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SOLVING THE LINEAR BALANCE EQUATION GLOBALLY

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20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

The linear balance equation, which is used in meteorology to relate the geopotential ϕ and the stream function Ψ, is studied. The analytic solutions to the homogeneous equation are analyzed and shown to have singularities which influence numerical integration. A new approach, using finite elements, is proposed.

In order to determine the wind field in the lower atmosphere over the entire globe one common technique is to solve the "linear balance equation",

$$f\nabla^2\Psi + \vec{\nabla}\Psi \cdot \vec{\nabla f} = g(\Phi) \tag{1}$$

where Ψ is the stress function whose gradient gives the wind field, f is the Coriolis function

$$f = 2\Omega \sin \phi , \tag{2}$$

$g(\Phi)$ is a known function of the (known) geopotential Φ, Ω is the earth's angular velocity and ϕ is the latitude (polar) angle measured from the equator. Equation (1) is to be solved on the surface of a sphere of radius a and so is written

$$\frac{2\Omega}{a^2} \sin \phi \left\{ \frac{1}{\cos \phi} \frac{\delta}{\delta\phi} (\cos \phi \frac{\delta \Psi}{\delta\phi}) + \frac{1}{\cos^2\phi} \frac{\delta^2\Psi}{\delta\lambda^2} \right\} + \frac{2\Omega \cos \phi}{a^2} \frac{\delta\Psi}{\delta\phi} = g(\Phi) . \tag{3}$$

In practical applications (3) is solved numerically, by finite difference methods, and the vanishing coefficient of the first term for small ϕ causes real numerical difficulties. (The question of whether (1) is an appropriate equation to solve near the equator (for small ϕ) is another argument to which we return later.) One conventional scheme to avoid these problems is to replace $\frac{2\Omega \sin \phi}{a^2}$ by a constant for small ϕ and in this way continue the solution of (3) for high latitudes to low latitudes by solving

$$f_o \nabla^2\Psi + \vec{\nabla}\Psi \cdot \vec{\nabla f} = g(\Phi) \tag{4}$$

for small angles ϕ . It is the purpose of this note to study the solutions of homogeneous versions of (1) and (4) in hopes of saying something about the solutions of the linear balance equation and methods for solving it.

We are interested in the homogeneous equation

$$\sin\phi \left\{ \frac{1}{\cos\phi} \frac{\delta}{\delta\phi} (\cos\phi \frac{\delta\Psi}{\delta\phi}) + \frac{1}{\cos^2\phi} \frac{\delta^2\Psi}{\delta\lambda^2} \right\} + \cos\phi \frac{\delta\Psi}{\delta\phi} = 0 \tag{5}$$

(where we have dropped the $2\Omega/a^2$) all the way around the globe, so we may expect periodicity on λ . We thus let

$$\Psi = e^{i\lambda n} V(\phi) \tag{6}$$

We also will change the independent variable ϕ by the obvious choice

$$x = \sin\phi , \tag{7}$$

and write $V(\phi) \equiv y(x)$. Then (5) becomes

$$x \left\{ \frac{d}{dx} \left[(1-x^2) \frac{dy}{dx}\right] - \frac{n^2}{1-x^2} y \right\} + (1-x^2) \frac{dy}{dx} = 0 \tag{8}$$

Guided by a suggestion in [1], we make one further change of variables

$$y(x) = (1-x^2)^{n/2} u(x) \tag{9}$$

to get

$$x(1-x^2) \frac{d^2u}{dx^2} + [1 - (2n+3)x^2] \frac{du}{dx} - (n^2+2n)xu = 0 . \tag{10}$$

Now (10) turns out to be an equation studied almost 100 years ago by Heun (see [1]), and is known as Heun's equation, or occasionally as Lamé's equation. (For reference we list Babster's parameters [1] for (10): $a = -1$, $b = 0$, $\alpha = n$, $\beta = n+2$, $\gamma = 1$, $\delta = n+1$, $\varepsilon = n+1$). It is known that (10) has regular singular points at $x = \pm 1$ and $x = \infty$, and $x = 0$. Using the method of Frobenius [2] we can generate modified power series about each of these points of the form

$$(x - x_0)^r \sum_{m=0}^{\infty} c_m(x - x_0)^m .$$

About the origin $x = 0$ (the equator) the indicial equation for r has a double root of zero. About the points $x = \pm 1$ (the North and South poles) the indicial equation has roots $r = 0$, $r = -n$.

