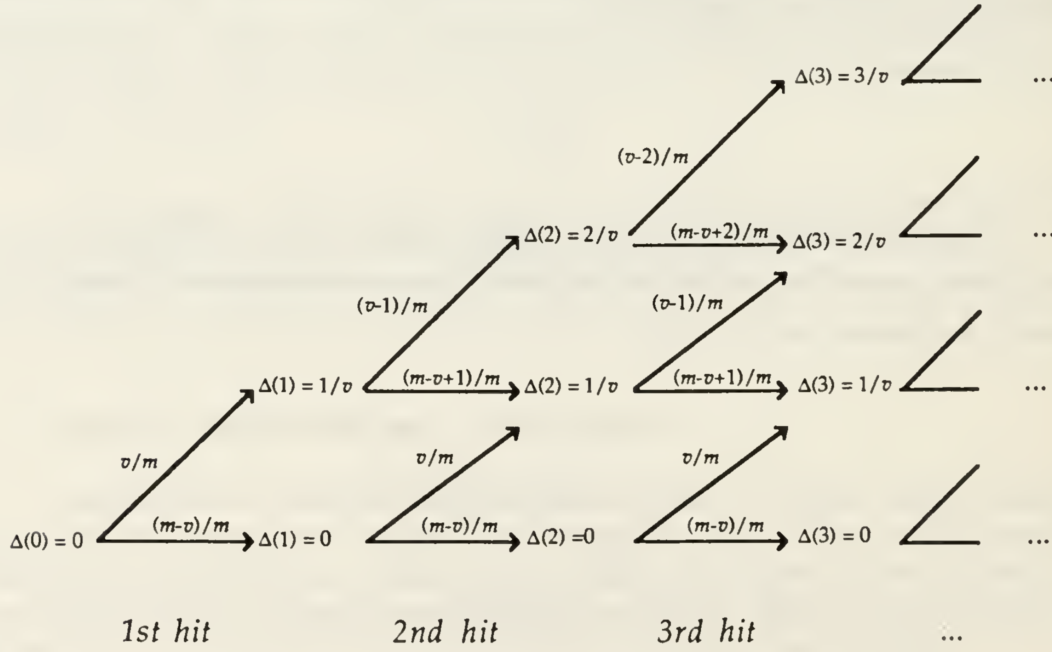
$$D(0) = 1 - (1-d)^0 = 0$$

$$D(1) = 1 - (1-d)^1 = d$$

for $d = 1/m$.

3. *Transition diagrams and equations*

The transition diagram for the aggregation of damage from one hit on the target can be extended as shown below. If $m = v$, the diagram reduces to the comparable diagram in Section 5 of [3]. The probability of an upward transition from any total damage state $\Delta(k) = v/v = 1$ is zero, so that all upward transitions beyond total damage become impossible.



Multiplying the transition probabilities along paths leading to a damage state and adding the products, it follows that

$$P[\Delta(2)=0] = \frac{m-v}{m}\cdot\frac{m-v}{m}$$
$$P[\Delta(2)=\tfrac{1}{v}] = \frac{v}{m}\cdot\frac{m-v+1}{m}+\frac{m-v}{m}\cdot\frac{v}{m}$$
$$P[\Delta(2)=\tfrac{2}{v}] = \frac{v}{m}\cdot\frac{v-1}{m}$$

so that

$$\begin{aligned} D(2) &= E\{\Delta(2)\} = 0\,P[\Delta(2)=0]+\tfrac{1}{v}P[\Delta(2)=\tfrac{1}{v}]+\tfrac{2}{v}P[\Delta(2)=\tfrac{2}{v}] \\ &= \frac{1}{v}\left(\frac{v}{m}\cdot\frac{m-v+1}{m}+\frac{m-v}{m}\cdot\frac{v}{m}\right)+\frac{2}{v}\left(\frac{v}{m}\cdot\frac{v-1}{m}\right) \\ &= \frac{2m-1}{m^2} = 1-\left(1-\frac{1}{m}\right)^2 \end{aligned}$$

Further

$$P[\Delta(3)=\tfrac{0}{v}] = \left(\frac{m-v}{m}\right)^3$$
$$P[\Delta(3)=\tfrac{1}{v}] = \left(\frac{m-v}{m}\right)^2\left(\frac{v}{m}\right)+\left(\frac{m-v}{m}\right)\left(\frac{v}{m}\right)\left(\frac{m-v+1}{m}\right)+\left(\frac{v}{m}\right)\left(\frac{m-v+1}{m}\right)^2$$
$$P[\Delta(3)=\tfrac{2}{v}] = \left(\frac{m-v}{m}\right)\left(\frac{v}{m}\right)\left(\frac{v-1}{m}\right)+\left(\frac{v}{m}\right)\left(\frac{m-v+1}{m}\right)\left(\frac{v-1}{m}\right)+\left(\frac{v}{m}\right)\left(\frac{v-1}{m}\right)\left(\frac{m-v+2}{m}\right)$$
$$P[\Delta(3)=\tfrac{3}{v}] = \left(\frac{v}{m}\right)\left(\frac{v-1}{m}\right)\left(\frac{v-2}{m}\right)$$

Then

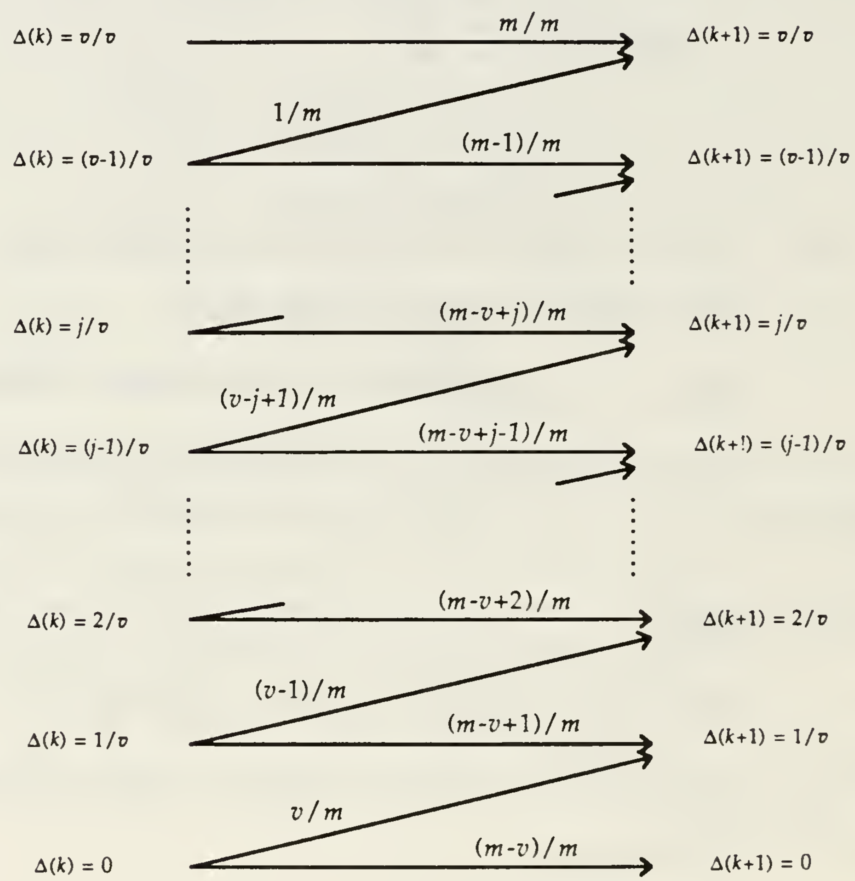
$$D(3) = E\{\Delta(3)\} = \frac{0}{v}P[\Delta(3)=\tfrac{0}{v}]+\frac{1}{v}P[\Delta(3)=\tfrac{1}{v}]+\frac{2}{v}P[\Delta(3)=\tfrac{2}{v}]+\frac{3}{v}P[\Delta(3)=\tfrac{3}{v}]$$

After some algebra, v drops out again and

$$D(3) = \frac{3m^2-3m+1}{m^3} = 1-\left(1-\frac{1}{m}\right)^3$$

The preceding computations of $D(1)$, $D(2)$, and $D(3)$ *indicate that the damage aggregation mechanism for the target is proportional with* $d = 1/m$. It remains to confirm the proportionality of $D(k)$ for an arbitrary k.

