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ON SINGULAR VALUES OF HANKEL OPERATORS OF FINITE RANK

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On singular values of Hankel operators of finite rank

W. B. Gragg† and L. Reichel ‡

Abstract

Let H be a Hankel operator defined by its symbol $\rho = \pi/\chi$ where χ is a monic polynomial of degree n and π is a polynomial of degree less than n. Then H has rank n. We derive a generalized Takagi singular value problem defined by two $n \times n$ matrices, such that its n generalized Takagi singular values are the positive singular values of H. If ρ is real, then the generalized Takagi singular value problem reduces to a generalized symmetric eigenvalue problem. The computations can be carried out so that the Lanczos method applied to the latter problem requires only $O(n \log n)$ arithmetic operations for each iteration. If π and χ are given in power form, then the elements of all $n \times n$ matrices required can be determined in $O(n^2)$ arithmetic operations.

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Keywords

Hankel operator, singular values, generalized Takagi singular value problem, generalized eigenvalue problem, Lanczos iterations

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1. Introduction

Let $H = [\eta_{j+k}]_{j,k=0}^{\infty}$ be a be a Hankel operator defined by its rational symbol $\rho = \pi/\chi$, where

$$\pi(\lambda) := \sum_{j=0}^{n-1} \pi_j \lambda^j \quad \text{and} \quad \chi(\lambda) := \sum_{j=0}^{n} \chi_j \lambda^j, \; \chi_n = 1. \tag{1.1}$$

We assume that π and χ have no common zeros. The elements η_j of H are then given by

$$\rho(\lambda) = \frac{\pi(\lambda)}{\chi(\lambda)} = \sum_{j=0}^{\infty} \eta_j \lambda^{-j-1}. \tag{1.2}$$

In order to simplify our presentation, we assume that the zeros $\{\lambda_k\}_{k=1}^{n}$ of χ are distinct. How our formulas need to be modified in order to remove this assumption is discussed in Remark 1.1 below. Hence ρ has a partial fraction decomposition

$$\rho(\lambda) =: \sum_{k=1}^{n} \frac{\alpha_k}{\lambda - \lambda_k}. \tag{1.3}$$

Expansion of the right hand side of (1.3) into a geometric series, and comparison with (1.2), yields

$$\eta_j = \sum_{k=1}^{n} \alpha_k \lambda_k^j. \tag{1.4}$$

We now express (1.4) in matrix form. Let

$$A := diag[\alpha_1, \alpha_2, \ldots \alpha_n] \in \mathbb{C}^{n\times n}, \tag{1.5}$$

$$\Lambda := diag[\lambda_1, \lambda_2, \ldots \lambda_n] \in \mathbb{C}^{n\times n}, \tag{1.6}$$

and introduce the Vandermonde matrix

$$V_0 := [\lambda_{k+1}^j]_{j,k=0}^{n-1} \in \mathbb{C}^{n\times n}. \tag{1.7}$$

Define

$$V := [V_j]_{j=0}^{\infty} \in \mathbb{C}^{\infty\times n}, \tag{1.8}$$

where

$$V_j := V_0 \Lambda^{jn}, \; j \geq 1. \tag{1.9}$$

Then (1.4) can be written as

$$H = V A V^T. \tag{1.10}$$

Let l^2 denote the vector space $\mathbb{C}^{\infty}$ equipped with the Euclidean norm.

Proposition 1.1. $H : l^2 \to l^2$ bounded $\Leftrightarrow |\lambda_k| < 1$ for $1 \le k \le n$.

Proof. The proposition holds independent of the multiplicity of the λ_k. In the present proof we assume that the λ_k are distinct. The proof for confluent λ_k is commented on in Remark 1.1.

Let $e_1 = [\varepsilon_j]_{j=0}^{\infty} \in \mathbb{C}^\infty$ be the axis vector with $\varepsilon_0 = 1$. Then

$$h = [\eta_j]_{j=0}^{\infty} := He_1 \in l^2 \Rightarrow \eta_j \to 0 \text{ as } j \to \infty \Rightarrow$$

$$|\lambda_k| < 1 \text{ for } 1 \le k \le n,$$

where the last implication follows from (1.4).

