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STUDIES ON DAMAGE AGGREGATION FOR WEAPONS SALVOS

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This document records three working studies from an ongoing investigation of models and methods for the prediction of the cumulative effect of weapons salvos. The first paper is about an extant formula for estimating the expected proportion of damage to an area target, which proves to be optimistic when compared to a plausible model for the effect of the salvo. The second paper describes an alternate formula which is conservative when compared to the same model. The third paper describes a basic case of an emerging family of target configuration and weapons impact scenarios which lead to the plausible model.

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Foreword

This document records three working studies from an ongoing investigation of models and methods for the prediction of the cumulative effect of weapons salvos. These papers are reproduced here in their entirety and in the chronological order in which they were produced.

The first paper in the sequence, *A comparison of an empirical rule for aggregating damage from a weapons salvo to a plausible model for the same purpose*, resulted from an examination of an extant formula for estimating the expected proportion of damage to an area target from a weapons salvo. Its conclusion is that the formula gives optimistic, and in some instances impossible, results when compared to a "plausible" model for the effect of the salvo.

The second paper, *Damage aggregation for a weapons salvo by an empirical rule related to the Poisson approximation to the binomial*, describes an alternative formula which is conservative when compared to the same "plausible" model.

The third paper, *A stochastic model for hit overlap in a weapons salvo directed against an area target that leads to a proportional mechanism for damage aggregation*, describes a basic case of an emerging family of target configuration and weapons impact scenarios which lead to the damage aggregation mechanism of the "plausible" model. The "plausible" damage aggregation mechanism is renamed the **proportional effects** mechanism in this paper.

The general setting for these studies is described in the first two paragraphs of the introduction to the first paper. The proportional effects (plausible) damage aggregation mechanism is derived in Section 3 of that paper. This information is recapitulated in the introduction to the subsequent papers.

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A comparison of an empirical rule for aggregating damage from a weapons salvo to a plausible model for the same purpose

1. Introduction

The scenario considered here is that a salvo of n weapons is launched against a target. The number of weapons that hit the target is a random variable N with possible values $0, 1, \dots, n$. Possible damage to the target is measured as a percentage (or proportion) of the whole ranging from 0% to 100%. The damage to a pristine target resulting from a single hit is a deterministic proportion d of the whole. The aggregate proportion of damage to the target from the salvo is a random variable D, the randomness in D resulting from the randomness in the number of hits N.

The end objective is to predict the expected proportion of damage $E(D)$ to the target resulting from the salvo. This prediction can involve modeling of the probability distribution of the number of hits N, and the way that damage aggregates as additional hits beyond the first are scored. For the immediate study, it is assumed that each weapon in the salvo hits independently with the same probability p, so that the number of hits N has a binomial probability distribution. This leaves the problem of how to aggregate the deterministic proportion of damage resulting from multiple hits on the target.

The empirical rule for predicting $E(D)$ is

$$\tilde{E}(D) = 1 - (1-d)^{E(N)}$$

where $E(N)$ is the expected number of hits on the target. This rule ducks the issue of aggregating deterministic damage from multiple hits and to some extent ducks the issue of modeling the probability distribution for the random number of hits on the target. We will first examine the workings of the empirical rule in the case of a salvo of size one, and then proceed to a comparison of the empirical rule to the output of a plausible model for deterministic damage aggregation.

2. For a salvo of size one

For a salvo of size one, i.e. a single weapon, there is either a hit with probability p, or a miss with probability 1-p. The consequences of the salvo are summarized in the following diagram.

Working Paper on Damage Aggregation
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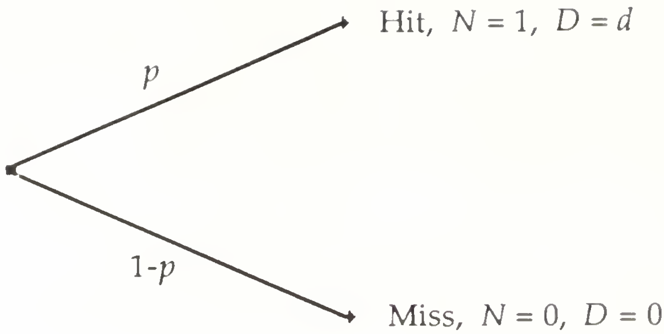
J. D. Esary April 1989



The expected number of hits is

$$E(N) \quad = \quad 1 \cdot p \; + \; 0 \cdot (1\text{-}p) \quad = \quad p$$

and the expected proportion of damage is

$$E(D) \quad = \quad d \cdot p \; + \; 0 \cdot (1\text{-}p) \quad = \quad pd$$

The issue then becomes how in this case does the empirical rule $\tilde{E}(D) \;=\; 1 - (1\text{-}d)^p$ for predicting $E(D)$ compare with the actual value $E(D) \;=\; pd$. *The following arguments show that* $\tilde{E}(D)$ *overestimates* $E(D)$ *for all values of* p *and* d.

We will hold p fixed and study the behavior of $\tilde{E}(D)$ and $E(D)$ as d varies from 0 to 1. First note that at $d = 0$ both $\tilde{E}(D)$ and $E(D)$ are 0. Then note that both $\tilde{E}(D)$ and $E(D)$ increase as d increases. This is evident by inspection of the two expressions, or from considering that the derivative of $\tilde{E}(D)$ is

$$\frac{d}{dd}\,\tilde{E}(D) \quad = \quad \frac{d}{dd}\{1 - (1\text{-}d)^p\} \quad = \quad \frac{p}{(1-d)^{1-p}}$$

and the derivative of $E(D)$ is

$$\frac{d}{dd}\,E(D) \quad = \quad \frac{d}{dd}(pd) \quad = \quad p$$

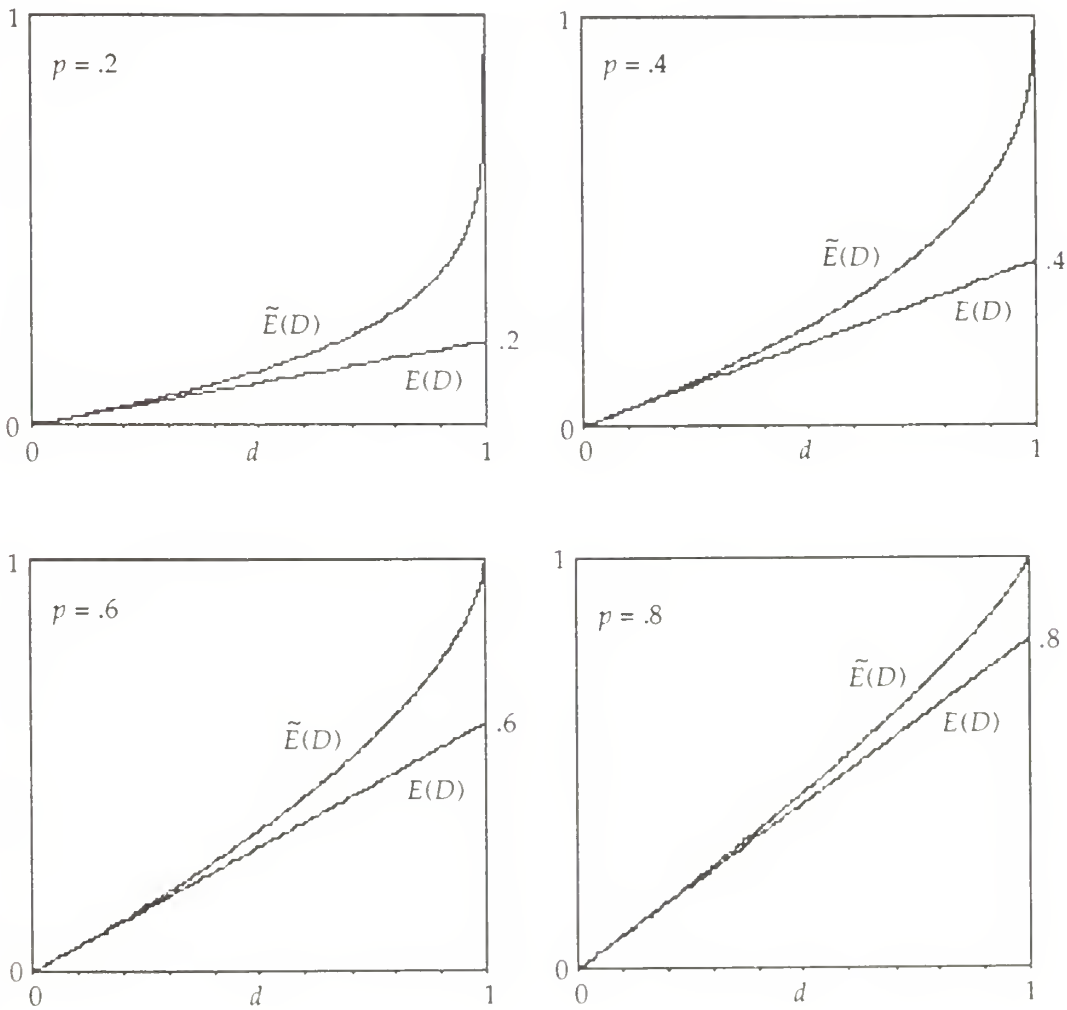
and that both derivatives are nonnegative. Also the derivative of $\tilde{E}(D)$ is larger than the derivative of $E(D)$, so that $\tilde{E}(D)$ increases faster than $E(D)$.