The general transition diagram, going from k hits to $k+1$ hits, appears below



It follows that

$$P[\Delta(k+1)=0] = \frac{m-v}{m} P[\Delta(k)=0]$$

and

$$P[\Delta(k+1)=\tfrac{j}{v}] = \frac{m-v+j}{m} P[\Delta(k)=\tfrac{j}{v}] + \frac{v-j+1}{m} P[\Delta(k)=\tfrac{j-1}{v}] \quad j = 1, \ldots, v$$

Using the preceding k to $k+1$ transition equations

$$D(k+1) = E\{\Delta(k+1)\} = \sum_{j=0}^{v} \tfrac{j}{v} P[\Delta(k+1) = \tfrac{j}{v}] = \tfrac{0}{v} P[\Delta(k+1) = \tfrac{0}{v}] + \sum_{j=1}^{v} \tfrac{j}{v} P[\Delta(k+1) = \tfrac{j}{v}]$$

$$= \frac{0}{v} \cdot \frac{m-v}{m} P[\Delta(k) = \tfrac{0}{v}] + \sum_{j=1}^{v} \frac{j}{v} \left\{ \frac{m-v+j}{m} P[\Delta(k) = \tfrac{j}{v}] + \frac{v-j+1}{m} P[\Delta(k) = \tfrac{j-1}{v}] \right\}$$

$$= \sum_{j=0}^{v} \frac{j}{v} \cdot \frac{m-v+j}{m} P[\Delta(k) = \tfrac{j}{v}] + \sum_{j=1}^{v} \frac{j}{v} \cdot \frac{v-j+1}{m} P[\Delta(k) = \tfrac{j-1}{v}]$$

$$= \sum_{j=0}^{v} \frac{j}{v} \cdot \frac{m-v+j}{m} P[\Delta(k) = \tfrac{j}{v}] + \sum_{j=0}^{v-1} \frac{j+1}{v} \cdot \frac{v-j}{m} P[\Delta(k) = \tfrac{j}{v}]$$

$$= \sum_{j=0}^{v} \frac{j}{v} \cdot \frac{m-v+j}{m} P[\Delta(k) = \tfrac{j}{v}] + \sum_{j=0}^{v} \frac{j+1}{v} \cdot \frac{v-j}{m} P[\Delta(k) = \tfrac{j}{v}] - \frac{v+1}{v} \cdot \underset{\uparrow}{\frac{v-v}{m}} P[\Delta(k) = \tfrac{v}{v}]$$

$$= \sum_{j=0}^{v} \frac{j}{v} \cdot \frac{m-v+j}{m} P[\Delta(k) = \tfrac{j}{v}] + \sum_{j=0}^{v} \frac{j+1}{v} \cdot \frac{v-j}{m} P[\Delta(k) = \tfrac{j}{v}]$$

since the term indicated by an arrow is zero. Continuing

$$D(k+1) = \sum_{j=0}^{v} \frac{j}{v} \cdot \frac{m-v+j}{m} P[\Delta(k) = \tfrac{j}{v}] + \sum_{j=0}^{v} \frac{j+1}{v} \cdot \frac{v-j}{m} P[\Delta(k) = \tfrac{j}{v}]$$

$$= \sum_{j=0}^{v} \left\{ \frac{j}{v} \cdot \frac{m-v+j}{m} + \frac{j+1}{v} \cdot \frac{v-j}{m} \right\} P[\Delta(k) = \tfrac{j}{v}]$$

$$= \sum_{j=0}^{v} \left\{ \frac{m-1}{m} \cdot \frac{j}{v} + \frac{1}{m} \right\} P[\Delta(k) = \tfrac{j}{v}]$$

$$= \frac{m-1}{m} \sum_{j=0}^{v} \frac{j}{v} P[\Delta(k) = \tfrac{j}{v}] + \frac{1}{m} \sum_{j=0}^{v} P[\Delta(k) = \tfrac{j}{v}]$$

$$= \frac{m-1}{m} E\{\Delta(k)\} + \frac{1}{m} \cdot 1 = \frac{m-1}{m} D(k) + \frac{1}{m}$$

The recursive relation

$$D(k+1) = \frac{m-1}{m} D(k) + \frac{1}{m}$$

which it turns out holds for $k = 0, 1, \ldots$, permits the verification of proportionality, since if

$$D(k) = 1 - \left(1 - \frac{1}{m}\right)^k$$

then

$$\begin{aligned} D(k+1) &= \frac{m-1}{m}\left\{1 - \left(1 - \frac{1}{m}\right)^k\right\} + \frac{1}{m} \\ &= \left(1 - \frac{1}{m}\right)\left\{1 - \left(1 - \frac{1}{m}\right)^k\right\} + \frac{1}{m} \\ &= \left(1 - \frac{1}{m}\right) - \left(1 - \frac{1}{m}\right)^{k+1} + \frac{1}{m} \\ &= 1 - \left(1 - \frac{1}{m}\right)^{k+1} \end{aligned}$$

and we have already verified proportionality for $k = 1, 2,$ and 3. *This completes the demonstration that the damage aggregation mechanism for the target is proportional with* $d = 1/m$.

6. Damage distributions

The k hits to $k+1$ hits transition equations

$$P[\Delta(k+1) = 0] = \frac{m-v}{m} P[\Delta(k) = 0]$$

and

$$P[\Delta(k+1) = \tfrac{j}{v}] = \frac{m-v+j}{m} P[\Delta(k) = \tfrac{j}{v}] + \frac{v-j+1}{m} P[\Delta(k) = \tfrac{j-1}{v}] \quad j = 1, \ldots, v$$

along with the initial condition $P[\Delta(0) = 0] = 1$, provide a means for the calculation of probability distributions for $\Delta(k)$, $k = 1, 2, \ldots$. As an example, the numerical table which follows for $\Delta(k)$ when $m = 20$, $v = 10$, and $u = 10$, was obtained by their recursive application.

k	1	2	3	4	5	6	7	8	9	10
$P[\Delta(k)=0]$	.5	.250	.125	.0625	.0312	.0156	.0078	.0039	.0020	.0010
$P[\Delta(k)=1/10]$	.5	.525	.414	.2901	.1908	.1206	.0741	.0447	.0265	.0156
$P[\Delta(k)=2/10]$		.225	.371	.4089	.3759	.3114	.2411	.1780	.1269	.0881
$P[\Delta(k)=3/10]$			.090	.2070	.2981	.3441	.3482	.3228	.2810	.2334
$P[\Delta(k)=4/10]$				.0315	.0945	.1705	.2398	.2897	.3158	.3194
$P[\Delta(k)=5/10]$					.0095	.0354	.0777	.1302	.1846	.2332
$P[\Delta(k)=6/10]$						.0024	.0108	.0280	.0550	.0901
$P[\Delta(k)=7/10]$							.0005	.0026	.0078	.0176
$P[\Delta(k)=8/10]$								.0001	.0004	.0016
$P[\Delta(k)=9/10$									.0000	.0000
$P[\Delta(k)=1]$										.0000

The algebraic structure of the probability distributions for the $\Delta(k)$'s is discussed in the appendix to this paper.