Conversely, assume that $|\lambda_k| < 1$ for $1 \le k \le n$. Then by (1.8) - (1.10) we obtain

$$\begin{aligned} \|H\|_2 \le \|A\|_2 \|V\|_2^2 &\le \|A\|_2 \|V_0\|_2^2 \|\sum_{j=0}^{\infty} (\Lambda^H \Lambda)^{nj}\|_2^2 \\ &= \|A\|_2 \|V_0\|_2^2 \|(I - (\Lambda^H \Lambda)^n)^{-1}\|_2^2. \quad \blacksquare \end{aligned}$$

We assume henceforth that $|\lambda_k| < 1$ for $1 \le k \le n$. Introduce

$$U := V\, V_0^{-1}, \tag{1.11}$$

$$H_0 := V_0 A V_0^T. \tag{1.12}$$

Then H_0 has rank n. We note , by comparing (1.12) with (1.10), that H_0 is the leading principal $n \times n$ submatrix of H. From (1.10) - (1.12) it follows that

$$H = U H_0 U^T. \tag{1.13}$$

The leading $n \times n$ submatrix of U is I_n, the $n \times n$ identity matrix. U therefore is of rank n and can be factored

$$U = QR, \quad Q \in \mathbb{C}^{\infty \times n}, \quad R \in \mathbb{C}^{n \times n},$$

where $Q^H Q = I_n$ and R is a nonsingular right triangular matrix. We obtain

$$\sigma_+(H) = \sigma_+(QRH_0R^TQ^T) = \sigma(RH_0R^T), \tag{1.14}$$

where σ denotes the set of singular values and σ_+ denotes the subset of the positive ones.

The $n \times n$ matrix RH_0R^T is complex symmetric. Takagi [Ta1], [Ta2] showed the existence of a complex symmetric singular value decomposition

$$RH_0R^T = W\Sigma W^T,\ W \in \mathbb{C}^{n \times n},\ \Sigma = diag[\sigma_1, \sigma_2, \dots \sigma_n], \tag{1.15}$$

where $W^H W = I_n$ and $\sigma_j > 0$ are the singular values of RH_0R^T. In Section 2 we present an elementary proof of the existence of this decomposition. Let $W = [w_1, w_2, \dots w_n]$, $w_j \in \mathbb{C}^n$. Then (1.15) can be written as the Takagi singular value problem

$$RH_0R^T \overline{w}_j = w_j \sigma_j, \quad w_j^H w_k = \delta_{jk}, \quad 1 \le j, k \le n, \tag{1.16}$$

where the bar denotes complex conjugation and δ_{jk} is Kronecker's δ function. The problems (1.15) - (1.16) could be solved by the algorithm described in [BGG], but this would require RH_0R^T to be explicitly computed. In order to avoid these matrix multiplications we let $v_j := R^H w_j$ and obtain from (1.16) the *generalized Takagi singular value problem*

$$H_0\bar{v}_j = (R^H R)^{-1} v_j \sigma_j, \quad v_j^H (R^H R)^{-1} v_k = \delta_{jk}, \ 1 \le j,k \le n. \tag{1.17}$$

The solution of (1.17) requires $(R^H R)^{-1}$ to be known. In Section 3 we show that

$$(R^H R)^{-1} = I - B_0 B_0^H, \tag{1.18}$$

where $B_0 \in \mathbb{C}^{n\times n}$ is a triangular Toeplitz matrix. The elements of B_0 and H_0 can be determined from the coefficients of π and χ in $O(n \log n)$ arithmetic operations by the fast Fourier transform (FFT) method. This is demonstrated in Section 4. Section 5 shows that

$$R^H R = \overline{T_1 M_0 T_1^H}, \quad T_1, M_0 \in \mathbb{C}^{n\times n}, \tag{1.19}$$

where T_1 and M_0 are Toeplitz matrices, and describes a numerical scheme for the computation of this factorization from (1.16) in $O(n^2)$ arithmetic operations. We also present a Hermitian factorization of $R^H R$ into $n \times n$ triangular matrices.

The factorization (1.19) may be of interest for the numerical solution of (1.17). Assume that the coefficients of π and χ are real valued. Then H_0, $(R^H R)^{-1} \in \mathbb{R}^{n\times n}$, and (1.17) reduces to a generalized symmetric eigenvalue problem. The Lanczos method ([Pa, Section 15.11], [ER]) would appear suitable for solving this eigenproblem for the following reason. Let $C \in \mathbb{C}^{n\times n}$ be a Hankel or Toeplitz matrix and let $v \in \mathbb{C}^n$ be arbitrary. It is well known that Cv can be computed in $O(n \log n)$ arithmetic operations using FFTs. Hence $H_0 v$, $(R^H R)^{-1} v$ and $(R^H R)v$ can be computed in $O(n \log n)$ arithmetic operations, where we use (1.18) - (1.19). Each iteration of the Lanczos algorithm given in [Pa, p.324] therefore requires only $O(n \log n)$ arithmetic operations.