Since both expressions start at 0 for $d = 0$ and $\tilde{E}(D)$ increases faster, it follows that $\tilde{E}(D)$ is greater than $E(D)$ for all values of d.

It is worth noting that at $d = 0$, $\tilde{E}(D)$ and $E(D)$ have the same slope, i.e.

$$\left.\frac{d}{dd}\tilde{E}(D)\right|_{d=0} = \left.\frac{d}{dd}E(D)\right|_{d=0} = p$$

and that while the slope of $E(D)$ continues at a constant p, the slope of $\tilde{E}(D)$ grows to infinity as d approaches 1. The common slope of $\tilde{E}(D)$ and $E(D)$ at $d = 0$ suggests that $\tilde{E}(D)$ can be a good approximation to $E(D)$ for small values of d. This issue is studied numerically in the following four plots.



The empirical rule $\tilde{E}(D)$ becomes absurd as d approaches 1. At d = 1 it says that expected damage to the target is total regardless of the probability that the single weapon scores a hit. The reality is that the expected proportion of damage can never exceed the hit probability.

Wayne Hughes [1] makes the observation—"A single missile attack is most attractive tactically when the damage $E(D)$ is thought to be high, that is when p and d together are high. We do not need to make a judgement as to what is "high" to conclude that the region of primary interest will be toward the right end of the plots, which is the region of greatest divergence between the empirical $\tilde{E}(D)$ and the more plausible $E(D)$ curves."

3. A plausible model for damage aggregation

The premise of the "plausible model" for damage aggregation is that *if the proportion of a pristine target that is damaged by a single hit is d, then each additional hit damages the same proportion d of that part of the target not previously damaged.* Thus if $D(k)$ is the aggregate proportion of damage to a pristine target from exactly k hits, then

$$D(0) = 0 = 1 - (1-d)^0$$

$$D(1) = d = 1 - (1-d)^1$$

$$D(2) = D(1) + d\{1 - D(1)\} = d + d(1-d) = 1 - (1-d)^2$$

$$\vdots$$

$$D(n) = D(n-1) + d\{1 - D(n-1)\} = 1 - (1-d)^{n-1} + d(1-d)^{n-1}$$

$$= 1 - (1-d)(1-d)^{n-1} = 1 - (1-d)^n$$

where n is the number of weapons in the salvo.

It is worth noting that the incremental proportion of damage from the k^{th} hit is, for $k = 1, \dots, n$,

$$D(k) - D(k-1) = \{1 - (1-d)^k\} - \{1 - (1-d)^{k-1}\} = d(1-d)^{k-1}$$

4. *For a salvo of size n*

For a salvo of size n in which each weapon hits independently with the same probability p, the random number of hits N has a binomial probability distribution, i.e.

$$P[N=k] = \binom{n}{k} p^k (1-p)^{n-k}, \quad k = 0, \dots, n$$

and $E(N) = np$.

Then the expected proportion of damage is

$$E(D) = \sum_{k=0}^{n} D(k)\, P[N=k] = \sum_{k=0}^{n} \{1 - (1-d)^k\}\, P[N=k]$$

$$= 1 - \sum_{k=0}^{n} (1-d)^k\, P[N=k] = 1 - \sum_{k=0}^{n} (1-d)^k \binom{n}{k} p^k (1-p)^{n-k}$$

$$= 1 - \sum_{k=0}^{n} \binom{n}{k} (1-d)^k p^k (1-p)^{n-k} = 1 - \sum_{k=0}^{n} \binom{n}{k} \{(1-d)p\}^k (1-p)^{n-k}$$

$$= 1 - \{(1-d)p + (1-p)\}^n = 1 - (1-pd)^n$$

and the empirical rule for predicting $E(D)$ is

$$\tilde{E}(D) = 1 - (1-d)^{np}$$

As in the case of a salvo of size one, $\tilde{E}(D)$ *overestimates* $E(D)$ *for all values of* p *and* d, since

$$\tilde{E}(D) \geq E(D) \Leftrightarrow 1 - (1-d)^{np} \geq 1 - (1-pd)^n \Leftrightarrow (1-pd)^n \geq (1-d)^{np}$$

$$\Leftrightarrow (1-pd) \geq (1-d)^p \Leftrightarrow 1 - (1-d)^p \geq pd$$

The inequality between $\tilde{E}(D)$ and $E(D)$ for a salvo of size n reduces to the inequality between $\tilde{E}(D)$ and $E(D)$ for a salvo of size one.

As before, we will hold p fixed and study the behavior of $\tilde{E}(D)$ and $E(D)$ as d varies from 0 to 1. First note that at $d = 0$ both $\tilde{E}(D)$ and $E(D)$ are 0. Then note that both $\tilde{E}(D)$ and $E(D)$ increase as d increases. This is evident by inspection of the two expressions, or from considering that the derivative of $\tilde{E}(D)$ is

$$\frac{d}{dd}\,\tilde{E}(D) \;=\; \frac{d}{dd}\{1 - (1-d)^{np}\} \;=\; np(1-d)^{np-1}$$

and the derivative of $E(D)$ is

$$\frac{d}{dd}\,E(D) \;=\; \frac{d}{dd}\{1 - (1-pd)^{n}\} \;=\; np(1-pd)^{n-1}$$

and that both derivatives are nonnegative. At $d = 0$ both derivatives reduce to np, so that $\tilde{E}(D)$ and $E(D)$ start with the same slope.

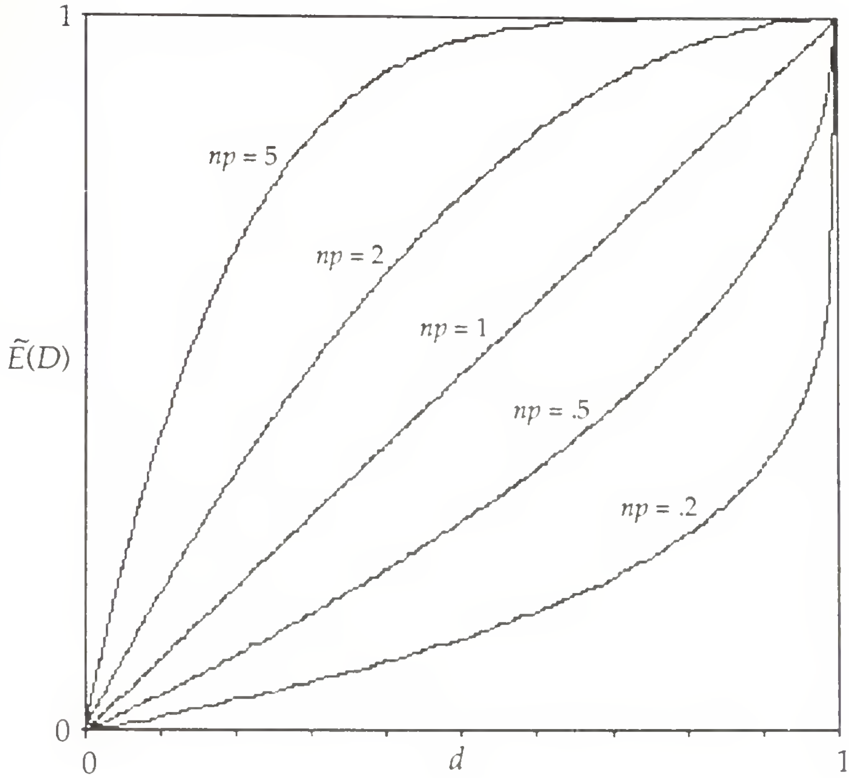
At $d = 1$, $\tilde{E}(D)$ becomes 1 while $E(D)$ is $1 - (1-p)^n$, the probability of scoring at least one hit. The probability of scoring at least one hit is a clear upper bound on $E(D)$, even when total destruction from a hit is assured. Also,

$$\left.\frac{d}{dd}\tilde{E}(D)\right|_{d=1} \;=\; \begin{cases} 0 & \text{if } np > 1 \\ 1 & \text{if } np = 1 \\ \infty & \text{if } np < 1 \end{cases}$$

while

$$\left.\frac{d}{dd}E(D)\right|_{d=1} \;=\; np(1-p)^{n-1}$$

As the values of the derivative of $\tilde{E}(D)$ at $d = 1$ suggest, there are three basic shapes that an $\tilde{E}(D)$ curve can assume. These shapes are illustrated in the following plot.