Appendix - Formulas for $\Delta(k)$ probabilities

The purpose of this appendix is to illustrate the algebraic structure of the probability distributions for the $\Delta(k)$'s (the proportions of damage resulting from k hits to an m celled target with v cells of value $1/v$ and m-v cells of value 0).

The following expressions for $P_k[j] = P_k[\Delta(k) = \frac{j}{v}]$ were obtained from the k hits to k+1 hits transition equations derived in Section 3 along with the initial condition $P_0[0] = 1$, using the software *Theorist* [4]. For this purpose these equations are more conveniently written as

$$P_{k+1}[0] = \frac{u}{m} P_k[0]$$

$$P_{k+1}[j] = \frac{u+j}{m} P_k[j] + \frac{v-j+1}{m} P_k[j-1] \qquad j = 1, \dots, v,$$

where $u = m\text{-}v$, the number of "unvaluable" cells in the target.

$$P_1[0] = \frac{u}{m} \qquad P_1[1] = \frac{v}{m}$$

$$P_2[0] = \frac{u^2}{m^2} \qquad P_2[1] = \frac{(2u+1)v}{m^2} \qquad P_2[2] = \frac{(v-1)v}{m^2}$$

$$P_3[0] = \frac{u^3}{m^3}$$

$$P_3[1] = \frac{(3u^2+3u+1)v}{m^3} \qquad P_3[2] = \frac{3(u+1)(v-1)v}{m^3} \qquad P_3[3] = \frac{(v-2)(v-1)v}{m^3}$$

$$P_4[0] = \frac{u^4}{m^4}$$

$$P_4[1] = \frac{(4u^3+6u^2+4u+1)v}{m^4}$$

$$P_4[2] = \frac{(6u^2+12u+7)(v-1)v}{m^4} \qquad P_4[3] = \frac{(4u+6)(v-2)(v-1)v}{m^4}$$

$$P_4[4] = \frac{(v-3)(v-2)(v-1)v}{m^4}$$

$$P_5[0] = \frac{u^5}{m^5}$$

$$P_5[1] = \frac{(5u^4+10u^3+10u^2+5u+1)v}{m^5}$$

$$P_5[2] = \frac{5(2u^3+6u^2+7u+3)(v-1)v}{m^5}$$

$$P_5[3] = \frac{(10u^2+30u+25)(v-2)(v-1)v}{m^5}$$

$$P_5[4] = \frac{(5u+10)(v-3)(v-2)(v-1)v}{m^5}$$

$$P_5[5] = \frac{(v-4)(v-3)(v-2)(v-1)v}{m^5}$$

Each $P_k[j]$ has the structure

$$P_k[j] = U_k[j] \cdot \frac{v(v-1)(v-2)\cdots(v-j+1)}{m^k} = U_k[j] \cdot \frac{v!}{m^k (v-j)!}$$

where $U_k[j]$ is a polynomial depending only on u, k, and j, a fact that can be confirmed by inspecting the transition diagram at the beginning of Section 3.

It appears that the generation of the $P_k[j]$'s *can be approached by seperately computing the terms* $\frac{v(v-1)(v-2)\cdots(v-j+1)}{m^k}$ *and the coefficients* $U_k[j]$ *using the modified version of the transition equations*

$$U_{k+1}[0] = U_k[0]$$

$$U_{k+1}[j] = (u+j)U_k[j] + U_k[j-1]$$

with the initial condition $U_0[0] = 1$. Whether this approach offers any advantage over direct numerical application of the basic transition equations is uncertain.

It should be noted that when all m cells are valuable so that $u = 0$, the polynomials $U_k[j]$ reduce to the coefficients $C_k[j]$ considered in the appendix to [3].

References

[1] J. D. Esary. *A comparison of an empirical rule for aggregating damage from a weapons salvo to a plausible model for the same purpose.* Working Paper on Damage Aggregation, Naval Postgraduate School, April 1989.

[2] J. D. Esary. *Damage aggregation for a weapons salvo by an empirical rule related to the Poisson approximation to the binomial.* Working Paper on Damage Aggregation, Naval Postgraduate School, April 1989.

[3] J. D. Esary. *A stochastic model for hit overlap in a weapons salvo directed against an area target that leads to a proportional mechanism for damage aggregation.* Working Paper on Damage Aggregation, Naval Postgraduate School, June 1990.

Note: The preceding three working papers are collected in the Naval Postgraduate School Technical Report *Studies in damage aggregation for weapons salvos*, NPS55-90-16, July 1990.

[4] *Theorist.* Presience Corporation, San Francisco, 1989.

A basic lemma on expected damage aggregation for cellular targets, and some of its applications

1. Introduction

This working paper is the fifth in a series about modeling the cumulative effect of weapons salvos directed against a target Its predecessors are references [1] - [4].

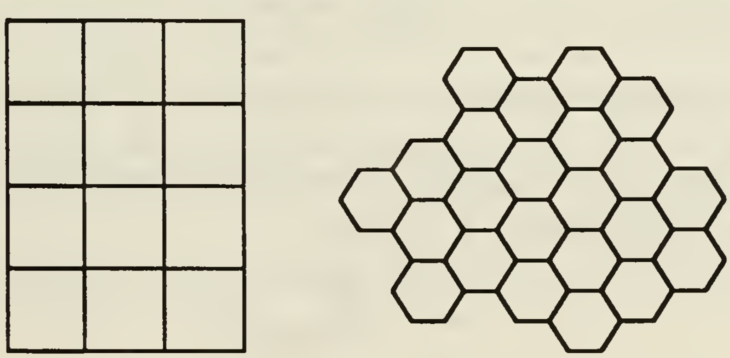
One topic so far considered has been mechanisms for aggregating the expected cumulative damage to a target as a function of the number of weapons that hit the target. Papers [1] and [2] assume what we have come to call a *proportional* damage aggregation mechanism as one of the foundations for an investigation of certain approximate formulas for expected overall damage from a salvo. Papers [3] and [4] are detailed looks at target configuration and weapons impact scenarios that lead to the *proportional* mechanism. The targets considered in [3] and [4] are *cellular* targets.

The purpose of the current paper is to present a basic lemma about expected cumulative damage to a cellular target. The lemma permits a simple demonstration of proportional damage aggregation in the targeting scenarios considered in [3] and [4]. Amongst its other applications, the lemma can be used to verify proportional damage aggregation for a wider range of targeting scenarios.

The terms *proportional* and *cellular* used in this introduction will be defined as they become pertinent in what follows.

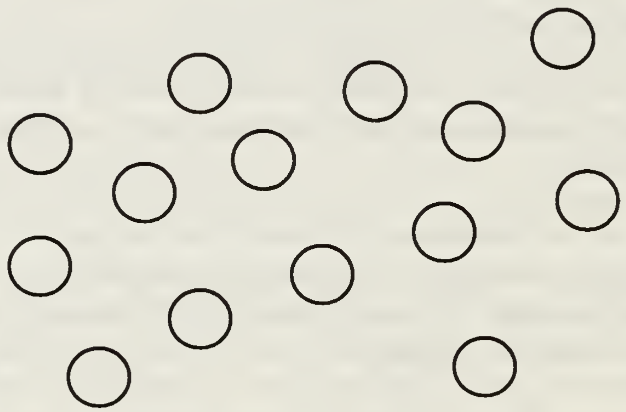
2. Cellular targets with a finite number of cells

Assume that a target is divided into a finite number m of **cells.** It is possible to visualize a cellular target as an area target as illustrated below.



Cellular Targets with $m = 12$ *and* $m = 23$ *cells*

Another possibility is to visualize a cellular target as a collection of isolated cells which might represent a formation of vehicles or aircraft, or a flight of incoming missiles.



A cellular target with $m = 13$ *cells*

The essential assumption about cellular targets is that a weapon that hits the target **impacts** on one or more of its cells. That is that the effect of the weapon, be it complete or partial cell damage, is upon some subset of cells selected by the weapon.