The first zero value for r at $x = 0$ means that there is a solution of (10) about $x = 0$ which is an ordinary power series. This solution turns out to be an even function of x and is given by

$$u = u_1(x) = \sum_{m=0}^{\infty} c_m x^{2m} \tag{11}$$

$$c_{m+1} = \frac{(m + {}^n/2)(m + 1 + {}^n/2)}{(m + 1)^2} c_m \tag{12}$$

The ratio test applied to (12) shows that (11) converges for $|x| < 1$, and the test fails for $|x| = 1$. A more subtle analysis shows that $u_1(x)$ converges for $|x| = 1$. The second zero value for r at $x = 0$ means that there is a logrithmic solution near $x = 0$, which can be found by the method of reduction of order [2]. This solution is given by

$$u_2(x) = u_1(x) \int^x \frac{(-1)^{n+1}\, dz}{z(1-z^2)^{n+1}\, u_1^2(z)} \, . \tag{13}$$

An analysis of (13) shows that $u_2(x)$ does have a logarithmic singularity at $x = 0$, <u>and</u>, since $u_1(x)$ is finite at $x = \pm 1$, $y_2(x)$ is singular there, behaving like $(1 \pm x)^{-n}$.

This is in agreement with the indicial equation for (10) at $x = \pm 1$, which says that there is one solution which behaves as a constant (the $r = 0$) and one which is singular like $(1 \pm x)^{-n}$. Thus the general solution to (8) can be written

$$y(x) = (1 - x^2)^{n/2} \left(c_1 u_1(x) + c_2 u_2(x)\right) \tag{14}$$

where $u_1(x)$ is finite everywhere. Thus $u_2(x)$ is singular at $x = 0$ and, at $x = \pm 1$, is more singular than $(1-x^2)^{-n/2}$. Thus the physically relevant version of (14) must have $c_2 \equiv 0$, so that

$$\Psi(\phi,\lambda) = \sum_{-\infty}^{\infty} e^{in\lambda} (\cos\phi)^{n/2} c_n u_1(\sin\phi) \tag{15}$$

where the c_n are the complex amplitudes of the longitudinal harmonics.

Equation (15) is the solution to the homogeneous equation (5) and is not really useful, in itself, for solving the linear balance equation (1). However, (14) tells us a great deat about solving (1) numerically. If one started at the poles or the equator ($x = \pm 1$ or $x = 0$) with a known (finite) value of Ψ then c_2 is necessarily zero. Then taking the exact solution, c_2 remains zero. However, integrating numerically, one does not have the exact solution, and the roundoff errors effectively introduce a small amount of u_2 into the solution. This would remain small if u_2 were bounded. But integrating towards the equator (or the poles) the unphysical solution u_2 must eventually dominate. In other words, no matter how careful an integration scheme is used, because of finite precision in any computer, there is no way to integrate the balance equation (1) from the poles to the equator and have a realistic solution at the equator.