That $\tilde{E}(D)$ is concave in d for $np \geq 1$ and convex in d for $np \leq 1$ can be confirmed analytically by examining the second derivative of $\tilde{E}(D)$ with respect to d.

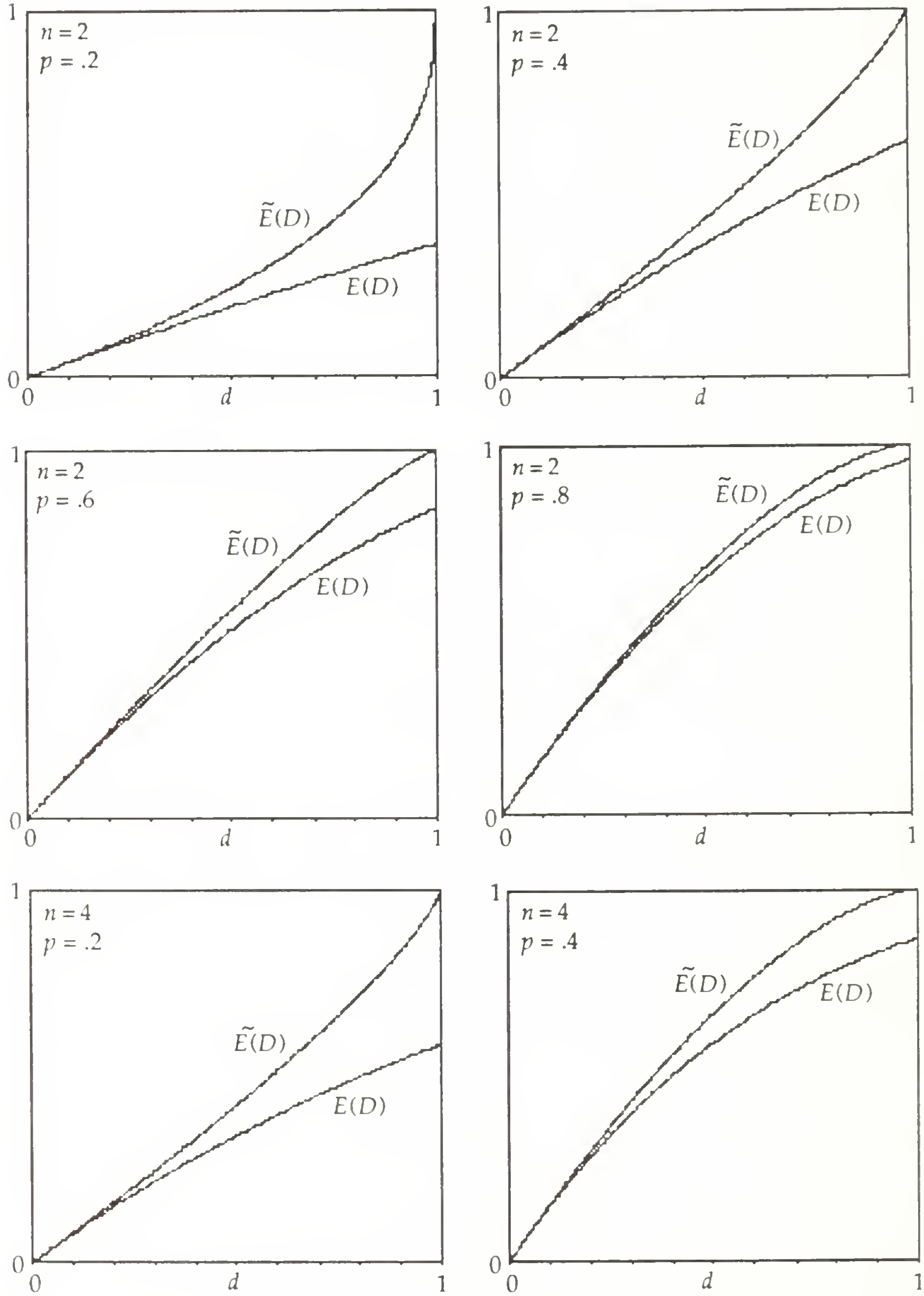
$$\frac{d^2}{dd^2}\tilde{E}(D) = -np(np-1)(1-d)^{np-2} \begin{cases} \leq 0 & \text{if } np \geq 1 \\ \geq 0 & \text{if } np \leq 1 \end{cases}$$

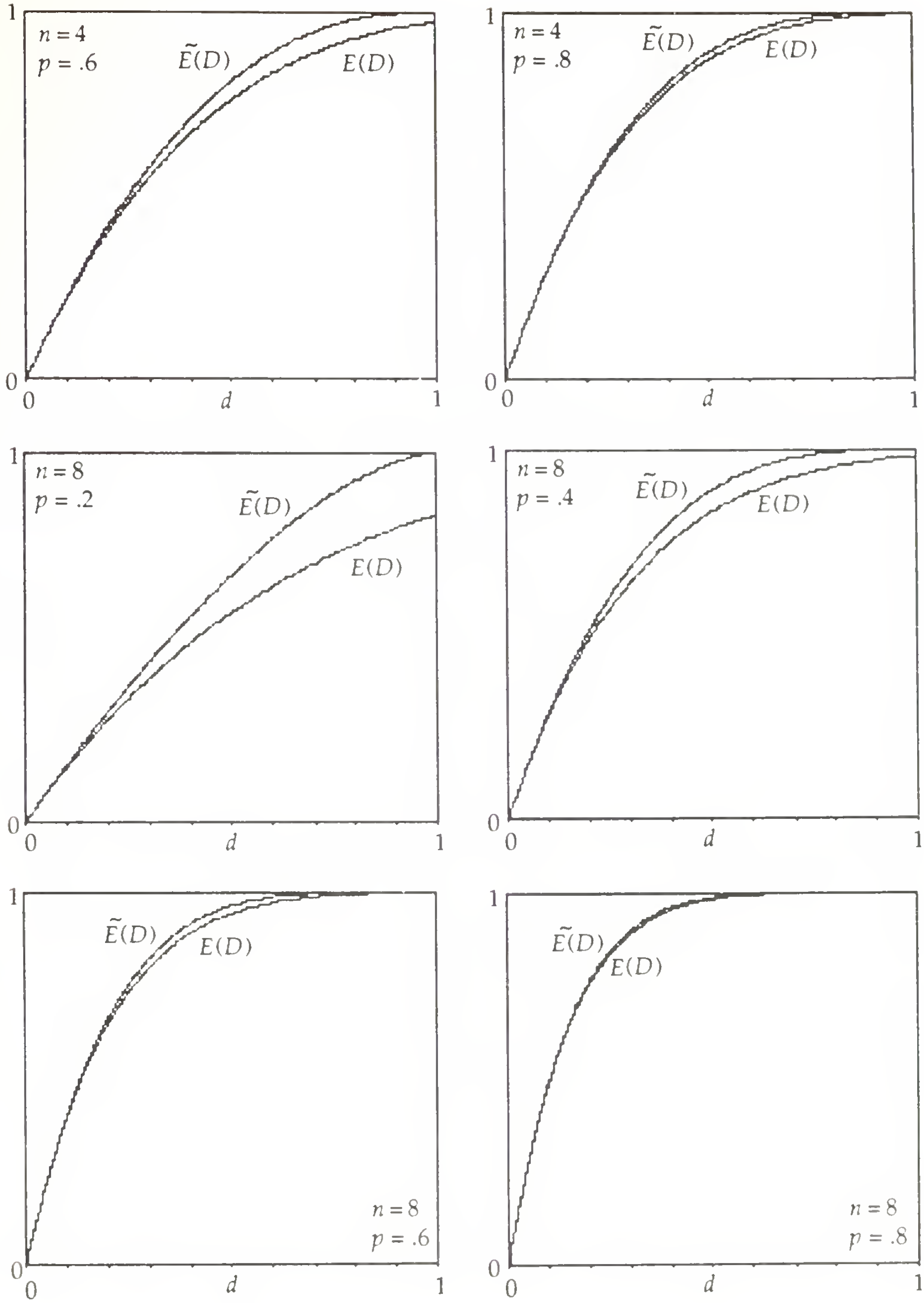
On the other hand, $E(D)$ is seen always to be concave in d by examination of its second derivative with respect to d.

$$\frac{d^2}{dd^2}E(D) = -np(n-1)(1-pd)^{n-2} \leq 0$$

It is reasonable to suspect that $\tilde{E}(D)$ will approximate $E(D)$ poorly for large values of d when $\tilde{E}(D)$ is convex and $E(D)$ is concave, i.e when the curves break oppositely from their initial common slope.