3. Cell target values

The target value of a cell represents the proportion of the target that is damaged by damaging that cell. In [3] the m cells had equal target value $1/m$. In [4] v of the m cells had target value $1/v$. and the remaining $m-v$ cells had target value zero.

In general assume that the i^{th} cell of a cellular target has a **target value** w_i , where $w_i \geq 0, i = 1, \dots, m,$ and

$$\sum_{i=1}^{m} w_i = 1$$

The proportion of damage to the target from k *weapon hits is the total target value of the cells that are damaged by the* k *hits.* If $\mathbf{C}(k)$ is the set of cells that are damaged as the result of k hits, and $\Delta(k)$ is the proportion of the target that is damaged by k hits, then

$$\Delta(k) = \sum_{i \in \mathbf{C}(k)} w_i$$

Another way of relating $\Delta(k)$ to cell target values is through the use of binary scoring variables

$$X_i(k) = \begin{cases} 1 & \text{if the } i^{\text{th}} \text{ cell is damaged as a result of } k \\ & \text{weapons hits on the target} \\ 0 & \text{if the } i^{\text{th}} \text{ cell remains undamaged as} \\ & \text{a result of } k \text{ weapons hits on the target} \end{cases} \qquad i = 1, \dots, m$$

Then

$$\Delta(k) = \sum_{i=1}^{m} w_i X_i(k)$$

Cells are **equally valued** if

$$w_1 = w_2 = \dots = w_m = \frac{1}{m}$$

that is, *if each cell has the same target value.* If cells are equally valued, then

$$\Delta(k) = \frac{1}{m} \sum_{i=1}^{m} X_i(k)$$

and the proportion of damage to the target from k hits is simply the ratio of the number of cells damaged by k hits to the total number of cells.

4. *Expected proportion of damage to a target*

In the presence of random cell impact choices by weapons that hit a cellular target, $\Delta(k)$ and $X_1(k), \dots, X_m(k)$ will be a random variables. Then the expected proportion of damage to the target $D(k)$ is

$$D(k) = E(\Delta(k)) = E\left(\sum_{i=1}^{m} w_i X_i(k)\right) = \sum_{i=1}^{m} w_i E\{X_i(k)\}$$

Since

$$E\{X_i(k)\} = 1 \cdot P[X_i(k) = 1] + 0 \cdot P[X_i(k) = 0] = \delta_i(k)$$

where $\delta_i(k) = P[X_i(k) = 1]$ is the probability that the i^{th} cell is damaged by k hits,

$$D(k) = \sum_{i=1}^{m} w_i \delta_i(k)$$

This simple relationship permits some interesting conclusions about damage aggregation for cellular targets It is the *basic lemma* of the title of this paper.

5. When cells are equally favored

Cells are **equally favored after** k **hits** if

$$\delta_1(k) = \delta_2(k) = \ldots = \delta_m(k) = \delta(k)$$

that is, if *each cell has the same probability of being damaged after* k *hits on the target*. Since $\delta_1(0) = \delta_2(0) = \ldots = \delta_m(0) = \delta(0) = 0$, cells are always equally favored after 0 hits.

If cells are equally favored after k hits, then

$$D(k) = \sum_{i=1}^{m} w_i \delta(k) = \delta(k) \sum_{i=1}^{m} w_i = \delta(k)$$

and *the expected proportion of damage to the target from* k *hits is the same as the common probability that a cell is damaged by* k *hits.*

Cells are **equally favored** if cells are equally favored after k hits for $k = 0, 1, 2, \ldots$. Thus *when cells are equally favored*

$$D(k) = \delta(k) \qquad k = 0, 1, 2, \ldots$$

6. The proportional damage aggregation mechanism

The term **proportional** damage aggregation **mechanism** refers in this context to a possible mode of growth for the expected proportion of damage $D(k)$ to a target as the result of an increasing number of hits k on the target. This mechanism was discussed in [1] as a "plausible model" for damage aggregation. Its derivation treats each $D(k)$ as an abstract deterministic proportion of damage to the target from k hits.

The premise of the proportional mechanism is that *if the proportion of a pristine target that is damaged by a single hit is* d, *then each additional hit damages the same proportion* d *of that part of the target not previously damaged.*

Thus if $D(k)$ is the aggregate proportion of damage to a pristine target from exactly k hits, then

$$D(0) = 0 = 1 - (1-d)^0$$

$$D(1) = d = 1 - (1-d)^1$$

$$D(2) = D(1) + d\{1 - D(1)\} = d + d(1-d) = 1 - (1-d)^2$$

$$\vdots$$

$$D(n) = D(n-1) + d\{1 - D(n-1)\} = 1 - (1-d)^{n-1} + d(1-d)^{n-1}$$

$$= 1 - (1-d)(1-d)^{n-1} = 1 - (1-d)^n$$

where n is the number of weapons in the salvo.

7. Application to the targeting scenarios of [3] *and* [4]

In [3] and [4] a weapon that hits a target impacts on exactly one of its cells. If the cell is previously undamaged, the weapon damages that cell. Thus $\Delta(1) = \frac{1}{m}$ with probability one, where m is the number of cells in the target. and

$$D(1) = E\{\Delta(1)\} = d = \frac{1}{m}$$

Subsequent impacts on the cell do no further damage to the cell and do not add to the damage to the target.

Also in [3] and [4] a weapon that hits a target makes a random choice of which cell in the target to impact, with each cell being given an equal chance. Weapons that hit the target choose their impact cells independently.

Under these assumptions the probability that a hit does not impact a particular cell is $1-\frac{1}{m}$, and the probability that k hits do not impact a particular cell is $\left(1-\frac{1}{m}\right)^k$, so that the probability that a particular cell is impacted and damaged is $1-\left(1-\frac{1}{m}\right)^k$. This means that *cells are equally favored* with $\delta(k) = 1-\left(1-\frac{1}{m}\right)^k$.

It follows that

$$D(k) = 1-\left(1-\frac{1}{m}\right)^k \qquad k = 1, 2, \ldots$$

so that the damage aggregation mechanism is proportional with $d = \frac{1}{m}$, confirming a conclusion reached in both [3] and [4].

Note that the target values of cells do not enter into the preceding argument.

8. *The targeting scenario* CA11UI

The elements of the targeting scenario **CA11UI** are:

(C) The target is *cellular* with m cells.

(A) Cell target values w_i are *arbitrary* subject only to the general constraints $w_i \geq 0, i = 1, \ldots, m$, and

$$\sum_{i=1}^{m} w_i = 1$$

(1) Each weapon that hits the target impacts on exactly *one* of its cells.

(1) *One* weapon impact on a cell damages the cell. Additional impacts on the cell do not cause further damage.

(U) A weapon that hits the target impacts its cells *uniformly* in the sense that each cell has the same probability $1/m$ of being impacted.

(I) Weapons that hit the target choose the cells to impact *independently*.

The targeting scenario of [3], **CE11UI**, where

(E) Cell target values are *equal* with each cell having value $1/m$.

replaces (A) above, and the targeting scenario of [4], **CZ11UI,** where

(Z) Of the m cells, v have target value $1/v$, and the remaining m-v have target value *zero.*

replaces (A) above, are special cases of **CA11UI**. The discussion of these scenarios in Section 7 applies more generally to show that in the CA11UI targeting scenario *damage aggregation is proportional with* $d = \frac{1}{m}$.

9. The targeting scenario CAr1UI

The elements of the targeting scenario **CAr1UI** are:

(C) The target is *cellular* with m cells.

(A) Cell target values w_i are *arbitrary* subject only to the general constraints $w_i \geq 0, i = 1, \ldots, m,$ and

$$\sum_{i=1}^{m} w_i = 1$$

(r) Each weapon that hits the target impacts on a subset of r of its cells.

