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PROPERTIES OF AN APPROXIMATE HAZARD TRANSFORM

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ABSTRACT

The calculation of the exact reliability of complex systems is a difficult and tedious task. Consequently simple approximating techniques have great practical value.

The hazard transform of a system is an invertible transformation of its reliability function which is convenient and useful in both applied and theoretical reliability work. A simple calculus for finding an approximate hazard transform for systems formed by series and parallel combinations of components is extended so that it can be used for any coherent system. The extended calculus is shown to lead to conservative approximations.

A first order version of the extended calculus is also discussed. This method of approximation is even more simple to use, but is not always conservative. Examples of its application indicate that it is capable of giving quite accurate results.

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1. INTRODUCTION

Suppose that two components perform independently and have probabilities p_1 and p_2 of doing so successfully, i.e. of *functioning*. If the components constitute a *series* system, i.e. a system that functions only if both of its components function, then the probability that the system functions is

$$p = h(p_1,p_2) = p_1p_2,$$

where $h(p_1,p_2)$ is a *reliability function*. With due regard for the limitations imposed by assuming that the components perform independently, the reliability function of a two component series system is a convenient summary of its stochastic properties. The convenience stems from the variety of interpretations that can be attached to the term functioning, for the components, and consequently for the system. For example, p_1 and p_2 could be component success probabilities for a mission that develops in time, or one of several types of component availabilities. In each case p, as computed from $h(p_1,p_2)$, will be the corresponding quantity for the system. The reliability functions of more complex systems have the same utility.

The quantities $u_1 = -log\ p_1, u_2 = -log\ p_2$ are the component *hazards*, and $u = -log\ p$ is the system hazard. For the two component series system

$$u = \eta(u_1,u_2) = u_1 + u_2,$$

where $\eta(u_1,u_2)$ is a *hazard transform*. The hazard transform of a system is equivalent to its reliability function, but for many purposes represents its properties in an even more convenient way. Examples of the application of hazard transforms are given in Esary, Marshall and Proschan (1970).

For the two component *parallel* system, i.e. the system that functions if either of its components functions, the reliability function is

$$h(p_1,p_2) = 1 - (1-p_1)(1-p_2),$$

and the hazard transform is

$$\eta(u_1,u_2) = -log\ \{1 - (1-e^{-u_1})(1-e^{-u_2})\}.$$

The contrast in complexity between the series and parallel hazard transforms is apparent. However, for small component hazards u_1,u_2, i.e. for large component success probabilities p_1,p_2,

$$\eta(u_1,u_2) \simeq u_1u_2$$

is a good approximation for the parallel system, and is conservative in that it overestimates system hazard and thus underestimates system reliability.

An *approximate hazard transform* η^* can be defined by:

$$\eta^*(u_1,u_2) = \eta(u_1,u_2) = u_1 + u_2 \quad \text{for two components in series,}$$

$$\eta^*(u_1,u_2) = u_1u_2 \quad \text{for two components in parallel.}$$

These definitions lead to a simple calculus for finding an approximate hazard transform for systems that can be formed using series and parallel

combinations of components. For example, the system with the reliability block diagram shown in Figure 1 has, following the above rules of computation, the approximate hazard transform

$$(1.1) \qquad \eta^*(u_1,\ldots,u_5) = (u_1u_2 + u_3)(u_4 + u_5)$$

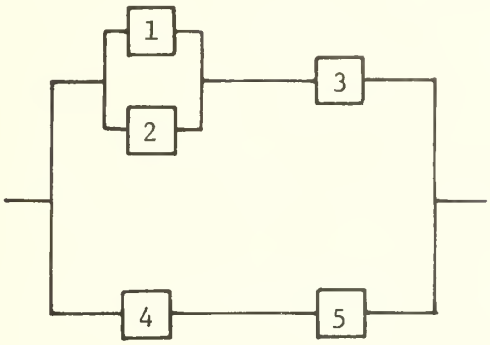

FIGURE 1

Rubinstein (1961) introduced this calculus as a step in deriving life test procedures for large systems, and considered (1965) more refined approximations for parallel and some other systems. The calculus is employed in the GUIDE MANUAL FOR RELIABILITY MEASUREMENT issued by the Navy Special Projects Office.