The computation of singular values of H is important in Hankel norm approximation problems of systems theory, such as the model reduction problem [Gl]. The approximation of functions by the Carathéodory - Fejér method yields another application [Gu], [Tr].

Other methods for reducing the singular value problem for H to a finite dimensional one have been described by Kung and Gutknecht [Gu] and Young [Yo]. These methods, however, do not preserve symmetry. Moreover, Young's approach requires generally $O(n^3)$ arithmetic operations to compute the matrices required.

Remark 1.1. Formulas (1.3) - (1.8) and the proof of Proposition 1.1 require distinct λ_k. This restriction can be removed. Assume first that $\lambda_1 = \lambda_2 = \ldots = \lambda_n$. Then (1.3) - (1.4) have to be replaced by

$$\rho(\lambda) =: \sum_{k=1}^{n} \frac{\alpha_k}{(\lambda - \lambda_1)^k}, \tag{1.3'}$$

$$\eta_j = \sum_{k=1}^{n} \frac{\alpha_k}{\lambda^k} \Big[\sum_{j=o}^{\infty} (\frac{\lambda_1}{\lambda})^j\Big]^k. \tag{1.4'}$$

In (1.5) A has to be substituted by the upper triangular Hankel matrix

$$A = [\alpha_{j+k+1}]_{j,k=0}^{n-1} \in \mathbb{C}^{n\times n}; \quad \alpha_p := 0, \quad p > n.$$

The matrix Λ in (1.6) has to be replaced by the Jordan matrix with all diagonal elements equal to λ_1 and all superdiagonal elements equal to one. The matrix V_0 in (1.7) need be replaced by the confluent Vandermonde matrix. For instance, we obtain for n = 3

$$\Lambda = \begin{bmatrix} \lambda_1 & 1 & \\ & \lambda_1 & 1 \\ & & \lambda_1 \end{bmatrix}, \qquad V_0 = \begin{bmatrix} 1 & & \\ \lambda_1 & 1 & \\ \lambda_1^2 & 2\lambda_1 & 1 \end{bmatrix}.$$

With A, Λ and V_0 modified as described, we define V_j and V by (1.8) - (1.9), U by (1.11) and H_0 by (1.12). Then (1.10) and (1.13) hold and H_0 is the leading principal $n \times n$ submatrix of H. Also (1.14) - (1.19) remain valid. Proposition 1.1 can be shown by replacing (1.4) by (1.4′), and by bounding the sum

$$\|\sum_{j=0}^{\infty}(\Lambda^H \Lambda)^{nj}\|_2^2$$

where Λ now is a Jordan matrix. This sum is bounded if $|\lambda_1| < 1$, and the proposition remains valid.

In general, when the λ_k are of arbitrary multiplicity, A in (1.5) has to be replaced by a block diagonal matrix, where each block is an upper triangular Hankel matrix. The blocks are of the same sizes as the multiplicities of the λ_k, and the number of blocks equals the number of distinct λ_k. Λ in (1.6) is replaced by a Jordan matrix with Jordan boxes of the same sizes as the multiplicities of the λ_k, and the number of boxes equal to the number of distinct λ_k. V_0 in (1.7) is replaced by an appropriate confluent Vandermonde matrix. With these changes (1.10) - (1.19) are valid, and so is Proposition 1.1. We omit the details since the numerical computations are independent of the multiplicity of the λ_k. ∎