(1) *One* impact on a cell damages the cell. Additional impacts on the cell do not cause further damage.

(U) A weapon that hits the target impacts its cells *uniformly* in the sense that each subset of r cells has the same probability $1/\binom{m}{r}$ of being impacted.

(I) Weapons that hit the target choose subsets of cells to impact *independently*.

The CAr1UI scenario is suggestive of the effect of cluster warheads on an area target.

In CAr1UI

$$D(1) = E\{\Delta(1)\} = d = \frac{r}{m}$$

The probability that a single hit on the target impacts a particular cell is

$$\frac{\binom{1}{1}\binom{m-1}{r-1}}{\binom{m}{r}} = \frac{\frac{(m-1)!}{(r-1)!(m-r)!}}{\frac{m!}{r!(m-r)!}} = \frac{r}{m}$$

so that the probability that a single hit does not impact a particular cell is $1-\frac{r}{m}$. The probability that k hits do not impact a particular cell is $\left(1-\frac{r}{m}\right)^k$, and the probability that a particular cell is impacted and damaged by k hits is $1-\left(1-\frac{r}{m}\right)^k$. Thus *the cells are equally favored* with

$$\delta(k) = 1-\left(1-\frac{r}{m}\right)^k \qquad k = 0, 1, 2, \ldots$$

It follows that

$$D(k) = 1-\left(1-\frac{r}{m}\right)^k \qquad k = 0, 1, 2, \ldots$$

so that *damage aggregation is proportional* with $d = \frac{r}{m}$.

9. The targeting scenario CA1cUI

The elements of the targeting scenario **CA1cUI** are:

(C) The target is *cellular* with m cells.

(A) Cell target values w_i are *arbitrary* subject only to the general constraints $w_i \geq 0, i = 1, \ldots, m,$ and

$$\sum_{i=1}^{m} w_i = 1$$

(1) Each weapon that hits the target impacts on exactly *one* of its cells.

(c) c weapon impacts on a cell are required to damage the cell. Additional impacts on the cell do not cause further damage.

(U) A weapon that hits the target impacts its cells *uniformly* in the sense that each cell has the same probability $1/m$ of being impacted.

(I) Weapons that hit the target choose the cells to impact *independently*.

In CA1cUI the probability that a single hit on the target impacts a particular cell is $\frac{1}{m}$. Out of k hits on the target, the probability that exactly j impact the cell is

$$\binom{k}{j}\frac{1}{m}^{j}\left(1-\frac{1}{m}\right)^{k-j}$$

and the probability that there are at least c impacts on the cell is

$$\sum_{j=c}^{k}\binom{k}{j}\frac{1}{m}^{j}\left(1-\frac{1}{m}\right)^{k-j}$$

Thus *cells are equally favored* with

$$D(k) = \delta(k) = \sum_{j=c}^{k} \binom{k}{j} \frac{1}{m}^{j} \left(1 - \frac{1}{m}\right)^{k-j} \qquad k = 0, 1, \ldots$$

This damage aggregation mechanism is *not* proportional, except in the special case that $c = 1$, when

$$D(k) = \sum_{j=1}^{k} \binom{k}{j} \frac{1}{m}^{j} \left(1 - \frac{1}{m}\right)^{k-j} = 1 - \sum_{j=0}^{0} \binom{k}{j} \frac{1}{m}^{j} \left(1 - \frac{1}{m}\right)^{k-j}$$
$$= 1 - \left(1 - \frac{1}{m}\right)^{k}$$

and the targeting scenario reduces to CA11UI.

References

[1] J. D. Esary. *A comparison of an empirical rule for aggregating damage from a weapons salvo to a plausible model for the same purpose.* Working Paper on Damage Aggregation, Naval Postgraduate School, April 1989.

[2] J. D. Esary. *Damage aggregation for a weapons salvo by an empirical rule related to the Poisson approximation to the binomial.* Working Paper on Damage Aggregation, Naval Postgraduate School, April 1989.

[3] J. D. Esary. *A stochastic model for hit overlap in a weapons salvo directed against an area target that leads to a proportional mechanism for damage aggregation.* Working Paper on Damage Aggregation, Naval Postgraduate School, June 1990.

Note: The preceding three working papers are collected in the Naval Postgraduate School Technical Report *Studies in damage aggregation for weapons salvos*, NPS55-90-16, July 1990.

[4] J. D. Esary. *Proportional damage aggregation for a cellular target with cells of zero value.* Working Paper on Damage Aggregation, Naval Postgraduate School, March 1991.

Comparisons of an empirical rule for expected damage aggregation from weapons salvos to models assuming a proportional damage aggregation mechanism and dependent weapons hit distributions.

1. Introduction

This working paper is a revisit to an issue considered in [1].

The scenario considered here is that a salvo of n weapons is launched against a target. The number of weapons that hit the target is a random variable N with possible values $0, 1, \dots, n$. Possible damage to the target is measured as a percentage (or proportion) of the whole ranging from 0% to 100%. The damage to a pristine target resulting from a single hit is a deterministic proportion d of the whole. The aggregate proportion of damage to the target from the salvo is a random variable D, the randomness in D resulting from the randomness in the number of hits N.

The end objective is to predict the expected proportion of damage $E(D)$ to the target resulting from the salvo. This prediction can involve modeling of the probability distribution of the number of hits N, and the way that damage aggregates as additional hits beyond the first are scored.

An empirical rule for predicting $E(D)$ is

$$\tilde{E}(D) = 1 - (1-d)^{E(N)}$$

where $E(N)$ is the expected number of hits on the target. This rule ducks the issue of aggregating deterministic damage from multiple hits and to some extent ducks the issue of modeling the probability distribution for the random number of hits on the target.

In [1] this empirical rule was compared to the result of assuming that each weapon in the salvo hits independently with the same probability p, so that the number of hits N has a binomial probability distribution, and that the deterministic damage aggregation resulting from multiple hits on the target is proportional. The premise of the **proportional** mechanism for damage aggregation is that *if the proportion of a pristine target that is damaged by a single hit is* d, *then each additional hit damages the same proportion* d *of that part of the target not previously damaged.* Thus if $D(k)$ is the aggregate proportion of damage to a pristine target from exactly k hits, then

$$D(0) = 0 = 1 - (1-d)^0$$

$$D(1) = d = 1 - (1-d)^1$$

$$D(2) = D(1) + d\{1 - D(1)\} = d + d(1-d) = 1 - (1-d)^2$$

$$\vdots$$

$$D(n) = D(n-1) + d\{1 - D(n-1)\} = 1 - (1-d)^{n-1} + d(1-d)^{n-1}$$

$$= 1 - (1-d)(1-d)^{n-1} = 1 - (1-d)^n$$

where n is the number of weapons in the salvo.

Here the empirical rule is compared to the result of assuming that *damage aggregation is proportional,* and that the *distribution of the number of hits* N *on the target is arbitrary.*

If $P[N=k]$, $k = 0, 1, \dots, n$, are the probabilities of exactly k hits on the target, then

$$E(D) = \sum_{k=0}^{n} D(k)P[N=k] = \sum_{k=0}^{n} \{1-(1-d)^k\}P[N=k]$$

$$= \sum_{k=0}^{n} P[N=k] - \sum_{k=0}^{n} (1-d)^k P[N=k]$$

$$= 1 - \sum_{k=0}^{n} (1-d)^k P[N=k]$$

For the binomial hit distribution considered in [1], $\tilde{E}(D)$ was shown to overestimate $E(D)$. In what follows $\tilde{E}(D)$ will be found to overestimate $E(D)$ for any hit distribution, with certain special hit distributions considered as examples.