Our purpose is to note a simple extension of the calculus to systems that cannot be formed using just series and parallel combinations of components, e.g. the system with the block diagram shown in Figure 2, *and to show that the extended calculus is conservative.*

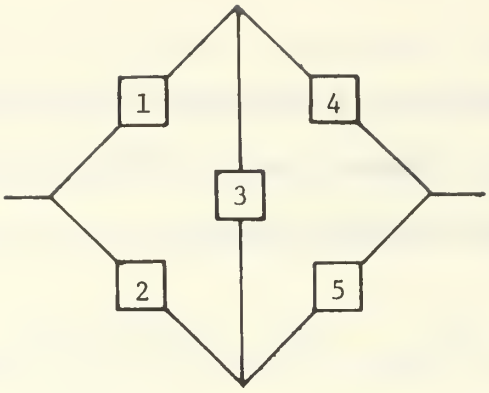

FIGURE 2

We also comment on a first order version of the extended calculus which is not necessarily conservative, but which gains in simplicity and can give quite accurate results.

2. AN APPROXIMATE HAZARD TRANSFORM

The approximate hazard transform we consider can be defined for the class of coherent systems. Systems describable by a reliability block diagram or by a fault tree using "and" and "or" gates are coherent. Alternately, the performance of the components in a system can be indicated by Bernoulli random variables $X_1,\ldots,X_n$, where $X_i = 1$ if the $i^{\underline{th}}$ component functions, $X_i = 0$ if the $i^{\underline{th}}$ component fails to function, and the performance of the system can be indicated by a *structure function* $\phi(\underset{\sim}{X}) = \phi(X_1,\ldots,X_n)$, where $\phi(\underset{\sim}{X}) = 1$ if the system functions, $\phi(\underset{\sim}{X}) = 0$ if the system fails to function. The system is *coherent* if ϕ is increasing in each of its coordinates and $\phi(1,\ldots,1) = 1$, $\phi(0,\ldots,0) = 0$, conditions which are clearly satisfied by systems described by block diagrams or fault trees.

If the components in a system perform independently, i,e. if $X_1,\ldots,X_n$ are independent, then the probability $p = P[\phi(\underset{\sim}{X}) = 1]$ that the system functions can be computed from the marginal probabilities $p_i = P[X_i = 1]$, $i = 1,\ldots,n$, that the components function. If not, then p depends on the joint distribution of $X_1,\ldots,X_n$. The *reliability function* h of a system describes the relationship between p and $p_1,\ldots,p_n$ in the case of independence. It is formally defined by

$$h(\underset{\sim}{p}) = h(p_1,\ldots,p_n) = P[\phi(\underset{\sim}{X}) = 1],$$

where $X_1,\ldots,X_n$ are independent and $P[X_i = 1] = p_i$, $0 \le p_i \le 1$, $i = 1,\ldots,n$.

It is common in the assurance disciplines to work with hazards rather than with probabilities. Examples are the "parts count" method in which component hazards are added to obtain a system hazard (in effect assuming that the system is series), and the practice of adding hazards over phases of a mission to obtain a mission hazard.

Recall that the component *hazards* are $u_i = -\log p_i$, so that $p_i = e^{-u_i}$, $i = 1,\ldots,n$, and the system *hazard* is $u = -\log p = -\log h(\underset{\sim}{p})$. A hazard is zero when the probability of functioning is one and increases to infinity as the probability of functioning decreases to zero, which to some extent makes the name appropriate. The *hazard transform* η of the system relates u to $u_1,\ldots,u_n$ and is defined by

$$\eta(\underset{\sim}{u}) = \eta(u_1,\ldots,u_n) = -\log h(\underset{\sim}{p}) = -\log h(e^{-u_1},\ldots,e^{-u_n}),$$

where $u_i \ge 0$, $i = 1,\ldots n$. Knowing the hazard transform of a system is equivalent to knowing its reliability function since

$$h(\underset{\sim}{p}) = e^{-\eta(\underset{\sim}{u})} = e^{-\eta(-\log p_1,\ldots,-\log p_n)},$$

The assumption that components perform independently is implicit in the definition of a hazard transform, just as it is in the definition of a reliability function.

The approximate hazard transform we consider can be conveniently introduced by first considering an approximation for the hazard transform of a parallel system and then extending the approximation to an arbitrary coherent system, using a representation of the system as a series of parallel subsystems related to its "minimal cuts."