The relative behavior of $\tilde{E}(D)$ and $E(D)$ in selected cases is shown in the following plots.





References

[1] Wayne P. Hughes, Capt., USN(Ret.), Chair of Tactical Analysis (Emeritus) and Adjunct Professor, Naval Postgraduate School.

Damage aggregation for a weapons salvo by an empirical rule related to the Poisson approximation to the binomial

1. Introduction

This working paper follows a previous working paper *"A comparison of an empirical rule for aggregating damage from a weapons salvo to a plausible model for the same purpose"* [1]. The material in the next three paragraphs of this introduction is summarized from the previous paper and is discussed in greater detail there.

The scenario considered is that a salvo of n weapons is launched against a target. The number of weapons that hit the target is a random variable N with possible values $0, 1, \ldots, n$. Possible damage to the target is measured as a percentage (or proportion) of the whole ranging from 0% to 100%. The damage to a pristine target resulting from a single hit is a deterministic proportion d of the whole. The aggregate proportion of damage to the target from the salvo is a random variable D, the randomness in D resulting from the randomness in the number of hits N.

The premise of the model for damage aggregation is that *if the proportion of a pristine target that is damaged by a single hit is d, then each additional hit damages the same proportion d of that part of the target not previously damaged.* Thus if $D(k)$ is the aggregate proportion of damage to a pristine target from exactly k hits, then

$$D(k) = 1 - (1-d)^k, \quad k = 0, \ldots, n$$

The objective is to predict the expected proportion of damage $E(D)$ to the target resulting from the salvo. This prediction can involve modeling of the probability distribution of the number of hits N. If it is assumed that each weapon in the salvo hits independently with the same probability p, so that the number of hits N has a binomial probability distribution, then

$$E(D) = 1 - (1-pd)^n$$

The purpose of this paper is to show that when N has a binomial distribution so that the expected number of hits is $E(N) = np$, the empirical rule

$$\hat{E}(D) = 1 - e^{-E(N)d}$$

is a conservative approximation to $E(D)$.

Like the empirical rule considered in the previous paper, which is an optimistic approximation to $E(D)$, this rule depends only on $E(N)$. Since the exact formula for $E(D)$ is simple in the binomial case, much of the interest in the empirical rules will focus on their behavior when the distribution of hits is not binomial.

2. Derivation of $\hat{E}(D)$

The familiar Poisson approximation to the binomial distribution arises in the case that the number of trials (salvo size) n approaches infinity, while the probability of success (hitting the target) p on a single trial approaches zero in such a way that the expected number of hits maintains the constant value

$$E(N) = np = \lambda$$

Then the binomial probability of exactly k hits approaches the Poisson probability

$$\mathrm{P}[N=k] = \frac{\lambda^k}{k!} e^{-\lambda}, \qquad k = 0, 1, \ldots$$

If the number of hits N is assumed to have a Poisson distribution with parameter $E(N) = \lambda$, then the expected damage to the target turns out to be

$$\hat{E}(D) = \sum_{k=0}^{\infty} D(k)\, \mathrm{P}[N=k] = \sum_{k=0}^{\infty} \{1 - (1-d)^k\} \frac{\lambda^k}{k!} e^{-\lambda}$$

$$= \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} e^{-\lambda} - \sum_{k=0}^{\infty} (1-d)^k \frac{\lambda^k}{k!} e^{-\lambda}$$

$$= 1 - e^{-\lambda} \sum_{k=0}^{\infty} \frac{\{(1-d)\lambda\}^k}{k!} = 1 - e^{-\lambda} e^{+(1-d)\lambda}$$

$$= 1 - e^{-d\lambda} = 1 - e^{-E(N)d}$$

3. *Properties of* $\hat{E}(D)$

That $\hat{E}(D)$ is a conservative approximation to $E(D)$ when the true distribution of N is binomial with $E(N) = np = \lambda$ follows from the fact that $(1-\frac{\lambda}{n}d)^n$ converges to $e^{-d\lambda}$ from below, so that $E(D) = 1 - (1-\frac{\lambda}{n}d)^n$ converges to $\hat{E}(D) = 1 - e^{-d\lambda}$ from above, or from the sequence of analytical comparisons

$$E(D) \geq \hat{E}(D) \Leftrightarrow 1 - (1-pd)^n \geq 1 - e^{-npd}$$

$$\Leftrightarrow e^{-npd} \geq (1-pd)^n \Leftrightarrow e^{-pd} \geq 1-pd$$

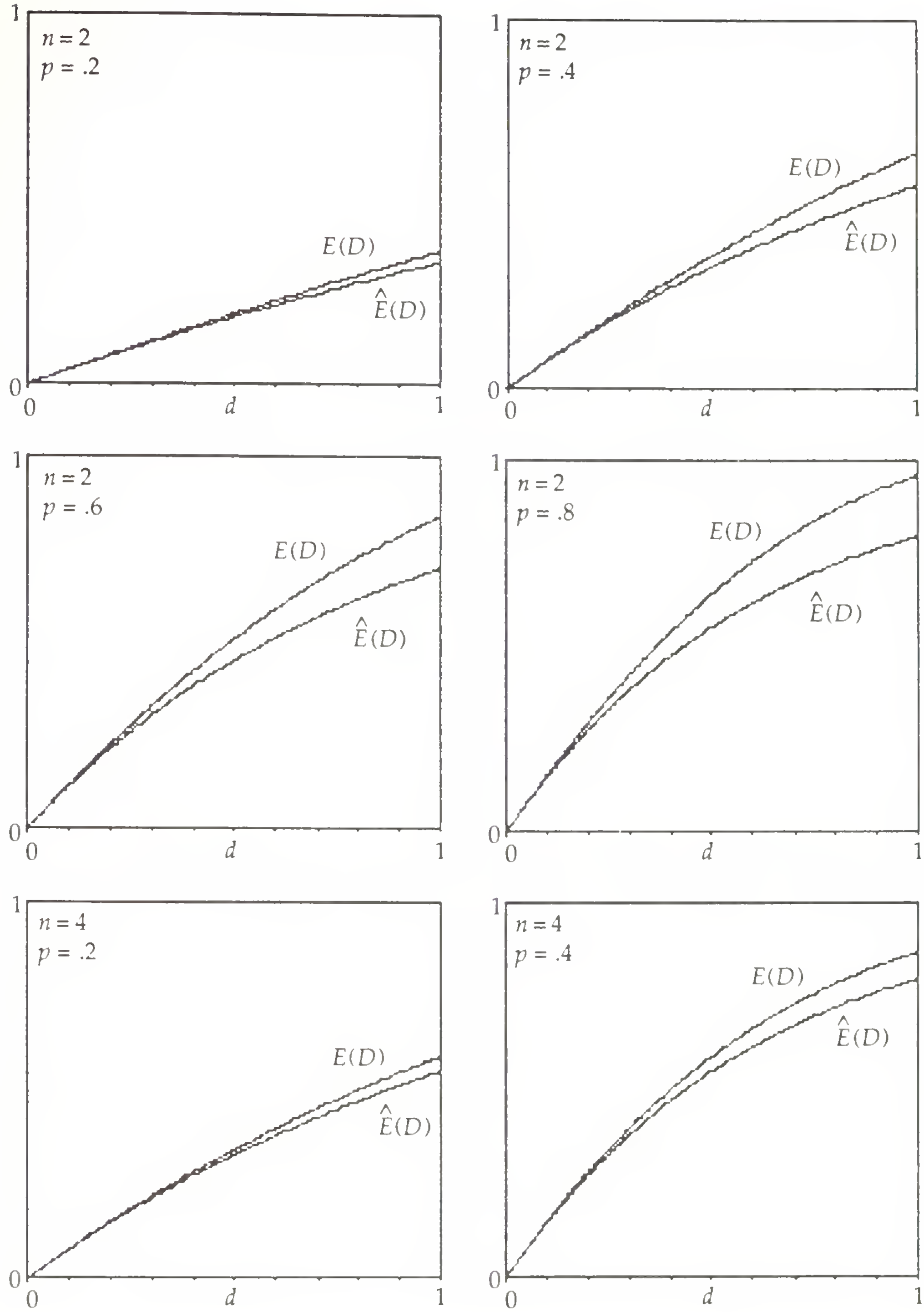
The final comparison is an application of a standard inequality about the first two terms of the power series expansion of e^{-x}.