2. *The all-or-nothing hit distribution*

Under the **all-or-nothing** hit distribution, either every weapon in the salvo hits the target or none do. The hit distribution can be represented as

$$P[N=n] = p \qquad P[N=0] = 1-p \qquad 0 \le p \le 1$$

Then

$$E(N) = 0 \cdot P[N=0] + n \cdot P[N=n] = 0 \cdot (1-p) + n \cdot p = np$$

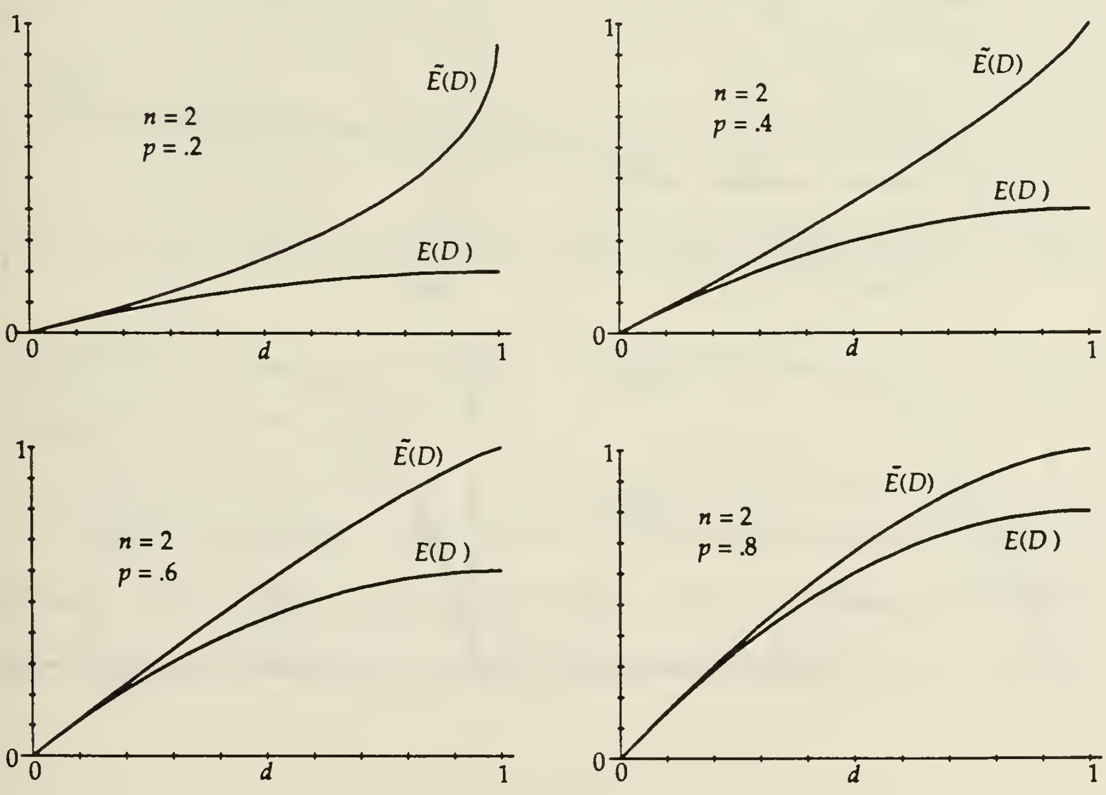
so that

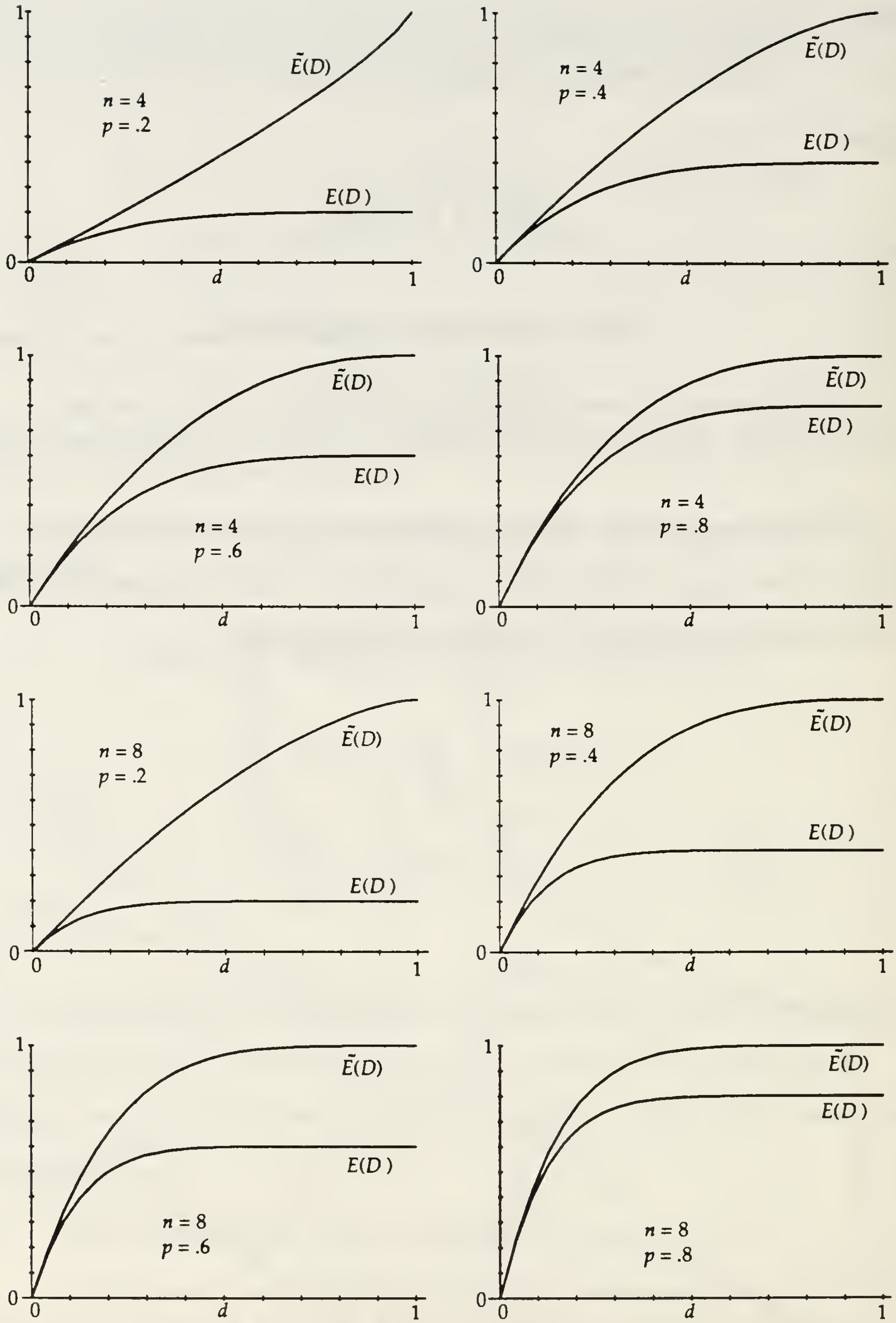
$$\tilde{E}(D) = 1 - (1-d)^{np}$$

and

$$\begin{aligned} E(D) &= D(0) \cdot P[N=0] + D(n) \cdot P[N=n] \\ &= 0 \cdot (1-p) + \{1 - (1-d)^n\} \cdot p \\ &= p\,\{1 - (1-d)^n\} \end{aligned}$$

Plots [6] comparing $\tilde{E}(D)$ to $E(D)$ for selected values of n and p follow. In each case $\tilde{E}(D) \geq E(D)$. A proof that $\tilde{E}(D)$ is always greater than $E(D)$ is best left to the case of an arbitrary hit distribution.