2. The Symmetric Singular Value Decomposition

In this section we present an elementary proof of Takagi's theorem, i.e. we show the existence of a symmetric singular value decomposition of a complex symmetric matrix. Let $C = C^T \in \mathbb{C}^{n\times n}$, and define $A, B \in \mathbb{R}^{n\times n}$ by $C := A + iB$, $i := \sqrt{-1}$. Then $A = A^T$ and $B = B^T$, so the matrix

$$\tilde{C} := \begin{bmatrix} A & B \\ B & -A \end{bmatrix}$$

is real and symmetric. Let $\{\sigma_j\}_{j=1}^r$ be the *positive* eigenvalues of $\tilde{C}$ and form

$$\Sigma := diag[\sigma_1, \sigma_2, ..., \sigma_r].$$

Let

$$\begin{bmatrix} A & B \\ B & -A \end{bmatrix}\begin{bmatrix} U \\ V \end{bmatrix} = \begin{bmatrix} U \\ V \end{bmatrix}\Sigma \tag{2.1}$$

with

$$U, V \in \mathbb{R}^{n\times r}$$

and

$$U^T U + V^T V = I_r.$$

Write (2.1) as

$$\begin{cases} AU + BV = U\Sigma \\ BU - AV = V\Sigma \end{cases} \tag{2.2}$$

and note that (2.2) also can be written as

$$\begin{cases} AV + B(-U) = V(-\Sigma) \\ BV - A(-U) = (-U)(-\Sigma), \end{cases}$$

i.e.

$$\begin{bmatrix} A & B \\ B & -A \end{bmatrix}\begin{bmatrix} V \\ -U \end{bmatrix} = \begin{bmatrix} V \\ -U \end{bmatrix}(-\Sigma) \tag{2.3}$$

with

$$V^T V + (-U)^T(-U) = I_r.$$

Hence $\tilde{C}$ has at least r negative eigenvalues. We could also have let σ_j be the negative eigenvalues of $\tilde{C}$ and then (2.3) would have given us positive ones. We therefore may assume that $\pm\sigma_1, \pm\sigma_2, ..., \pm\sigma_r$ are all the nonzero eigenvalues of $\tilde{C}$.

Since eigenvectors associated with distinct eigenvalues of a real symmetric matrix are orthogonal, we have

$$0 = [V^T, -U^T]\begin{bmatrix} U \\ V \end{bmatrix} = V^T U - U^T V.$$

The spectral resolution of $\tilde{C}$ is thus

$$\begin{bmatrix} A & B \\ B & -A \end{bmatrix} = \begin{bmatrix} U & V \\ V & -U \end{bmatrix}\begin{bmatrix} \Sigma & \\ & -\Sigma \end{bmatrix}\begin{bmatrix} U^T & V^T \\ V^T & -U^T \end{bmatrix},$$

which yields

$$\begin{cases} A = U\Sigma U^T - V\Sigma V^T \\ B = V\Sigma U^T + U\Sigma V^T. \end{cases}$$

Therefore

$$C = A + iB = U\Sigma U^T - V\Sigma V^T + i(V\Sigma U^T + U\Sigma V^T)$$
$$= (U + iV)\Sigma(U^T + iV^T) = W\Sigma W^T = \sum_{k=1}^{r} \sigma_k w w_k^T,$$

where

$$U + iV =: W = [w_1, w_2, ..., w_r], \; w_k \epsilon \; \mathbb{C}^n.$$

Moreover

$$W^H W = (U^T - iV^T)(U + iV) = (U^T U + V^T V) + i(U^T V - V^T U) = I_r.$$

If $r < n$ then one may replace Σ by

$$\Sigma_0 := diag[\sigma_1, \sigma_2, ..., \sigma_r, 0, ..., 0] \; \epsilon \; \mathbb{R}^{n \times n}$$

and W by

$$W_0 := [w_1, w_2, ..., w_r, w_{r+1}, ... w_n] \; \epsilon \; \mathbb{C}^{n \times n},$$

where $w_{r+1}, ..., w_n \epsilon \; \mathbb{C}^n$ are chosen so that $W_0^H W_0 = I_n$. ∎

3. A Simple Expression for $(R^H R)^{-1}$

In this section we derive (1.18). Introduce the Frobenius matrix

$$F := [e_2, e_3, \ldots, e_n, -f] \in \mathbb{C}^{n\times n},$$

where

$$e_j := [\delta_{1j}, \delta_{2j}, \ldots, \delta_{nj}]^T \in \mathbb{R}^n, \quad 2 \le j \le n, \qquad (3.1)$$
$$f := [\chi_0, \chi_1, \ldots, \chi_{n-1}]^T \in \mathbb{C}^n.$$

Then F is the companion matrix of χ and

$$F^T V_0 = V_0 \Lambda. \qquad (3.2)$$

Throughout this section V_0 and Λ are defined by (1.6) - (1.7) if the λ_k are distinct. For confluent λ_k we modify V_0 and Λ according to Remark 1.1. The following lemma shows that

$$G := \overline{R^H R} \qquad (3.3)$$

satisfies a Stein equation. This will enable us to obtain a simple expression for G^{-1} by an application of the Sherman-Morrison-Woodbury formula.