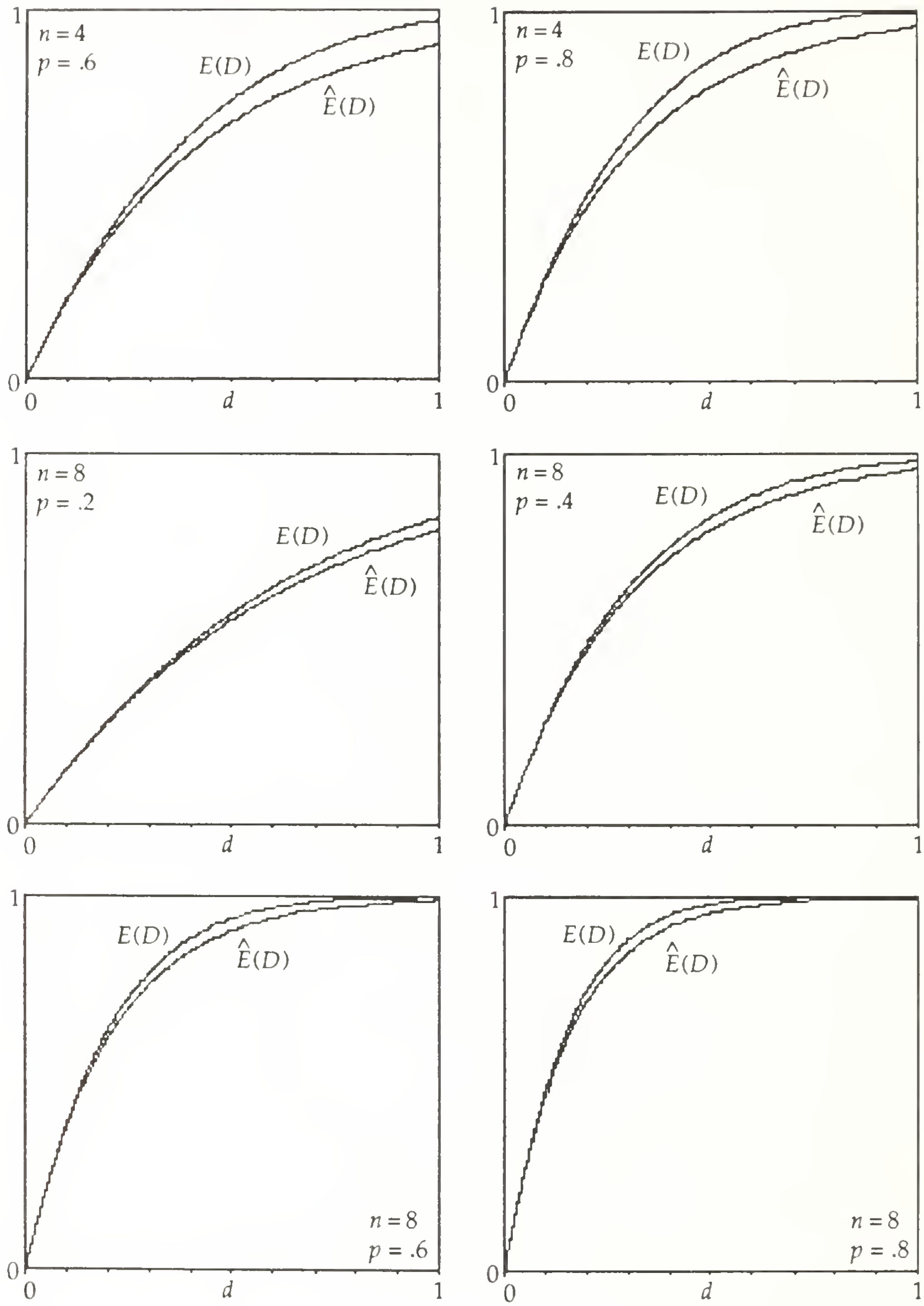
As in the previous paper, we can hold n and p fixed and study the behavior of $\hat{E}(D)$ as d varies from 0 to 1. At $d = 0$, $\hat{E}(D) = 0$. The derivative of $\hat{E}(D)$ is

$$\frac{d}{dd}\hat{E}(D) = np\, e^{-npd}$$

This derivative reduces to np at $d = 0$, so that $E(D)$, $\hat{E}(D)$, and the empirical rule considered in the previous paper, all begin at zero with the same slope. It is clear by inspection that $\hat{E}(D)$ is increasing and concave.

The relative behavior of $\hat{E}(D)$ and $E(D)$ in selected cases is shown in the following plots.


n = 2
p = .2
E(D)
Ê(D)
n = 2
p = .4
n = 2
p = .6
n = 2
p = .8
n = 4
p = .2
n = 4
p = .4
d
0
1



References

[1] J. D. Esary *A comparison of an empirical rule for aggregating damage from a weapons salvo to a plausible model for the same purpose.* Working Paper on Damage Aggregation, Naval Postgraduate School, April, 1989.

z.A stochastic model for hit overlap in a weapons salvo directed against an area target that leads to a proportional mechanism for damage aggregation

1. Introduction

This working paper follows two previous working papers *"A comparison of an empirical rule for aggregating damage from a weapons salvo to a plausible model for the same purpose"* [1] and *"Damage aggregation for a weapons salvo by an empirical rule related to the Poisson approximation to the binomial"* [2]. The material in the next three paragraphs of this introduction is in part summarized from [1] and is discussed in greater detail there.

The scenario that has been considered is that a salvo of n weapons is launched against a target. The number of weapons that hit the target is a random variable N with possible values $0, 1, \dots, n$. Possible damage to the target is measured as a percentage (or proportion) of the whole ranging from 0% to 100%. The damage to a pristine target resulting from a single hit is a deterministic proportion d of the whole. The aggregate proportion of damage to the target from the salvo is a random variable D, the randomness in D resulting from the randomness in the number of hits N.

The premise of the model for damage aggregation has been that *if the proportion of a pristine target that is damaged by a single hit is d, then each additional hit damages the same proportion d of that part of the target not previously damaged.* Thus if $D(k)$ is the aggregate proportion of damage to a pristine target from exactly k hits, then

$$D(k) \; = \; 1 - (1-d)^k, \qquad k = 0, \dots, n$$

This mechanism for aggregating the cumulative effect of hits has been referred to as *plausible* in the previous papers. *A more descriptive terminology would be to call it a* **proportional effects** *mechanism.*

The objective has been to predict the expected proportion of damage $E(D)$ to the target resulting from the salvo. Since

$$E(D) \; = \; \sum_{k=0}^{n} D(k) \, P[N=k]$$

where $P[N{=}k]$ is the probability of exactly k hits from the salvo, this prediction involves modeling the probability distribution of the number of hits N. So far it has been assumed that each weapon in the salvo hits independently with the same probability p, so that the number of hits N has a binomial probability distribution. The impact of this assumption on $E(D)$ is discussed in [1].

Effects mechanisms can be derived from assumptions about the geometry of the target, the coverage of the weapon, and the probabilities of hitting locations within the target area. Then the proportion of damage from k hits becomes a random variable $\Delta(k)$, and with $D(k) = E\{\Delta(k)\}$

$$E(D) = \sum_{k=0}^{n} E\{\Delta(k)\}P[N{=}k] = \sum_{k=0}^{n} D(k)P[N{=}k]$$

Viewing the model at this deeper level of detail can provide a better picture of its applicability.

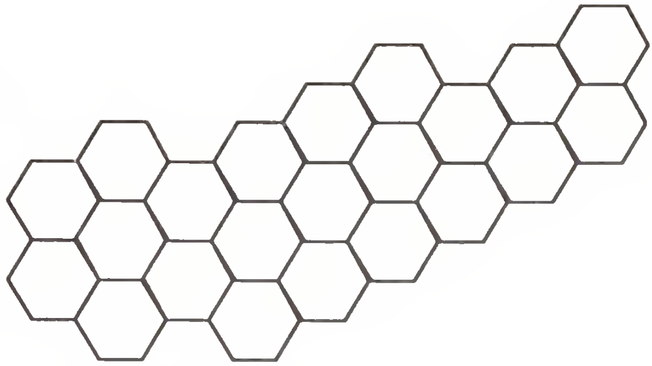
The purpose of this working paper is to present one scenario for weapons overlap on an area target which leads to a proportional effects mechanism in the sense that

$$D(k) = E\{\Delta(k)\} = 1 - (1-d)^k, \quad k = 0, \dots, n$$

for a pertinent value of d and to examine the probability distributions for $\Delta(k)$ that it implies. *Probability distributions for $\Delta(k)$ can be combined with probability distributions for N to obtain deeper rooted probability distributions for D, the proportion of damage to the target.* Given a probability distribution for D, one can set a damage threshold sufficient to meet a tactical goal and predict the probability that the salvo will damage the target in the sense that D achieves the established threshold.

2. A cellular area target scenario

Suppose that an area target is divided into m disjoint cells. Each cell represents a portion of the target which would be damaged by a single weapon which impacts on that cell, so that m nonoverlapping hits (one on each cell) would exactly suffice to totally damage the target.