The n component *parallel* system functions if at least one of its components functions, so that its structure function is

$$\phi(\underset{\sim}{X}) = 1 - \prod_{i=1}^{n}(1-X_i),$$

and its reliability function is

$$h(\underset{\sim}{p}) = P[\phi(\underset{\sim}{X}) = 1] = 1 - P[X_1 = 0,\ldots,X_n = 0]$$

$$= 1 - \prod_{i=1}^{n} P[X_i = 0] = 1 - \prod_{i=1}^{n}(1-p_i).$$

Its hazard transform is

$$(2.1) \qquad \eta(\underset{\sim}{u}) = -\log h(e^{-u_1},\ldots,e^{-u_n}) = -\log\ [1 - \prod_{i=1}^{n}(1-e^{-u_i})].$$

An *approximate hazard transform* for the n component parallel system is

$$(2.2) \qquad \eta^*(\underset{\sim}{u}) = \prod_{i=1}^{n} u_i\ ,\quad \underset{\sim}{u} \geq \underset{\sim}{0},$$

where $\underset{\sim}{0} = (0,\ldots,0)$ and $\underset{\sim}{u} \geq \underset{\sim}{0}$ means $u_i \geq 0$, $i = 1,\ldots,n$. The

approximation is based on the power series expansion for $\eta(\underset{\sim}{u})$.

The following lemma shows, for parallel systems, that: (a) the exact and approximate hazard transforms agree for perfectly reliable components, (b) the approximate hazard transform is conservative, i.e. it indicates greater hazard (less reliability) than the exact transform, and (c) the accuracy of the approximate hazard transform decreases as the component hazards increase. The lemma extends an observation of Rubinstein (1965, Appendix B).

Lemma 2.1 *For an* n *component parallel system the following comparisons exist between the hazard transform* η *and the approximate hazard transform* η^*:

(a) $\eta^*(\underset{\sim}{0}) = \eta(\underset{\sim}{0}) = 0$.

(b) $\eta^*(\underset{\sim}{u}) \geq \eta(\underset{\sim}{u})$, $\underset{\sim}{u} \geq \underset{\sim}{0}$.

(c) $\eta^*(\underset{\sim}{u}) - \eta(\underset{\sim}{u})$ *is increasing with respect to* u_i, $i = 1,\ldots,n$.

Proof. It is immediate that $\eta^*(\underset{\sim}{0}) = 0$ and that $\eta(\underset{\sim}{0}) = -\log 1 = 0$. Thus (a) holds. That (b) holds follows from (a) and (c). To verify (c) note that

$$\frac{\partial}{\partial u_i}\{\eta^*(\underset{\sim}{u}) - \eta(\underset{\sim}{u})\} = \prod_{j\neq i} u_i - \frac{e^{-u_i}\prod_{j\neq i}(1-e^{-u_j})}{1 - \prod_{j=1}^{n}(1-e^{-u_j})}.$$

Since $u_j \geq 1-e^{-u_j}$ so that $\prod_{j\neq i} u_j \geq \prod_{j\neq i}(1-e^{-u_j})$, and

$1 - \prod_{j=1}^{n}(1-e^{-u_j}) \geq e^{-u_i}$ (the reliability of a parallel system is not less than the reliability of one of its components), it follows that

$\frac{\partial}{\partial u_i}\{\eta^*(\underset{\sim}{u}) - \eta(\underset{\sim}{u})\} \geq 0$. Thus (c) holds. □

The assumption that the components perform independently is crucial to the comparisons of Lemma 2.1, as is shown by the following example.

Example 2.2 Suppose the components in a two component parallel system perform dependently in the strong, positive sense that $X_1 = X_2 = X$, i.e. if one component fails, so does the other. Then $\phi(X_1,X_2) = 1 - (1-X_1)(1-X_2) = X$, and

$$p = P[\phi(X_1,X_2) = 1] = P[X = 1] = p_1 = p_2,$$

where $p_1 = P[X_1 = 1]$ and $p_2 = P[X_2 = 1]$. The system hazard is $u = -\log p$ and the component hazards are $u_1 = -\log p_1 = u$, $u_2 = -\log p_2 = u$. If one now tries to approximate $\eta = u$ by $\eta^* = u_1u_2 = u^2$, only part (a) of Lemma 2.1 remains valid. In particular if $0 < u < 1$, then $\eta^* < \eta$, i.e. the approximation is not conservative. □

In a coherent system those combinations of components whose failure is just enough to cause a system failure are called minimal cuts. More precisely, a set of components K is a *cut* if $X_i = 0$, $i \in K$ and $X_i = 1$, $i \notin K$ implies $\phi(\underset{\sim}{X}) = 0$, and K is a *minimal cut* if no proper subset of K is also a cut. For example, the system shown in Figure 1 has the four minimal cuts

$$K_1 = \{1,2,4\}, \quad K_2 = \{1,2,5\}, \quad K_3 = \{3,4\}, \quad K_4 = \{3,5\}.$$

and the system shown in Figure 2 also has four minimal cuts, but these are

$$K_1 = \{1,2\}, \quad K_2 = \{1,3,5\}, \quad K_3 = \{2,3,4\}, \quad K_4 = \{4,5\}.$$

We will denote the minimal cuts of a coherent system by $K_1,\ldots,K_k$, where k is the number of minimal cuts.