Lemma 3.1. G is the unique solution of the Stein equation

$$X - F^n X F^{nH} = I_n, \; X \in \mathbb{C}^{n\times n}. \qquad (3.4)$$

Proof. By (1.8), (1.9) and (1.11) we obtain

$$R^H R = U^H U = \sum_{k=0}^{\infty} V_0^{-H} (\Lambda^{nk})^H V_0^H V_0 \Lambda^{nk} V_0^{-1}, \qquad (3.5)$$

and (3.2) yields now

$$G = \sum_{k=0}^{\infty} F^{nk} (F^{nk})^H. \qquad (3.6)$$

The series in (3.5) - (3.6) converge because $|\lambda_k| < 1$ for all k. Substitution of (3.6) into (3.4) shows that G solves (3.4). The unicity follows from $|\lambda_k| < 1$ for all k. The latter can be seen by a similarity transform of F^n to Schur triangular form. ∎

Introduce the cyclic downshift operator in $\mathbb{C}^{2n}$

$$E := [e_2, e_3, \ldots, e_n, e_1] \in \mathbb{C}^{2n\times 2n},$$

where

$$e_j := [\delta_{1j}, \delta_{2j}, \ldots, \delta_{2n,j}]^T \in \mathbb{R}^{2n}. \qquad (3.7)$$

Let

$$t := [\chi_0, \chi_1, \ldots, \chi_n, 0, 0, \ldots, 0]^T \in \mathbb{C}^{2n},$$

and define the Toeplitz matrix T of parallelogram form

$$T := [t, Et, E^2t, ..., E^{n-1}t] \in \mathbb{C}^{2n\times n}. \tag{3.8}$$

Let T_0 be the leading $n \times n$ submatrix of T, and let T_1 be the trailing $n \times n$ submatrix of T. Then T_0 is a left triangular Toeplitz matrix, and T_1 is a unit right triangular Toeplitz matrix.

Example 3.1. Let $n = 3$. Then

$$T = \begin{bmatrix} \chi_0 & & \\ \chi_1 & \chi_0 & \\ \chi_2 & \chi_2 & \chi_0 \\ \chi_3 & \chi_2 & \chi_1 \\ & \chi_3 & \chi_2 \\ & & \chi_3 \end{bmatrix}, \qquad T_0 = \begin{bmatrix} \chi_0 & & \\ \chi_1 & \chi_0 & \\ \chi_2 & \chi_1 & \chi_0 \end{bmatrix}, \qquad T_1 = \begin{bmatrix} \chi_3 & \chi_2 & \chi_1 \\ & \chi_3 & \chi_2 \\ & & \chi_3 \end{bmatrix},$$

where we note that $\chi_3 = 1$. ∎

Lemma 3.2. Let T_0 and T_1 be defined as above. Then

$$T_0^H T_0 + T_1^H T_1 = T_0 T_0^H + T_1 T_1^H. \tag{3.9}$$

Proof. Let $N := T^H T = T_0^H T_0 + T_1^H T_1$. We first show that N is a Toeplitz matrix. Let e_j be defined by (3.1). Then by (3.8) we have for $1 \leq j, k \leq n$,

$$e_j^T \ N e_k = e_j T^H T \ e_k = t^H (E^H)^{j-1} E^{k-1} t = t^H E^{k-j} t,$$

where we have used that $E^H = E^{-1}$. We next define the reversal matrix

$$J := [e_n, e_{n-1}, ..., e_1] \in \mathbb{R}^{n\times n}.$$

Toeplitz matrices are counter symmetric, i.e. $N = JN^T J$. Using that N is counter symmetric and Hermitian yields

$$\begin{aligned} T_0^H T_0 + T_1^H T_1 &= N = JN^T J = J\overline{N} J = J(T_0^T \overline{T_0} + T_1^T \overline{T_1})J \\ &= JT_0^T J \cdot J\overline{T_0} J + JT_1^T J \cdot J\overline{T_1} J = T_0 T_0^H + T_1 T_1^H. \end{aligned}$$ ∎

The next lemma presents a Gaussian factorization of F^n in terms of T_0 and T_1. This will be used together with Lemma 3.1 to express G^{-1} in terms of T_0 and T_1.