A cellular area target with $m = 22$

However suppose that each hit from the salvo impacts a randomly chosen cell within the area independently of the cells impacted by the other hits. Then if k hits impact j different cells, the proportion of the target which is damaged is

$$\Delta(k) = \frac{j}{m}$$

If there are no hits ($k = 0$), then $\Delta(0) = 0$ and $D(0) = E\{\Delta(0)\} = 0$. If there is only one hit ($k = 1$), then $\Delta(1) = 1/m$ and $D(1) = E\{\Delta(1)\} = 1/m$. Thus

$$D(0) = 1 - (1-d)^0 = 0$$

$$D(1) = 1 - (1-d)^1 = d$$

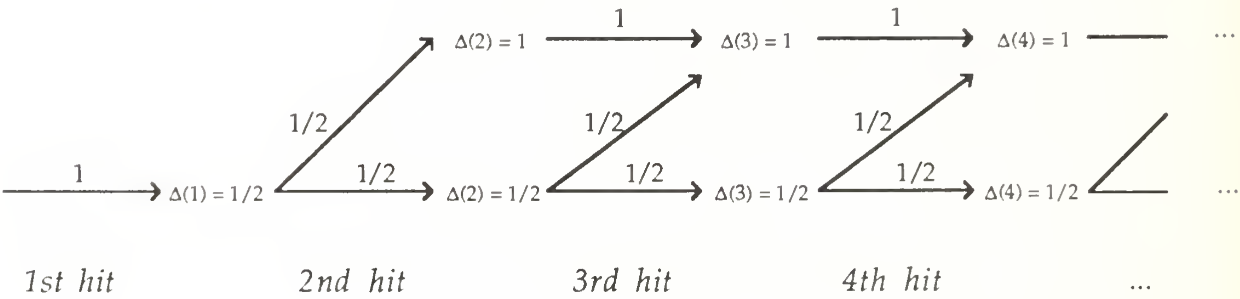
for $d = 1/m$. These formulas for $D(0)$ and $D(1)$ hold regardless of the number of cells in the target.

3. The single cell target

Suppose that a target has only one cell ($m = 1$). Then the target is totally damaged by the first hit, and each subsequent hit has no additional effect. Thus $\Delta(k) = 1/m = 1$, and $D(k) = E\{\Delta(k)\} = 1 = 1 - (1-d)^k$ for $d = 1/m = 1$, $k = 1, 2, \ldots$. *Thus a single cell model leads, in a trivial way, to a proportional effects mechanism.*

4. The two cell target

Suppose that a target has two cells ($m = 2$). Then the possible proportions of the target that can be damaged after a sequence of hits are $1/m = 1/2$ and $2/m = 1$. The aggregation of damage from successive hits can be represented by the simple transition diagram below



where the possible proportions of damage after each hit are indicated by the corresponding values of $\Delta(k)$, the arrows indicate the possible transitions from each damage state to subsequent damage states, and the probabilities of such transitions are shown by labels on the arrows.

The probability that the target is in damage state $\Delta(k) = 1$ or $\Delta(k) = 1/2$ after k hits can be computed by multiplying transition probabilities together along each path leading to the state and then adding up the products. It follows that

$$P[\Delta(1) = 1/2] = 1$$

so that

$$D(1) = E\{\Delta(1)\} = (1/2)\, P[\Delta(1) = 1/2] = 1/2 = 1 - \{1-(1/2)\}^1$$

as previously observed. Continuing

$$P[\Delta(2) = 1/2] = 1/2$$
$$P[\Delta(2) = 1] = 1/2$$

so that

$$D(2) = E\{\Delta(2)\} = (1/2)\, P[\Delta(2) = 1/2] + (1)\, P[\Delta(2) = 1]$$
$$= (1/2)\,(1/2) + (1)\,(1/2) = 3/4 = 1 - \{1-(1/2)\}^2$$

Similarly

$$P[\Delta(3) = 1/2] = 1/4$$
$$P[\Delta(3) = 1] = 3/4$$

so that

$$D(3) = E\{\Delta(3)\} = (1/2)\, P[\Delta(3) = 1/2] + (1)\, P[\Delta(3) = 1]$$
$$= (1/2)\,(1/4) + (1)\,(3/4) = 7/8 = 1 - \{1-(1/2)\}^3$$

In general

$$P[\Delta(k) = 1/2] = (1/2)^{k-1}$$
$$P[\Delta(k) = 1] = 1 - (1/2)^{k-1}$$

so that

$$D(k) = E\{\Delta(k)\} = (1/2)\, P[\Delta(k) = 1/2] + (1)\, P[\Delta(k) = 1]$$
$$= (1/2)\,\{1/2\}^{k-1} + (1)\,\{1-(1/2)\}^{k-1}$$
$$= 1 - (1/2)^k = 1 - \{1-(1/2)\}^k$$

Thus

$$D(k) = 1 - (1-d)^k \qquad k = 0, 1, \ldots$$

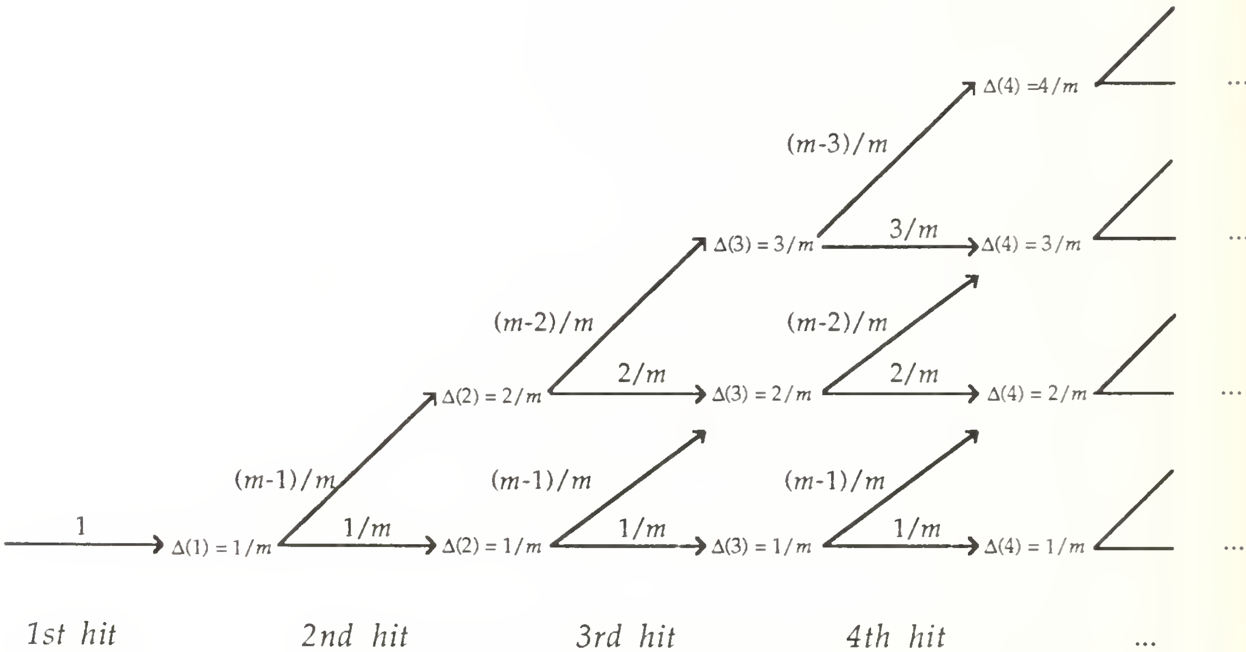
for $d = 1/m = 1/2$ *showing that the two cell model leads to a proportional effects mechanism.*

In the two cell case the probability distribution for $\Delta(k)$ has been particularly easy to compute. That distribution is tabulated below for the first few values of k.

k	0	1	2	3	4	5	6	7	8
$P[\Delta(k) = 1/2]$	0	1	1/2	1/4	1/8	1/16	1/32	1/64	1/128
$P[\Delta(k) = 1]$	0	0	1/2	3/4	7/8	15/16	31/32	63/64	127/128

5. *The m cell target*

Now consider a target with an arbitrary number m of cells. for the The transition diagram for the aggregation of damage from successive hits begins as shown below. The probability of an upward transition from any total damage state $\Delta(k) = m/m = 1$ is zero, so that all upward transitions beyond total damage become impossible.



Again multiplying the transition probabilities along paths leading to a damage state and adding the products, it follows that

$$P[\Delta(1) = \tfrac{1}{m}] = 1$$

so that

$$D(1) = E\{\Delta(1) = \tfrac{1}{m} P[\Delta(1) = \tfrac{1}{m}] = \tfrac{1}{m} = 1 - \left(1 - \tfrac{1}{m}\right)^1$$

as previously observed.