Any coherent system can be represented in terms of its minimal cuts by forming, for each minimal cut, a parallel subsystem from the components in the cut, and then connecting the parallel subsystems in series. Formally the system structure function ϕ satisfies

$$\phi(\underset{\sim}{X}) = \coprod_{j=1}^{k} \phi_j(\underset{\sim}{X}) , \tag{2.3}$$

where $\phi_j(\underset{\sim}{X}) = 1 - \coprod_{i\in K_j}(1-X_i)$, $j = 1,\ldots,k$, are the structure functions of the parallel subsystems corresponding to the minimal cuts of the system. More graphically, the minimal cuts of a coherent system determine a particular way of drawing its block diagram. For example, the block diagram shown in Figure 3 is equivalent to the block diagram shown in Figure 1, and the block diagram of Figure 4 is equivalent to the block diagram of Figure 2.

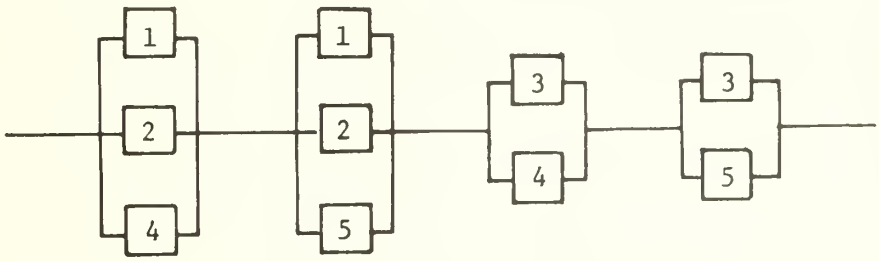

FIGURE 3

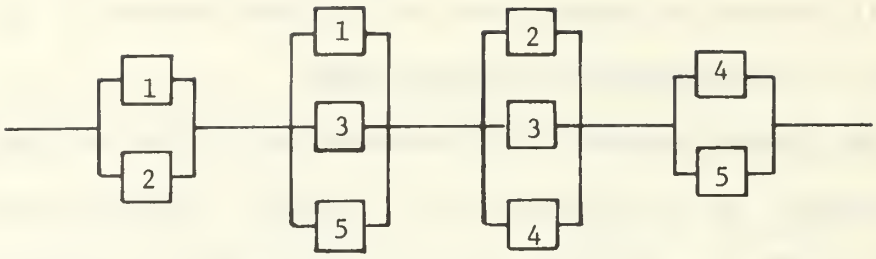

FIGURE 4

An important thing to note about the minimal cut representation of a coherent system is that the same component can appear in several minimal cuts. So even though the components perform independently, the parallel subsystems corresponding to the minimal cuts in general do not. Thus, if $h_j(\underset{\sim}{p}) = 1 - \prod_{i\in K_j}(1-p_i)$, $j = 1,\ldots,k$, are the reliability functions of the parallel subsystems, then the relationship $h(\underset{\sim}{p}) = \prod_{j=1}^{k} h_j(\underset{\sim}{p})$ which would hold if the parallel systems performed independently is *not necessarily valid.* What is true is that

$$(2.4) \qquad h(\underset{\sim}{p}) \geq \prod_{j=1}^{k} h_j(\underset{\sim}{p}) , \quad \underset{\sim}{0} \leq \underset{\sim}{p} \leq \underset{\sim}{1},$$

(Esary and Proschan, 1963, Theorem 4.1). The function

$$(2.5) \qquad h_{MC}(\underset{\sim}{p}) = \prod_{j=1}^{k} h_j(\underset{\sim}{p}) , \quad \underset{\sim}{0} \leq \underset{\sim}{p} \leq \underset{\sim}{1},$$

is the *minimal cut lower bound* on the exact reliability function h.

Recall that $\eta(\underset{\sim}{u}) = -log\ h(\underset{\sim}{p})$ is the exact system hazard transform. Define the *minimal cut upper bound* on η by

$$(2.6) \qquad \eta_{MC}(\underset{\sim}{u}) = -log\ h_{MC}(\underset{\sim}{p}) = \sum_{j=1}^{k} n_j(\underset{\sim}{u}) , \quad \underset{\sim}{u} \geq \underset{\sim}{0},$$

where $\eta_j(\underset{\sim}{u}) = -\log h_j(\underset{\sim}{p})$, $j = 1,\ldots,k$. The bound is obtained by finding the exact hazard transform for each of the parallel subsystems in the minimal cut representation for the system and adding these together as if the parallel systems performed independently. The following lemma is largely a restatement of inequality (2.4).