n = 4
p = .2
Ẽ(D)
E(D)
n = 4
p = .4
Ẽ(D)
E(D)
n = 4
p = .6
Ẽ(D)
E(D)
n = 4
p = .8
Ẽ(D)
E(D)
n = 8
p = .2
Ẽ(D)
E(D)
n = 8
p = .4
Ẽ(D)
E(D)
n = 8
p = .6
Ẽ(D)
E(D)
n = 8
p = .8
Ẽ(D)
E(D)
d
0
1

3. *The uniform hit distribution*

Under the **uniform** hit distribution every possible number of hits on the target, from no hits to all hits, is equally probable. The hit distribution is represented by

$$P[N{=}k] = \frac{1}{n+1} \qquad\qquad k = 0, 1, \dots, n$$

Then

$$\begin{aligned} E(N) &= \sum_{k=0}^{n} k\,P[N=k] = \sum_{k=0}^{n} \frac{k}{n+1} = \frac{1}{n+1}\sum_{k=0}^{n} k \\ &= \frac{1}{n+1}\,\frac{n(n+1)}{2} = \frac{n}{2} \end{aligned}$$

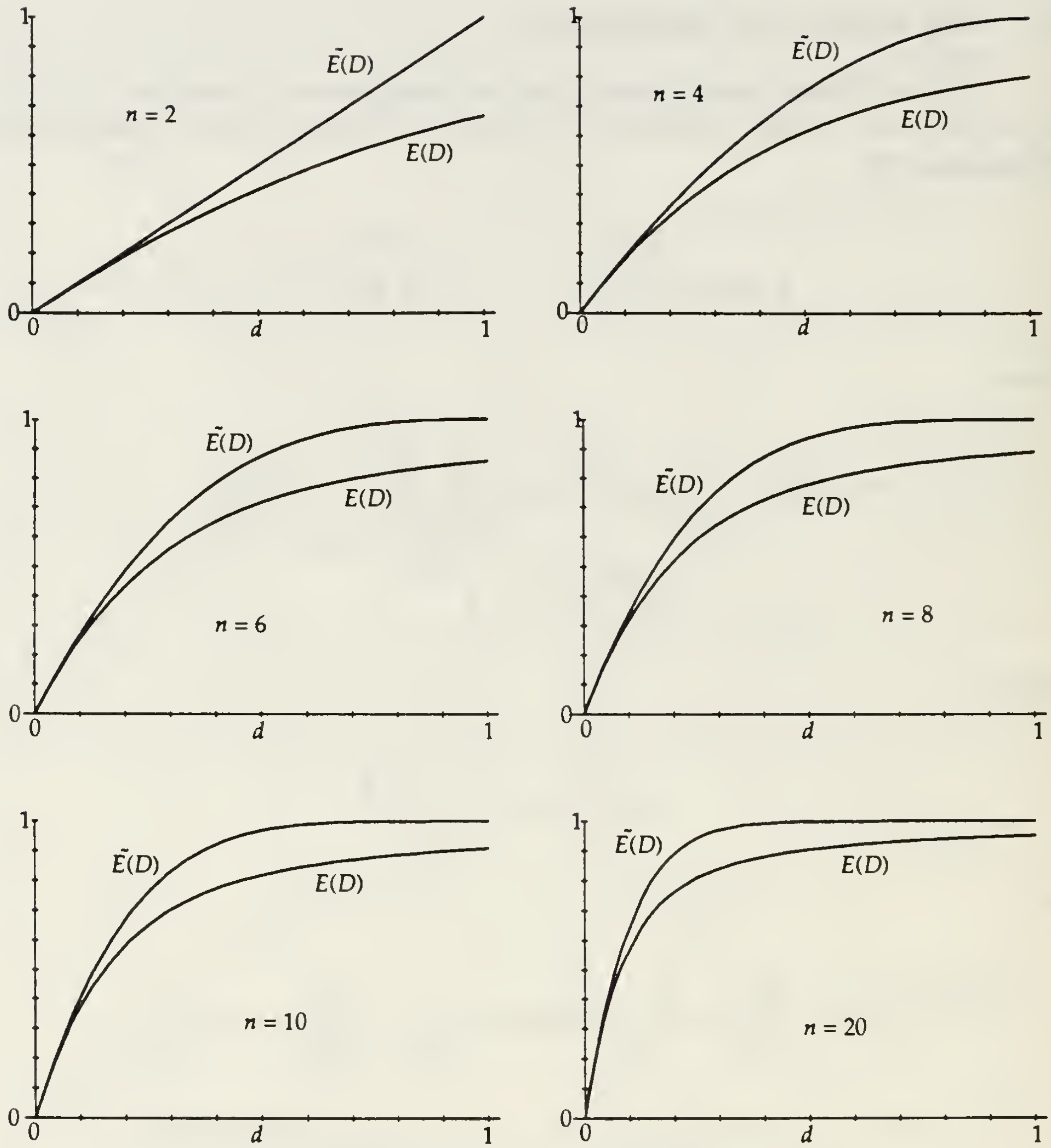
so that

$$\tilde{E}(D) = 1-(1-d)^{\frac{n}{2}}$$

and

$$\begin{aligned} E(D) &= \sum_{k=0}^{n} \{1-(1-d)^k\}P[N=k] = 1-\sum_{k=0}^{n}(1-d)^k\frac{1}{n+1} \\ &= 1-\frac{1}{n+1}\sum_{k=0}^{n}(1-d)^k = 1-\frac{1}{n+1}\left\{\frac{1-(1-d)^{n+1}}{1-(1-d)}\right\} \\ &= 1-\frac{1-(1-d)^{n+1}}{(n+1)d} \end{aligned}$$

Plots [6] comparing $\tilde{E}(D)$ to $E(D)$ for selected values of n follow. In each case $\tilde{E}(D) \geq E(D)$. Again a proof that $\tilde{E}(D)$ is always greater than $E(D)$ is best left to the case of an arbitrary hit distribution.



4. The r-certain hit distribution

Under the **r-certain** hit distribution, it is assured that from the n weapons in the salvo exactly r will hit the target. The hit distribution is represented by