This has been instinctively recognized, a common solution method for the Northern Hemisphere models [3] has been to stop integrating (1) at about 20 degrees latitude and instead solve

$$2\Omega k\nabla^2\Psi + \vec{\nabla} f \cdot \vec{\nabla}\Psi = g(\Phi) \tag{16}$$

where k is $\sin(20^\circ)$. What can one say about the homogeneous version of (16)? Using the same substitutions as above we obtain, for the homogeneous equation,

$$(1-x^2)u'' - \frac{1}{k}\left[x^2 + 2k(n+1)x - 1\right]u' - \frac{n}{k}(x + kn + n)u = 0 \tag{17}$$

Equation (17) is not a standard equation, but is amenable to a standard power series expansion about the origin

$$u = \sum a_m x^m. \tag{18}$$

We then obtain the recursion relations

$$\begin{aligned} 2a_2 &= -\frac{1}{k}a_1 + n(n+1)a_0 \\ 6a_3 &= -\frac{2}{k}a_2 + (n^2 + 3n + 2)\,a_1 + \frac{n}{k}a_0 \end{aligned} \tag{19}$$

$$m(m-1)a_m = -\frac{(m-1)}{k}a_{m-1} + \left[n^2 + (2m-3)n + (m^2-3m-2)\right]a_{m-2} + \frac{(n+m-3)}{k}a_{m-3}\ .$$

These involved equations show two things. First, if k is small (considerably less than 1), they tend to simplify, and this becomes a "singular perturbation" difference equation. (This is an almost unknown area which the author intends to investigate later.) Secondly, we note that no solution of (17) is inherently odd or even, in contrast to both solutions of (18) which are even (see (11) and (13)). Thus, although any function can always be written as the sum of an odd and an even function

$$f(x) \equiv \frac{1}{2}\left\{f(x) + f(-x)\right\} + \frac{1}{2}\left\{f(x) - f(-x)\right\},$$

numerical integration of (16) will always introduce some odd elements of the solution. Since the equation one is really trying to solve is inherently even, these portions of the solution are unphysical. Since all solutions of (17) are bounded at $x = 0$, the unwanted portion of the solution does not grow drastically, so this is not as serious as trying to integrate (8). However, it is expected that a singular perturbation analysis of the difference equations (19) would show that the smaller the value of k, the more closely the solution to (17) will display the purely even character which (8) has.

The non-homogenous versions (1) and (16) could be solved in power series in $x = \sin \phi$ and Fourier series in λ. The resultant double series show exactly the features above. With care, obviously an analytic solution in terms of a double series is possible. However, it certainly would be difficult to get reasonable answers from such an approach. The purpose of this exercise was to demonstrate that the exact solutions have inherent problems and thus any attempt to directly integrate the equations will give difficulty.

The author would like to suggest an alternate approach to this problem which ought to be more stable, and which, in other applications, is often faster - the finite element method. This method has been applied to some elementary meteorological problems [4]. The basic idea is as follows: instead of taking an infinite series valid over the entire domain, one expands Ψ as a collection of low order polynomials - a different polynomial for each grid rectangle in some discretization of the globe. The coefficients for such a ploynomial are determined by an integration of the trial solution, times suitable functions, over the entire globe. The justification for the computations is either an orthogonalization one (Galerkin) or a variational calculus one (Rayleigh-Ritz). The latter is more in keeping with the use of NVA (numerical variational analysis) now used in meteorology, although it will cause a slight problem later.

For a given function $\Phi(\phi,\lambda)$ it is easy to see that (1) is the Euler equation for

$$I(\Psi)= \iint_D \left\{f(\vec{\nabla}\Psi)^2 + 2g(\Phi)\Psi\right\} a^2 \cos\phi \, d\phi \, d\lambda \quad . \tag{20}$$

where D is the portion of the globe of interest. If we superimpose on D a set of grid points and connect these to cover D with triangles (as opposed to the rectangles normally used in finite difference solutions of (1)) we have a set of domains D_i. In <u>each</u> of these domains we express Ψ as a simple function Ψ_i (a low order polynomial) whose coefficients are to be determined. <u>Some</u> of the coefficients are determined by the need for continuity of Ψ and some of its derivatives across the boundaries of D_i. The remaining coefficients are determined by minimizing (20). The result is a matrix equation for these coefficients which is similar, but <u>not</u> identical, to the finite difference equations for (1). In particular, the fact that one is integrating (20) over an area means that the vanishing of f on a line (the equator) causes less problem than before.