Lemma 2.3 *For a coherent system the following comparisons exist between the hazard transform* η *and* η_{MC}, *the minimal cut upper bound on* η:

(a) $\eta_{MC}(\underset{\sim}{0}) = \eta(\underset{\sim}{0}) = 0$.

(b) $\eta_{MC}(\underset{\sim}{u}) \geq \eta(\underset{\sim}{u})$, $\underset{\sim}{u} \geq \underset{\sim}{0}$.

Proof. That $\eta_{MC}(\underset{\sim}{0}) = 0$ follows from (2.6) and part (a) of Lemma 2.1. If $\underset{\sim}{u} = \underset{\sim}{0}$, then $\underset{\sim}{p} = (e^{-0},\ldots,e^{-0}) = \underset{\sim}{1}$, and $\eta(\underset{\sim}{0}) = -\log h(\underset{\sim}{1}) = -\log 1 = 0$. Thus (a) holds. That (b) holds follows from inequality (2.4), since $\eta_{MC}(\underset{\sim}{u}) = -\log h_{MC}(\underset{\sim}{p}) \geq -\log h(\underset{\sim}{p}) = \eta(\underset{\sim}{u})$. □

Remark 2.4 In contrast with part (b) of Lemma 2.1, part (b) of Lemma 2.3 remains valid, in essence, when component performances are positively dependent in a sense called "association" (Esary, Proschan, and Walkup, 1967 and Esary-Proschan, 1970). However, making use of this fact in the calculation of conservative approximate hazard transforms requires finding suitable modifications, under whatever degree of association is specified, to the approximation (2.2) for parallel systems. □

The *approximate hazard transform* that we consider for an arbitrary coherent system with minimal cuts $K_1,\ldots,K_k$ is

$$\eta^*(\underset{\sim}{u}) = \sum_{j=1}^{k} \eta^*_j(\underset{\sim}{u}), \quad \underset{\sim}{u} \geq \underset{\sim}{0}, \tag{2.7}$$

where $\eta^*_j(\underset{\sim}{u}) = \coprod_{i\in K_j} u_i$, $j = 1,\ldots,k$. The approximation is obtained by finding the approximate hazard transform for each of the parallel subsystems in the minimal cut representation for the system and then adding the results as if the parallel systems performed independently. Thus approximations at two different levels are involved. For example, the approximate hazard transform for the system of Figures 1 and 3 is

$$(2.8) \qquad \eta^*(u_1,\ldots,u_5) = u_1u_2u_4+u_1u_2u_5+u_3u_4+u_3u_5$$

and for the system of Figures 2 and 4

$$(2.9) \qquad \eta^*(u_1,\ldots,u_5) = u_1u_2+u_1u_3u_5+u_2u_3u_4+u_4u_5.$$

The following theorem shows that the approximate hazard transform for a coherent system is: (a) exact for perfectly reliable components, and (b) conservative.

Theorem 2.5 *For a coherent system the following comparisons exist between the hazard transform* η, *the minimal cut upper bound* η_{MC} *on the hazard transform, and the approximate hazard transform* η^*:

(a) $\eta^*(\underset{\sim}{0}) = \eta_{MC}(\underset{\sim}{0}) = \eta(\underset{\sim}{0}) = 0.$

(b) $\eta^*(\underset{\sim}{u}) \geq \eta_{MC}(\underset{\sim}{u}) \geq \eta(\underset{\sim}{u}), \quad \underset{\sim}{u} \geq \underset{\sim}{0}.$

Proof. That $\eta^*(0) = 0$ is immediate from (2.7). That $\eta_{MC}(\underset{\sim}{0}) = \eta(\underset{\sim}{0}) = 0$ is part (a) of Lemma 2.3. Thus (a) holds. To show (b) recall that $\eta_{MC}(\underset{\sim}{u}) \geq \eta(\underset{\sim}{u})$ is part (b) of Lemma 2.3, and note that from (2.6), (2.7), and part (b) of Lemma 2.1

$$\eta^*(\underset{\sim}{u}) = \sum_{j=1}^{k}\eta^*_j(\underset{\sim}{u}) \geq \sum_{j=1}^{k}\eta_j(\underset{\sim}{u}) = \eta_{MC}(\underset{\sim}{u}).$$

Thus (b) holds. □

3. A CALCULUS FOR APPROXIMATE HAZARD TRANSFORMS

One of the most attractive features of the approximate hazard transform (2.7) is the simple calculus by which it can be computed for certain important categories of coherent systems, notably the class of systems that can be formed by successive series and parallel combinations of subsystems with non-overlapping sets of components, i.e. the class of *simple* systems considered by Lomnicki (1973). The essential ideas of this calculus are described in Section 1, and its application to the system of Figure 1 is illustrated. Note that the approximate hazard transform for that system given in (1.1) agrees with the approximate hazard transform given in (2.8).