Lemma 3.3.

$$F^n = -T_0 T_1^{-1}. \tag{3.10}$$

Proof. We first show that

$$[T_0^T, T_1^T] \begin{bmatrix} V_0 \\ \\ V_0 \Lambda^n \end{bmatrix} = 0. \tag{3.11}$$

Let e_j be defined by (3.7) and assume for the moment that the λ_k are distinct. Then

$$e_j^T \ [T_0^T, T_1^T] \begin{bmatrix} V_0 \\ \\ V_0 \Lambda^n \end{bmatrix} e_k = \chi(\lambda_k) \lambda_k^{j-1} \tag{3.12}$$

and the right hand side vanishes for $1 \le j, k \le n$. If the λ_k are confluent, then the right hand side expression of (3.12) contains derivatives of $\chi(\lambda)$ evaluated at λ_k. The right hand side of (3.12), however, still vanishes and (3.11) holds.

We now write (3.11) as

$$T_0^T V_0 + T_1^T V_0 \Lambda^n = 0$$

and apply (3.2). This shows (3.10). ■

We are now in a position to show (1.18). By (3.4) G satisfies

$$G = I + F^n G \, F^{nH}$$

and an application of the Sherman-Morrison-Woodbury formula yields

$$G^{-1} = (I + F^n G \, F^{nH})^{-1} = I - F^n (G^{-1} + F^{nH} F^n)^{-1} F^{nH}. \tag{3.13}$$

We now determine an expression for

$$Y := I - G^{-1}. \tag{3.14}$$

Substitute Y and (3.10) into (3.13) to obtain

$$Y = T_0 (T_0^H T_0 + T_1^H T_1 - T_1^H Y T_1)^{-1} T_0^H. \tag{3.15}$$

In order to determine a simple expression for Y from (3.15) we need the following observation, which is also central to Section 4. T_0 and T_1^{-H} are both left triangular $n \times n$ Toeplitz matrices. Multiplication of T_0 with T_1^{-H} can be identified with polynomial multiplication, see [He1, Section 1.3] and Section 4. Since multiplication of polynomials commutes, we obtain

$$T_0 T_1^{-H} = T_1^{-H} T_0. \tag{3.16}$$

From the correspondence between polynomials and left triangular Toeplitz matrices it also follows that $T_0 T_1^{-H}$ is a left triangular Toeplitz matrix.

Lemma 3.4. Equation (3.15) has the unique solution

$$Y = T_1^{-H} T_0 T_0^H T_1^{-1} = T_0 T_1^{-H} T_1^{-1} T_0^H. \tag{3.17}$$

Proof. Unicity follows from (3.14) and that (3.4) has a unique solution. From (3.16) we obtain

$$T_1^{-H} T_0 T_0^H T_1^{-1} = T_0 T_1^{-H} T_1^{-1} T_0^H. \tag{3.18}$$

Now substitute

$$Y = T_1^{-H} T_0 T_0^H T_1^{-1}$$

into (3.15). We obtain

$$T_1^{-H} T_0 T_0^H T_1^{-1} = T_0 (T_0^H T_0 + T_1^H T_1 - T_0 T_0^H)^{-1} T_0^H. \tag{3.19}$$

An application of (3.9) reduces (3.19) to (3.18). The latter has already been shown to be valid. Therefore (3.17) solves (3.15). ∎

Let

$$B_0 := \overline{T}_0 T_1^{-T} = T_1^{-T} \overline{T}_0. \tag{3.20}$$

Then B_0 is a left triangular $n \times n$ Toeplitz matrix. By (3.14) and (3.17)

$$G^{-1} = I - \overline{B}_0 B_0^T = I - B_0^T \overline{B}_0.$$

From (3.3) it now follows that

$$(R^H R)^{-1} = I - B_0 B_0^H. \tag{3.21}$$

4. Computation of H_0 and B_0

We summarize some results in [He 1, Section 1.3] and [He 2, Section 13.9] in order to show that the elements of H_0 and B_0 can be computed in $O(n \log n)$ arithmetic operations from the coefficients χ_j of χ and π_j of π, see (1.1). To a polynomial or power series

$$\varsigma(\lambda) := \sum_{j=0}^{n-1} \varsigma_j \lambda^j + O(\lambda^n)$$

we associate the left triangular $n \times n$ Toeplitz matrix

$$Z = [\varsigma_{j-k}]_{j,k=0}^{n-1}, \qquad \varsigma_j = 0 \text{ for } j < 0,$$

and we write $\varsigma \to Z$. If $\xi(\lambda)$ is a polynomial and X a left triangular $n \times n$ Toeplitz matrix such that $\xi \to X$, then it is easily seen that $\varsigma\xi \to ZX$. In particular, ZX is a left triangular $n \times n$ Toeplitz matrix. From $\xi\varsigma = \varsigma\xi$ and $\xi\varsigma \to XZ$ is follows that $ZX = XZ$.