Continuing

$$P[\Delta(2) = \tfrac{1}{m}] = \frac{1}{m}$$
$$P[\Delta(2) = \tfrac{2}{m}] = \frac{m-1}{m}$$

so that

$$D(2) = E\{\Delta(2) = \frac{1}{m}P[\Delta(2) = \tfrac{1}{m}] + \frac{2}{m}P[\Delta(2) = \tfrac{2}{m}] = \frac{1}{m}\left(\frac{1}{m}\right) + \frac{2}{m}\left(\frac{m-1}{m}\right)$$
$$= \frac{2m-1}{m^2} = 1 - \left(1 - \frac{1}{m}\right)^2$$

Further

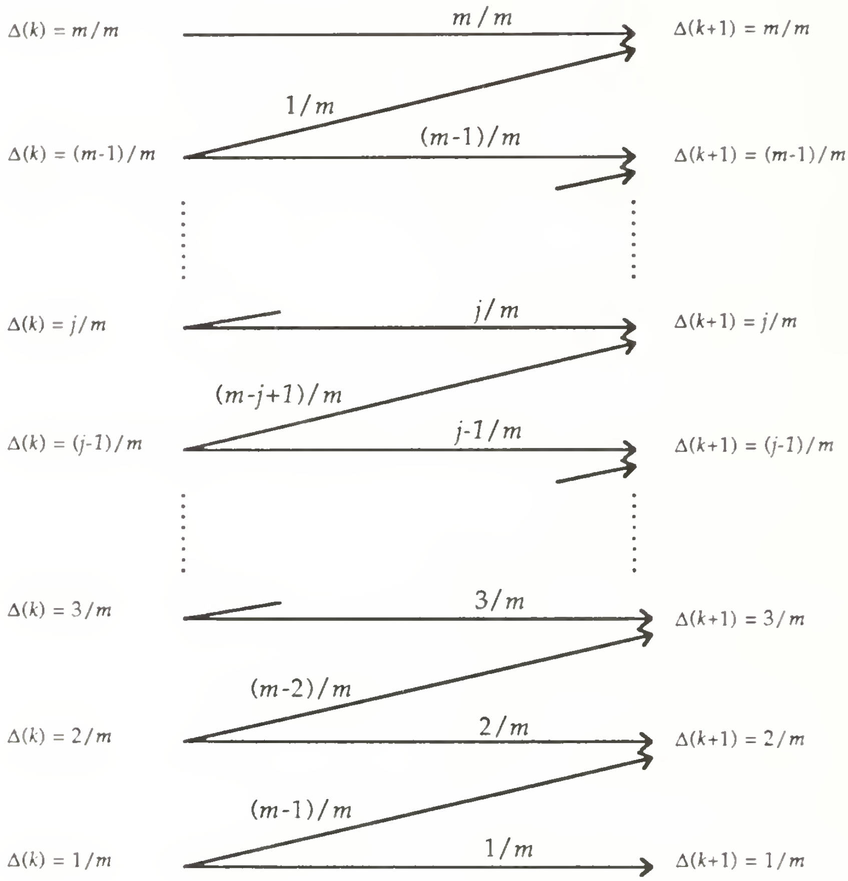
$$P[\Delta(3) = \tfrac{1}{m}] = \left(\frac{1}{m}\right)^2 = \frac{1}{m^2}$$
$$P[\Delta(3) = \tfrac{2}{m}] = \left(\frac{m-1}{m}\right)\left(\frac{2}{m}\right) + \left(\frac{1}{m}\right)\left(\frac{m-1}{m}\right) = \frac{3(m-1)}{m^2}$$
$$P[\Delta(3) = \tfrac{3}{m}] = \left(\frac{m-1}{m}\right)\left(\frac{m-2}{m}\right) = \frac{(m-1)(m-2)}{m^2}$$

so that

$$D(3) = E\{\Delta(3)\} = \frac{1}{m}P[\Delta(3) = \tfrac{1}{m}] + \frac{2}{m}P[\Delta(3) = \tfrac{2}{m}] + \frac{3}{m}P[\Delta(3) = \tfrac{3}{m}]$$
$$= \frac{1}{m}\left(\frac{1}{m^2}\right) + \frac{2}{m}\left(\frac{3(m-1)}{m^2}\right) + \frac{3}{m}\left(\frac{(m-1)(m-2)}{m^2}\right) = \frac{3m^2 - 3m + 1}{m^3}$$
$$= 1 - \left(1 - \frac{1}{m}\right)^3$$

The preceding computations of $D(1)$, $D(2)$, and $D(3)$ *indicate that the damage aggregation mechanism for the* m *cell target is proportional with* $d = 1/m$. It remains to confirm the proportionality of $D(k)$ for an arbitrary k.

The general transition diagram, going from k hits to $k+1$ hits, appears below



It follows that

$$P[\Delta(k+1) = \tfrac{1}{m}] \;=\; \frac{1}{m} P[\Delta(k) = \tfrac{1}{m}]$$

and

$$P[\Delta(k+1) = \tfrac{j}{m}] \;=\; \frac{j}{m} P[\Delta(k) = \tfrac{j}{m}] + \frac{m-j+1}{m} P[\Delta(k) = \tfrac{j-1}{m}] \qquad j = 2, \ldots, m$$

Using the preceding k to $k+1$ transition equations

$$D(k+1) = E\{\Delta(k+1)\} = \sum_{j=1}^{m} \frac{j}{m} P[\Delta(k+1) = \tfrac{j}{m}] = \frac{1}{m} P[\Delta(k+1) = \tfrac{1}{m}] + \sum_{j=2}^{m} \frac{j}{m} P[\Delta(k+1) = \tfrac{j}{m}]$$

$$= \frac{1}{m} \cdot \frac{1}{m} P[\Delta(k) = \tfrac{1}{m}] + \sum_{j=2}^{m} \frac{j}{m} \left\{ \frac{j}{m} P[\Delta(k) = \tfrac{j}{m}] + \frac{m-j+1}{m} P[\Delta(k) = \tfrac{j-1}{m}] \right\}$$

$$= \sum_{j=1}^{m} \frac{j}{m} \cdot \frac{j}{m} P[\Delta(k) = \tfrac{j}{m}] + \sum_{j=2}^{m} \frac{j}{m} \cdot \frac{m-j+1}{m} P[\Delta(k) = \tfrac{j-1}{m}]$$

$$= \sum_{j=1}^{m} \frac{j}{m} \cdot \frac{j}{m} P[\Delta(k) = \tfrac{j}{m}] + \sum_{j=1}^{m-1} \frac{j+1}{m} \cdot \frac{m-j}{m} P[\Delta(k) = \tfrac{j}{m}]$$

$$= \sum_{j=1}^{m} \frac{j}{m} \cdot \frac{j}{m} P[\Delta(k) = \tfrac{j}{m}] + \sum_{j=1}^{m} \frac{j+1}{m} \cdot \frac{m-j}{m} P[\Delta(k) = \tfrac{j}{m}] - \frac{m+1}{m} \cdot \underset{\uparrow}{\frac{m-m}{m}} P[\Delta(k) = \tfrac{m}{m}]$$

$$= \sum_{j=1}^{m} \frac{j}{m} \cdot \frac{j}{m} P[\Delta(k) = \tfrac{j}{m}] + \sum_{j=1}^{m} \frac{j+1}{m} \cdot \frac{m-j}{m} P[\Delta(k) = \tfrac{j}{m}]$$

since the term indicated by an arrow is zero. Continuing

$$D(k+1) = \sum_{j=1}^{m} \frac{j}{m} \cdot \frac{j}{m} P[\Delta(k) = \tfrac{j}{m}] + \sum_{j=1}^{m} \frac{j+1}{m} \cdot \frac{m-j}{m} P[\Delta(k) = \tfrac{j}{m}]$$

$$= \sum_{j=1}^{m} \left\{ \frac{j}{m} \cdot \frac{j}{m} + \frac{j+1}{m} \cdot \frac{m-j}{m} \right\} P[\Delta(k) = \tfrac{j}{m}]$$

$$= \sum_{j=1}^{m} \left\{ \frac{m-1}{m} \cdot \frac{j}{m} + \frac{1}{m} \right\} P[\Delta(k) = \tfrac{j}{m}]$$

$$= \frac{m-1}{m} \sum_{j=1}^{m} \frac{j}{m} P[\Delta(k) = \tfrac{j}{m}] + \frac{1}{m} \sum_{j=1}^{m} P[\Delta(k) = \tfrac{j}{m}]$$

$$= \frac{m-1}{m} E\{\Delta(k)\} + \frac{1}{m} \cdot 1 = \frac{m-1}{m} D(k) + \frac{1}{m}$$