For the purposes of this section we will denote a system by a couple (C,ϕ), where C is the set of components used in forming the system (from the mathematical viewpoint C is a set of indices i used to label the components) and ϕ is the structure function of the system. We will be considering coherent systems (C,ϕ) formed from two coherent subsystems, (C_1,ϕ_1) and (C_2,ϕ_2). In this situation $C = C_1 \cup C_2$, i.e. the system component set consists of all the components appearing in either subsystem. The subsystems have non-overlapping or *disjoint* component sets if $C_1 \cap C_2 = \emptyset$, where $\emptyset$ is the empty set, i.e. no component appears in both subsystems. The reason for being interested in subsystems with disjoint component sets is that *if all the components in* C *perform independently and* C_1, C_2 *are disjoint, then the subsystems perform independently.*

The system (C,ϕ) is a *series* combination of the subsystems (C_1,ϕ_1) and $(C_2\phi_2)$ if

(3.1) $\phi(\underset{\sim}{X}) = \phi_1(\underset{\sim}{X})\phi_2(\underset{\sim}{X})$, for all realizations of $\underset{\sim}{X}$,

where $\underset{\sim}{X}$ is the vector of component performance indicators X_i, $i \in C$. If C_1 and C_2 are disjoint, then ϕ_1 depends only on those X_i such that $i \in C_1$, and ϕ_2 depends only on those X_i such that $i \in C_2$. Similarly, (C,ϕ) is a *parallel* combination of (C_1,ϕ_1) and (C_2,ϕ_2) if

(3.2) $\phi(\underset{\sim}{X}) = 1 - \{1-\phi_1(\underset{\sim}{X})\}\{1-\phi_2(\underset{\sim}{X})\}$, for all realizations of $\underset{\sim}{X}$.

It is easy to check that a series or parallel combination of coherent subsystems is a coherent system. Graphically, these definitions correspond to being able to display the block diagram for the system as a series or parallel combination of the block diagrams for the subsystems. The following proposition describes how approximate hazard transforms can be computed, for series and parallel combinations of subsystems with disjoint component sets, to obtain results that agree with (2.7).

Proposition 3.1 *Suppose a coherent system* (C,ϕ) *is a combination of the coherent subsystems* (C_1,ϕ_1) *and* (C_2,ϕ_2) *where* C_1 *and* C_2 *are disjoint. Let* η^* *be the approximate hazard transform for* (C,ϕ) *and* η^*_1,η^*_2 *be the approximate hazard transforms for* (C_1,ϕ_1) *and* (C_2,ϕ_2). *Then for all* $\underset{\sim}{u} \geq \underset{\sim}{0}$:

(a) $\eta^*(\underset{\sim}{u}) = \eta^*_1(\underset{\sim}{u}) + \eta^*_2(\underset{\sim}{u})$, *if the combination is series.*

(b) $\eta^*(\underset{\sim}{u}) = \eta^*_1(\underset{\sim}{u})\,\eta^*_2(\underset{\sim}{u})$, *if the combination is parallel.*

Proof. Suppose (C_1,ϕ_1) has the minimal cuts $K_{11},\ldots,K_{1k_1}$ and (C_2,ϕ_2) has the minimal cuts $K_{21},\ldots,K_{2k_2}$.

(a) If the combination is series, then since C_1 and C_2 are disjoint,

(C,ϕ) has $k = k_1+k_2$ minimal cuts, namely

$$K_1 = K_{11},\ldots,K_{k_1} = K_{1k_1},K_{k_1+1} = K_{21},\ldots,K_{k_1+k_2} = K_{2k_2}.$$

From (2.7)

$$\eta^*(\underset{\sim}{u}) = \sum_{j=1}^{k}\prod_{i\in K_j} u_i = \sum_{j_1=1}^{k_1}\prod_{i\in K_{1j_1}} u_i + \sum_{j_2=1}^{k_2}\prod_{i\in K_{2j_2}} u_i$$

$$= \eta^*_1(\underset{\sim}{u}) + \eta^*_2(\underset{\sim}{u}) ,$$

so (a) holds.