$$P[N=r] = 1 \qquad \text{for some } r \qquad 0 \le r \le n$$

Then

$$E(N) = r \cdot P[N=r] = r \cdot 1 = r$$

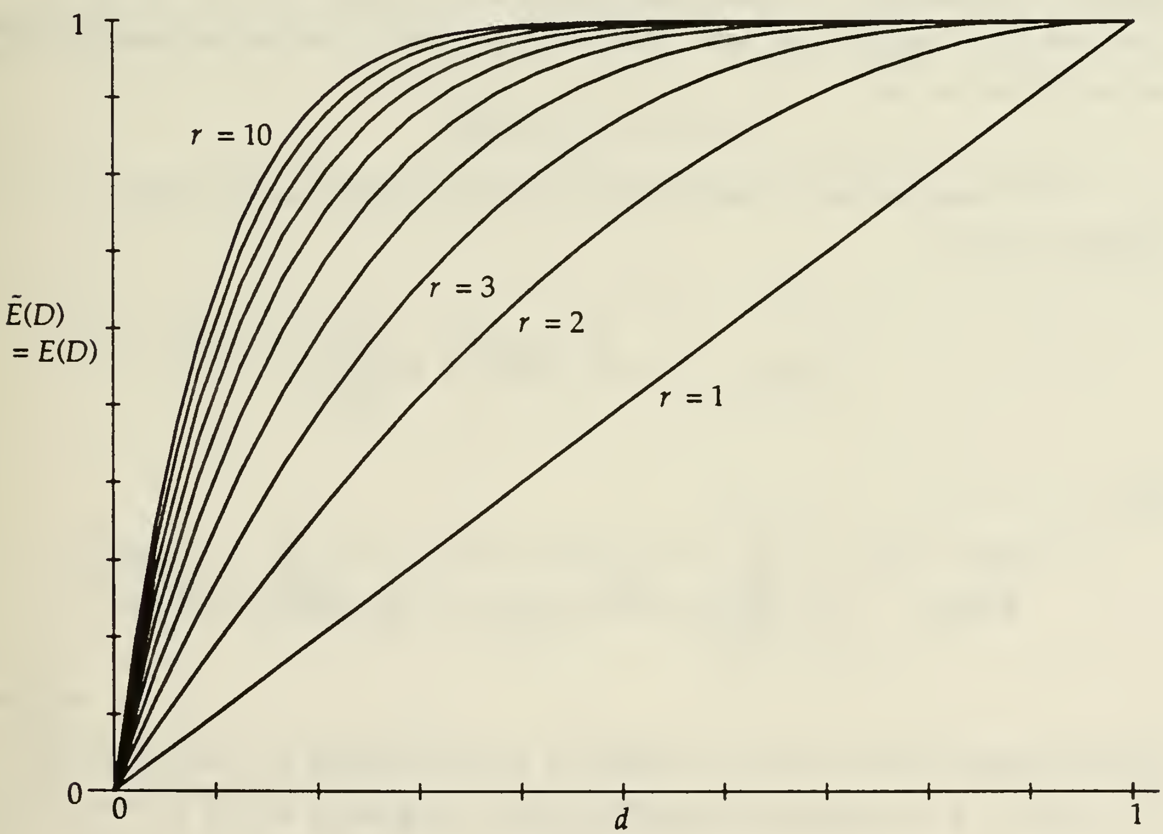
so that

$$\tilde{E}(D) = 1 - (1-d)^r$$

and

$$E(D) = D(r) \cdot P[N=r] = D(r) \cdot 1 = 1 - (1-d)^r$$

It appears that $\tilde{E}(D)$ and $E(D)$ depend only on the guaranteed number of hits r, and not on the salvo size n. A plot [6] of $\tilde{E}(D) = E(D)$ for selected values of r follows.



5. The initial and terminal values of $\tilde{E}(D)$ and $E(D)$

$\tilde{E}(D) = 1-(1-d)^{E(N)}$ and $E(D) = 1-\sum_{k=0}^{n}(1-d)^k P[N=k]$ have the same value at $d = 0$ for any hit distribution, since

$$\tilde{E}(D)\big|_{d=0} = 1-(1-0)^{E(N)} = 1-1 = 0$$

and

$$E(D)\big|_{d=0} = 1-\sum_{k=0}^{n}(1-0)^k P[N=k] = 1-1 = 0$$

Both are consistent with the principle that *if the proportion of the target damaged by a single hit is zero, then the proportion of the target damaged by a salvo should be zero.*

On the other hand, the values of $\tilde{E}(D)$ and $E(D)$ at $d = 1$ are different, since

$$\tilde{E}(D)\big|_{d=1} = 1-(1-1)^{E(N)} = 1-0 = 1$$

and

$$E(D)\big|_{d=1} = 1-\sum_{k=0}^{n}(1-1)^k P[N=k] = 1-P[N=0] = P[N \geq 1]$$

The principle here is that *if the target is totally damaged by a single hit, then the expected proportion of the target damaged by a salvo should be zero if no hit is scored or one if at least one hit is scored,* i.e. $0 \cdot P[N=0] + 1 \cdot P[N \geq 1]$.

6. The initial slope of $\tilde{E}(D)$ and $E(D)$

Inspection of the plots associated with each of the three preceding hit distributions suggests that $\tilde{E}(D)$ and $E(D)$, viewed as functions of d, have the same derivative at $d = 0$. The following computations confirm this observation in general and show that the shared initial slope of both functions is equal to $E(N)$, the expected number of hits on the target.

For $\tilde{E}(D) = 1 - (1-d)^{E(N)}$,

$$\frac{d}{dd}\tilde{E}(D) = -E(N)\cdot(1-d)^{E(N)-1}\cdot(-1) = E(N)\cdot(1-d)^{E(N)-1}$$

so that

$$\frac{d}{dd}\tilde{E}(D)\Big|_{d=0} = E(N)$$

For $E(D) = 1 - \sum_{k=0}^{n}(1-d)^k P[N=k]$,

$$\frac{d}{dd}E(D) = -\sum_{k=0}^{n} k\cdot(1-d)^{k-1}(-1)\cdot P[N=k] = \sum_{k=0}^{n} k\cdot(1-d)^{k-1}\cdot P[N=k]$$

so that

$$\frac{d}{dd}E(D)\Big|_{d=0} = \sum_{k=0}^{n} k\,P[N=k] = E(N)$$

7. A proof that $\tilde{E}(D)$ is greater than or equal to $E(D)$

A general proof that $\tilde{E}(D) \geq E(D)$ can be based on the inequality between the arithmetic mean and the geometric mean of nonnegative numbers
$a_1, a_2, \ldots, a_n$

$$\frac{a_1 + a_2 + \cdots + a_n}{n} \geq \left(a_1 a_2 \cdots a_n\right)^{\frac{1}{n}}$$

in its extended form

$$p_1 a_1 + p_2 a_2 + \cdots + p_n a_n \geq a_1^{p_1} a_2^{p_2} \cdots a_n^{p_n}$$

where $a_k \geq 0$, $0 \leq p_k \leq 1$, $k = 1, \ldots, n$, and $p_1 + p_2 + \cdots + p_n = 1$. The proof applies to an arbitrary hit distribution.

Since

$$E(D) = 1 - \sum_{k=0}^{n} (1-d)^k P[N=k]$$

and

$$\tilde{E}(D) = 1 - (1-d)^{E(N)} = 1 - (1-d)^{\sum_{k=0}^{n} kP[N=k]} = 1 - \prod_{k=0}^{n} (1-d)^{kP[N=k]}$$

then $\tilde{E}(D) \geq E(D)$ if, and only if,

$$\sum_{k=0}^{n} (1-d)^k P[N=k] \geq \prod_{k=0}^{n} (1-d)^{kP[N=k]}$$

Letting $p_k = P[N{=}k]$, and $a_k = (1\text{-}d)^k$, $k = 0, \ldots, n$, the required inequality becomes

$$\sum_{k=0}^{n} a_k p_k \geq \prod_{k=0}^{n} a_k^{p_k}$$

References

[1] J. D. Esary. *A comparison of an empirical rule for aggregating damage from a weapons salvo to a plausible model for the same purpose.* Working Paper on Damage Aggregation, Naval Postgraduate School, April 1989.

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[4] J. D. Esary. *Proportional damage aggregation for a cellular target with cells of zero value.* Working Paper on Damage Aggregation, Naval Postgraduate School, March 1991.

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[6] *Mathematica* (1.2.2 for Macintosh). Wolfram Research, Inc., Champaign, IL.

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Naval Undersea Warfare Engineering Center
Keyport, WA 98345-0580

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Department of Operations Research
Naval Postgraduate School
Monterey, CA 93943-5000

Samuel H. Parry..1
Department of Operations Research
Naval Postgraduate School
Monterey, CA 93943-5000

Peter Purdue...1
Department of Operations Research
Naval Postgraduate School
Monterey, CA 93943-5000

Rex N. Randolph...1
Weapons Effectiveness Branch (Code 3517)
Naval Weapons Center
China Lake, CA 93555

Roger Stemp...1
Department of Computer Science
Naval Postgraduate School
Monterey, CA 93943-5000

William G. Tonkin..1
Target Vulnerability Office (Code 35104)
c/o Code 3517
Naval Weapons Center
China Lake, CA 93555

US Army AMSAA...1
Attn: AMXSY-GA (Will Brooks)
Aberdeen Proving Grounds, MD 21005