The recursive relation

$$D(k+1) = \frac{m-1}{m} D(k) + \frac{1}{m}$$

which it turns out holds for $k = 0, 1, \ldots$, permits the verification of proportionality, since if

$$D(k) = 1 - \left(1 - \frac{1}{m}\right)^k$$

then

$$\begin{aligned} D(k+1) &= \frac{m-1}{m}\left\{1 - \left(1 - \frac{1}{m}\right)^k\right\} + \frac{1}{m} \\ &= \left(1 - \frac{1}{m}\right)\left\{1 - \left(1 - \frac{1}{m}\right)^k\right\} + \frac{1}{m} \\ &= \left(1 - \frac{1}{m}\right) - \left(1 - \frac{1}{m}\right)^{k+1} + \frac{1}{m} \\ &= 1 - \left(1 - \frac{1}{m}\right)^{k+1} \end{aligned}$$

and we have already verified proportionality for $k = 1, 2$, and 3. *This completes the demonstration that the damage aggregation mechanism for the m cell target is proportional with $d = 1/m$.*

6. Damage distributions for the m cell target

The k hits to $k+1$ hits transition equations provide a means for the calculation of probability distributions for $\Delta(k)$, $k = 1, 2, \ldots$. For this purpose these equations are perhaps more conveniently written as

$$P[\Delta(k+1) = \tfrac{1}{m}] = \frac{1}{m} P[\Delta(k) = \tfrac{1}{m}]$$

$$P[\Delta(k+1) = \tfrac{j}{m}] = \frac{j}{m} P[\Delta(k) = \tfrac{j}{m}] + \left(1 - \frac{j-1}{m}\right) P[\Delta(k) = \tfrac{j-1}{m}] \qquad j = 2, 3, \ldots, m$$

The numerical table which follows for $\Delta(k)$ when $m = 3$ was obtained by their recursive application.

k	1	2	3	4	5	6	7	8	9	10
$P[\Delta(k) = 1/3]$	1	1/3	1/9	1/27	1/81	1/243	1/729	1/2187	1/6561	1/19683
$P[\Delta(k) = 2/3]$	0	2/3	6/9	14/27	30/81	62/243	126/729	254/2187	510/6561	1022/19683
$P[\Delta(k) = 1]$	0	0	2/9	12/27	50/81	180/243	180/729	1932/2187	6050/6561	18660/19683

Distributions for $\Delta(1)$, $\Delta(2)$, and $\Delta(3)$ were obtained in algebraic form in Section 5. They are repeated in the table which follows. The distribution for $\Delta(4)$ was obtained from the $k = 3$ to $k = 4$ transition equations.

k	1	2	3	4
$P[\Delta(k) = 1/m]$	1	$\frac{1}{m}$	$\frac{1}{m^2}$	$\frac{1}{m} \cdot \frac{1}{m^2} = \frac{1}{m^3}$
$P[\Delta(k) = 2/m]$	0	$\frac{m-1}{m}$	$\frac{3(m-1)}{m^2}$	$\frac{2}{m} \cdot \frac{3(m-1)}{m^2} + \left(1 - \frac{1}{m}\right) \cdot \frac{1}{m^2} = \frac{7(m-1)}{m^3}$
$P[\Delta(k) = 3/m]$	0	0	$\frac{(m-1)(m-2)}{m^2}$	$\frac{3}{m} \cdot \frac{(m-1)(m-2)}{m^2} + \left(1 - \frac{2}{m}\right) \cdot \frac{3(m-1)}{m^2} = \frac{6(m-1)(m-2)}{m^3}$
$P[\Delta(k) = 4/m]$	0	0	0	$\frac{4}{m} \cdot 0 + \left(1 - \frac{3}{m}\right) \frac{(m-1)(m-2)}{m^2} = \frac{(m-1)(m-2)(m-3)}{m^3}$
$P[\Delta(k) = 5/m]$	0	0	0	0
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

In the preceding tables it is understood that $P[\Delta(0) = 0] = 1$ and $P[\Delta(k) = 0] = 0, k = 1, 2, \ldots$.

Recursive application of the transition equations computes that $P[\Delta(k) = j/m] = 0$ whenever j is greater than m. The formulas in the algebraic table reduce to 0 whenever j is greater than m.

A possible algorithm for automating the computation of distributions for the $\Delta(k)$'s is discussed in the appendix to this paper.

Appendix - A posssible algorithm for computing damage probability distributions.

The purpose of this appendix is to describe an approach to generating probability distributions for the $\Delta(k)$'s (the proportions of damage to an m celled target resulting from k hits) which has some potential for efficiency and the control of numerical error.

The following expressions for $P_k[\frac{j}{m}] = P[\Delta(k) = \frac{j}{m}]$ were derived from the recursion equations at the beginning of Section 6, with the initial conditions $P_1[\frac{1}{m}] = 1$ and $P_1[\frac{j}{m}] = 0, j = 2, 3, \ldots$, using the symbolic algebra software *Theorist* [3].

$$P_2[\tfrac{1}{m}] = \frac{1}{m} \qquad P_2[\tfrac{2}{m}] = \frac{m-1}{m}$$

$$P_3[\tfrac{1}{m}] = \frac{1}{m^2} \qquad P_3[\tfrac{2}{m}] = 3\frac{m-1}{m^2} \qquad P_3[\tfrac{3}{m}] = \frac{[m-2][m-1]}{m^2}$$

$$P_4[\tfrac{1}{m}] = \frac{1}{m^3} \qquad P_4[\tfrac{2}{m}] = 7\frac{m-1}{m^3}$$

$$P_4[\tfrac{3}{m}] = 6\frac{[m-2][m-1]}{m^3} \qquad P_4[\tfrac{4}{m}] = \frac{[m-3][m-2][m-1]}{m^3}$$

$$P_5[\tfrac{1}{m}] = \frac{1}{m^4} \qquad P_5[\tfrac{2}{m}] = 15\frac{m-1}{m^4} \qquad P_5[\tfrac{3}{m}] = 25\frac{[m-2][m-1]}{m^4}$$

$$P_5[\tfrac{4}{m}] = 10\frac{[m-3][m-2][m-1]}{m^4} \qquad P_5[\tfrac{5}{m}] = \frac{[m-4][m-3][m-2][m-1]}{m^4}$$

$$P_6[\tfrac{5}{m}] = \frac{1}{m^5} P_6[\tfrac{2}{m}] = 31\frac{m-1}{m^5} \qquad P_6[\tfrac{3}{m}] = 90\frac{[m-2][m-1]}{m^5}$$

$$P_6[\tfrac{4}{m}] = 65\frac{[m-3][m-2][m-1]}{m^5} \qquad P_6[\tfrac{5}{m}] = 15\frac{[m-4][m-3][m-2][m-1]}{m^5}$$

$$P_6[\tfrac{6}{m}] = \frac{[m-5][m-4][m-3][m-2][m-1]}{m^5}$$

Each $P_k[\frac{j}{m}]$ has the structure

$$P_k[\tfrac{j}{m}] = C_k[j] \cdot \frac{(m-1)(m-2)\cdots(m-j+1)}{m^{k-1}} = C_k[j] \cdot \frac{m!}{m^k (m-j)!}$$

where $C_k[j]$ is a coefficient depending only on k and j, a fact that can be confirmed by inspecting the m celled transition diagram at the beginning of Section 5.

It appears that the generation of the $P_k[\frac{j}{m}]$*'s can be approached by seperately computing the terms* $\frac{(m-1)(m-2)\cdots(m-j+1)}{m^{k-1}}$ *and the coefficients* $C_k[j]$ *using the modified version of the transition equations*

$$C_{k+1}[1] = C_k[1]$$

$$C_{k+1}[j] = jC_k[j] + C_k[j-1]$$

with the initial conditions $C_1[1] = 1$ and $C_1[j] = 0, j = 2, 3, \ldots$.

The following table of coefficients $C_k[j]$ was derived from the preceding coefficient transition equations.

k	2	3	4	5	6	7	8	9	10
$C_k[1]$	1	1	1	1	1	1	1	1	1
$C_k[2]$	1	3	7	15	31	63	127	255	511
$C_k[3]$	0	1	6	25	90	301	966	3025	9330
$C_k[4]$	0	0	1	10	65	350	1701	7770	34105
$C_k[5]$	0	0	0	1	15	140	1050	6951	42525
$C_k[6]$	0	0	0	0	1	21	266	2646	22827
$C_k[7]$	0	0	0	0	0	1	28	462	5880
$C_k[8]$	0	0	0	0	0	0	1	36	750
$C_k[9]$	0	0	0	0	0	0	0	1	45
$C_k[10]$	0	0	0	0	0	0	0	0	1

References

[1] J. D. Esary. *A comparison of an empirical rule for aggregating damage from a weapons salvo to a plausible model for the same purpose.* Working Paper on Damage Aggregation, Naval Postgraduate School, April 1989.

[2] J. D. Esary. *Damage aggregation for a weapons salvo by an empirical rule related to the Poisson approximation to the binomial.* Working Paper on Damage Aggregation, Naval Postgraduate School, April 1989.

[3] *Theorist.* Presience Corporation, San Francisco, 1989.

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