(b) If the combination is parallel, then since C_1 and C_2 are disjoint, (C,ϕ) has $k = k_1k_2$ minimal cuts, namely all $K_j = K_{1j_1}\cup K_{2j_2}$ where $j = 1,\ldots,k$ as $j_1 = 1,\ldots,k_1$ and $j_2 = 1,\ldots,k_2$. From (2.7)

$$\eta^*(\underset{\sim}{u}) = \sum_{j=1}^{k}\prod_{i\in K_j} u_i = \sum_{j_1=1}^{k_1}\sum_{j_2=1}^{k_2}\prod_{i\in K_{1j_1}\cup K_{2j_2}} u_i$$

$$= \sum_{j_1=1}^{k_1}\sum_{j_2=1}^{k_2}\prod_{i\in K_{1j_1}} u_i\cdot\prod_{i\in K_{2j_2}} u_i$$

$$= \sum_{j_1=1}^{k_1}\prod_{i\in K_{1j_1}} u_i \cdot \sum_{j_2=1}^{k_2}\prod_{i\in K_{2j_2}} u_i = \eta^*_1(\underset{\sim}{u})\eta^*_2(\underset{\sim}{u}) ,$$

so (b) holds. □

It is clear that if a system is formed by a sequence of series or parallel combinations of *modules*, i.e. subsystems with non-overlapping component sets, then parts (a) and (b) of Proposition 3.1 can be applied in the same sequence to evaluate its approximate hazard transform. Thus the proposition defines a calculus which is applicable to simple systems.

Remark 3.2 Proposition 3.2 can be extended to show that whenever a module (cf. Birnbaum and Esary, 1965) occurs in a coherent system, the approximate

transform for the system can be found by first finding the approximate hazard transform for the module and then proceeding as if the module were a component in the larger system with the hazard given by its approximate transform. □

A specialized application of the approximate hazard transform is found in the GUIDE MANUAL FOR RELIABILITY MEASUREMENT (Section 3.1.6). All components are assumed to have exponential life distributions, i.e. for a mission of duration t, $p_i = e^{-\lambda_i t}$, $i = 1,\ldots,n$. The mission length is taken as the unit of time measurement, and the component failure rates λ_i are scaled accordingly. In this way the mission reliability of a component has the simple form $p_i = e^{-\lambda_i}$. The component mission hazard is then $u_i = \lambda_i$. Under these conventions the calculus used in the manual is a special case of the approximate hazard transform calculus.

4. THE FIRST-ORDER APPROXIMATION

The approximate hazard transform η^* is always conservative. In this section we describe a *first-order approximate hazard transform* η^1 which is easier to compute than η^*, sometimes more accurate than η^*, but is not always conservative.

The approximate hazard transform η^* is based on all minimal cuts of the system, i.e. η^*, computed from (2.7), is the sum of the products of component hazards over each minimal cut. The first-order approximation is computed in a similar manner but using only the minimal cuts having the smallest number of components. The procedure is best illustrated by an example.

Recall that the system shown in Figures 2 and 4 has four minimal cuts. Two minimal cuts, $K_1 = \{1,2\}$ and $K_4 = \{4,5\}$, contain two components. The other cuts contain more than two components. The first-order approximation for this system is based only on cuts K_1 and K_2, i.e.

$$\eta^1(u_1,\ldots,u_5) = u_1u_2+u_4u_5. \tag{4.1}$$

The procedure for the first-order approximation, then, is to identify the smallest minimal cuts and to sum the products of component hazards over those cuts. By ignoring the larger minimal cuts, the first-order approximation tends to "correct" for the conservative error in the approximate hazard transform. Of course the "correction" may be too large, i.e. the first-order approximation is not always conservative.

If a component does not appear in any of the smallest minimal cuts, e.g. component 3 of Figures 2 and 4, then the first-order approximation implicitly treats the component as being perfectly reliable. If the reliability of such components is no less than the reliability of the components in the smallest minimal cuts, then the first-order approximation usually gives rather good results.

5. NUMERICAL COMPARISONS

In this section we compute system hazard using the approximate hazard transform η^* and the first-order approximation η^1, and compare the results with the true system hazard η for selected examples. The comparison is shown in terms of "percent error" where

$$\% \text{ error} = \frac{(\eta^1 \text{ or } \eta^*) - \eta}{\eta} \times 100.$$

In the examples, and in the experience of the authors with similar examples involving small systems, the accuracy of the approximations usually decreases with decreasing component reliabilities. If the reliabilities are greater than 0.9, the approximations usually are in error by less than 20%.