Assume that $\varsigma_0 \neq 0$ and let $1/\varsigma \to Z'$. Then $1/\varsigma \cdot \varsigma \to I$, $Z'Z$ and ZZ'. We obtain $Z' = Z^{-1}$ and therefore Z^{-1} is a left triangular Toeplitz matrix.

Example 4.1. We have $\chi \to T_0$. Let

$$\tilde{\chi}(\lambda) := \lambda^n \, \overline{\chi}(1/\lambda) = \sum_{j=0}^{n} \overline{\chi}_{n-j} \lambda^j. \tag{4.1}$$

Then $\tilde{\chi} \to T_1^H$ and the Blaschke product

$$\frac{\chi}{\tilde{\chi}} \to T_0 T_1^{-H} = \overline{B}_0. \tag{4.2}$$

■

Now let $\xi(\lambda)$ and $\varsigma(\lambda)$ be arbitrary polynomials such that $\varsigma(0) \neq 0$. Henrici [He2, Theorem 13.9e] shows that the first n coefficients in the MacLaurin expansion of $\xi(\lambda)/\varsigma(\lambda)$ can be computed in $O(n \log n)$ multiplications. The proof uses FFT. It is easily seen that the number of additions also is $O(n \log n)$.

From $\chi_n = 1$ and (4.1) we obtain $\tilde{\chi}(0) \neq 0$. Hence, the first n terms in the MacLaurin expansion of $\chi/\tilde{\chi}$ can be computed in $O(n \log n)$ arithmetic operations. By (4.2) therefore $\overline{T_0 T_1^{-H}} = B_0$ can be computed in $O(n \log n)$ arithmetic operations.

Because $\lambda^n \chi(1/\lambda) \neq 0$ for $\lambda = 0$, we can compute the first n terms in the MacLaurin expansion of

$$\frac{\lambda^n \pi(1/\lambda)}{\lambda^n \chi(1/\lambda)} = \sum_{j=0}^{n-1} \eta_j \lambda^{j+1} + O(\lambda^n)$$

in $O(n \log n)$ arithmetic operations. This shows that H_0 can be computed in $O(n \log n)$ arithmetic operations.

5. A Factorization of $R^H R$

It follows from (3.3) and (3.20) - (3.21) that

$$G^{-1} = \overline{(R^H R)}^{-1} = I - \overline{B_0 \; B_0^H} = I - T_1^{-H} T_0 T_0^H T_1^{-1}, \tag{4.1}$$

and therefore

$$T_1^H G^{-1} T_1 = T_1^H T_1 - T_0 T_0^H =: M_0^{-1}. \tag{4.2}$$

The expression defining M_0^{-1} is a Gohberg-Semencul formula for the inverse of an $n \times n$ Toeplitz matrix, see, e.g., [Io, Theorem 18.2, p. 152]. We denote this Toeplitz matrix by M_0. From the left hand expression of (4.2) and the nonsingularity of T_1 and R it follows that M_0 is Hermitian and positive definite. The desired factorization of $R^H R$ is

$$R^H R = \overline{T_1 \; M_0 \; T_1^H}.$$

We will now show how M_0 can be computed. The computation involves running the Levinson algorithm backwards.