Experiments run with problems in stress-strain mechanics indicate that using linear functions for the Ψ_i, or perhaps quadratics, will give excellent accuracy. The author has not yet run such an experiment for (20).

A question with using (20)(referred to before) is that physically (1) may not be the best equation to use from the poles to the equator. Equation (1) is actually

$$f\nabla^2\Psi + \vec{\nabla}f \cdot \vec{\nabla}\Psi = \nabla^2\Phi \tag{21}$$

This is really one equation on 2 dependent variables Ψ and Φ. In the high latitudes Φ is assumed known and one solves for Ψ, as in (1). In the equatorial latitudes however, it is believed [5] that Ψ is better known and one uses (21) to solve for Φ . The variational formulation of (21) for Φ is

$$I_2(\Phi) = \iint\limits_{D_2} \left[(\nabla\vec{\Phi})^2 + 2h(\Psi)\Phi\right] a^2\cos\phi d\phi d\lambda \tag{22}$$

where $h(\Psi)$ is the left side of (21). Then one could take the domain for (20) to be down to some reasonable latitude, say 20 degrees N, and then use (22) through the equator to 20°S., then use (20) down to the South pole. The difficulty is that Ψ must be precisely specified on the boundary (20°) for (20) and Φ there likewise for (22). That is not very realistic.

Haltiner has proposed [5] an iterative method to avoid the effects of inaccuracies in specifying the boundary conditions. It basicly involves solving three problems in overlapping regions. It involves solving for Ψ from, say, 40°N. to 40°S., using some reasonable approximation (say Φ/f) for Ψ on the boundaries. This computation is done by using the observed winds to find the vorticity ζ and solving $\nabla^2\Psi = \zeta$. The resultant values of Ψ at, say , 20° are used to solve (1) for Ψ from the poles to 20° . Using these values of Ψ from, say, 30° to 20° and the values of Ψ from $\pm 20^\circ$ to 0° from step one, one can then solve for Φ from 30°N. to 30°S. using the linear (or non-linear) balance equation. Because of the inherent character of the Laplacean as a smoothing operator, this overlapping procedure should give more reasonable answers.

Another approach which minimizes this dichotomy is to use a finite element-Galerkin approach, a non-variational method. This requires that we use (21), the differential equation, rather than (20) or (22). One then subdivides the entire globe into D_i and takes Φ <u>and</u> Ψ to be low order polynomials in each D_i . In those domains where Φ is believed known, the coefficients of Ψ_i are specified. Then the coefficient of the remaining functions are determined by insisting that, for all i ,

$$\iint_{D_i} \left\{ f\nabla^2\psi_i + \vec{\nabla\psi}_i \cdot \vec{\nabla} f - \nabla^2\Phi_i \right\} p_i \, dA = 0 \tag{23}$$

where the p_i are the low order basis polynomials. This is nearly equivalent to the previous two integrals, but conceptially places both functions Ψ and Φ on the same basis. The result of (23) is again a linear matrix system for the coefficients of the Φ_i and Ψ_i , the polynomial functions which represent Φ and Ψ in each grid area.

A potential advantage to this approach is that there is no need whatsoever to make the grid spacing rectangular or uniform, because there is no finite differences whatsoever. The Φ_i and Ψ_i are explicit functions, and are differentiated exactly by hand. By coverting to the x , λ coordinate system $(x = \sin\phi)$, the coefficients in (23) are polynomials, so that the computations and integrations are elementary and in fact are exact. Thus the computational errors are only in the use of the Φ_i and Ψ_i , and in the data itself.

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[2] W. E. Boyce and Richard C. DiPrima, Elementary Differential Equations and Boundary Value Problems, J. Wiley and Sons, N. Y. (1969).

[3] William Elias, personal communication.

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