The formulas for the true system hazard are not exhibited in the examples. Some appreciation of the need for approximations can be gained by working them out. The motivation for the approximations discussed here is not only that numerical calculations are less tedious, but also that a formula for numerical calculations, or other purposes, can be derived with relative ease.

Example 5.1 Suppose the components in the system of Figures 1 and 3 have independent times to failure, exponentially distributed with parameters $\lambda_1,\ldots,\lambda_5$. Then for a mission of duration t, $p_i = e^{-\lambda_i t}$ and $u_i = \lambda_i t$, $i = 1,\ldots,5$. The approximations to the system hazard for the mission are

$$\eta^1 = u_3u_4 + u_3u_5 = \lambda_3 t\lambda_4 t + \lambda_3 t\lambda_5 t = \lambda_3(\lambda_4+\lambda_5)t^2,$$

and

$$\eta^* = (\lambda_1\lambda_2 t^2 + \lambda_3 t)(\lambda_4 t + \lambda_5 t)$$

[cf. (1.1)]. Some numerical comparisons for this system are tabulated below.

Case 1 $\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = \lambda_5 = 0.10$

		Percent Error	
t	η	η^1	η^*
0.2	.0008	1.2	2.4
0.4	.0031	2.2	6.0
0.6	.0070	3.3	9.5
0.8	.0122	4.5	12.8
1.0	.0189	5.8	16.4

Case 2 $\lambda_1 = \lambda_2 = \lambda_3 = 0.20, \quad \lambda_4 = \lambda_5 = 0.10$

		Percent Error	
t	η	η^1	η^*
0.2	.0016	--	4.4
0.4	.0064	0.8	8.8
0.6	.0142	1.6	13.8
0.8	.0249	2.7	19.1
1.0	.0385	4.0	24.7

In Case 2, when $t = 1.0$, $p_i = 0.82$, $i = 1,2,3$. This low component reliability adversely affects the accuracy of the approximate hazard transform. □

Example 5.2 Let components 1,2,3,4 in the system of Figures 2 and 4 have independent times to failure with the distributions shown below. Let component 3 be a "one-shot" device which, independently, either functions for a mission of any duration or is failed from the outset.

Component	Distribution	Parameters	Component Hazards for a Mission of Duration t
1	EXPONENTIAL	λ	λt
2	EXPONENTIAL	λ	λt
3	BERNOULLI	p_3	$-log\ p_3$
4	WEIBULL	μ,α	μt^{α}
5	WEIBULL	μ,α	μt^{α}

For a mission of duration t

$$\eta^1 = \lambda^2 t^2 + \mu^2 t^{2\alpha}$$

[cf. (4.1)], and

$$\eta^* = \lambda^2 t^2 - (log\ p_3)\ 2\lambda\mu t^{\alpha+1} + \mu^2 t^{2\alpha}$$

[cf. (2.9)]. Some numerical comparisons are tabulated below.

Case 1 $\lambda = 0.10$, $p_3 = 0.90$, $\mu = 0.10$, $\alpha = 2$

		Percent Error	
t	η	η^1	η^*
0.2	.0004	-1.5	2.4
0.4	.0019	-2.7	4.3
0.6	.0050	-2.5	6.5
0.8	.0106	-1.1	9.1
1.0	.0197	+1.5	12.2

Case 2 $\lambda = 0.10$, $p_3 = 0.95$, $\mu = 0.15$, $\alpha = 2$

		Percent Error	
t	η	η^1	η^*
0.2	.0006	0.1	2.5
0.4	.0030	0.7	4.8
0.6	.0081	2.3	7.4
0.8	.0179	5.2	10.8
1.0	.0342	9.5	15.2

In Case 1 the first-order approximation is not conservative for $t \leq 0.8$, i.e. system reliability is *over*estimated. Note that the component ignored by the approximation, component 3, is the least reliable component in the system for $t \leq 1.0$. □

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The calculation of the exact reliability of complex systems is a difficult and tedious task. Consequently simple approximating techniques have great practical value.

The hazard transform of a system is an invertible transformation of its reliability function which is convenient and useful in both applied and theoretical reliability work. A simple calculus for finding

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an approximate hazard transform for systems formed by series and parallel combinations of components is extended so that it can be used for any coherent system. The extended calculus is shown to lead to conservative approximations.

A first order version of the extended calculus is also discussed. This method of approximation is even more simple to use, but is not always conservative. Examples of its application indicate that it is capable of giving quite accurate results.