Consider the related Gohberg-Semencul formula, see, e.g., [Io, Theorem 18.1, p. 148] or [AG],

$$M_1^{-1} = \begin{bmatrix} \chi_n & \chi_{n-1} & \cdots & \chi_0 \\ & \ddots & & \vdots \\ & & & \chi_{n-1} \\ & & & \chi_n \end{bmatrix}^H \begin{bmatrix} \chi_n & \chi_{n-1} & \cdots & \chi_0 \\ & \ddots & & \vdots \\ & & & \chi_{n-1} \\ & & & \chi_n \end{bmatrix} - \begin{bmatrix} 0 & & & \\ \chi_0 & \ddots & & \\ \chi_1 & & & \\ \vdots & & & \\ \chi_{n-1} & \cdots & \chi_1 \; \chi_0 & 0 \end{bmatrix} \begin{bmatrix} 0 & & & \\ \chi_0 & \ddots & & \\ \chi_1 & & & \\ \vdots & & & \\ \chi_{n-1} & \cdots & \chi_1 \; \chi_0 & 0 \end{bmatrix}^H \tag{4.3}$$

where the four triangular Toeplitz matrices define the inverse of an $(n+1) \times (n+1)$ Hermitian Toeplitz matrix. Denote this Toeplitz matrix by M_1. Then M_0 is the leading principal $n \times n$ submatrix of M_1, see [Io, Theorems 18.1 - 18.2].

Let $R_1 := [\rho_{jk}]_{j,k=0}^n \; \epsilon \; \mathbb{C}^{(n+1)\times(n+1)}$ be the unit right triangular matrix, and let $D_1 := diag[\delta_0, \delta_1, ..., \delta_n]$ be the diagonal matrix such that

$$R_1^H \; M_1 \; R_1 = D_1. \tag{4.4}$$

Given $M_1 = [\mu_{j-k}]_{j,k=0}^{n}$, the matrices R_1 and D_1 can be computed by the Levinson algorithm, and by comparing R_1 with (4.3) one finds that

$$\rho_{jn} = \chi_j,\ 0 \le j \le n \text{ and } \delta_n = \chi_n,$$

see, e.g., [AG]. We now apply the recursion formula in Levinson's algorithm backwards in order to determine R_1 and D_1 from the last column of R_1 and δ_n. Then the recursion formula is used forwards to determine M_0. We will also obtain a Hermitian factorization of $R^H R$ into triangular matrices.

Backward Levinson algorithm

input: $[\rho_{jn}]_{j=0}^{n}, \delta_n$; output: R_1, D_1, Schur parameters $\{\gamma_j\}_{j=1}^{n}$ of M_0;

for $k := n, n-1, n-2, ..., 1$ **do**

$\gamma_k := \rho_{ok}$; $\rho_{k-1,k-1} := 1$;

for $j := 1, 2, ...,$ integer part$(\frac{k}{2})$ **do**

solve for $\rho_{j-1,k-1}$ and $\rho_{k-1-j,k-1}$ the linear system of equations

$$\begin{bmatrix} 1 & \gamma_k \\ \overline{\gamma}_k & 1 \end{bmatrix} \begin{bmatrix} \rho_{j-1,k-1} \\ \overline{\rho}_{k-1-j,k-1} \end{bmatrix} = \begin{bmatrix} \rho_{j,k} \\ \overline{\rho}_{k-j,k} \end{bmatrix};$$

$\delta_{k-1} := (\delta_k/(1-|\gamma_k|))/(1+|\gamma_k|)$;

Levinson recursion for computing $M_0 = [\mu_{j-k}]_{j,k=0}^{n-1}$

input: $R_1, D_1, \{\gamma_j\}_{j=1}^{n}$; output: $\{\mu_j\}_{j=0}^{n-1}$;

$\mu_0 := \delta_0$; $\mu_1 := -\delta_0\overline{\gamma}_1$;

for $k := 1, 2, ..., n-1$ **do**

$\mu_{k+1} := -\delta_k\overline{\gamma}_{k+1} - \sum_{j=1}^{k} \mu_j \overline{\rho}_{j-1,k}$;

Hence M_0, R_1, and D_1 are computed in $O(n^2)$ arithmetic operations from the coefficients of χ. Let R_0 and D_0 denote the $n \times n$ leading principal submatrices of R_1 and D_1 respectively. Similarly to (4.4) we have

$$R_0^H M_0 R_0 = D_0. \tag{4.5}$$

Because M_0 is positive definite, so is D_0. $D_0^{1/2}$ can therefore easily be computed. We obtain from

(4.1) - (4.2) and (4.5), with $\hat{R} := D_0^{1/2} R_0^{-1}$,

$$R^H R = (\hat{R} T_1^H)^T \overline{(\hat{R} T_1^H)}. \tag{4.6}$$

The right hand side of (4.6) is a Hermitian factorization into triangular matrices. It can be computed in $O(n^2)$ arithmetic operations from the coefficients of χ.

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