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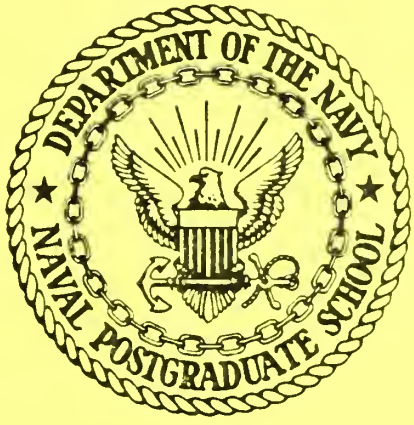

ON THE COMPUTATION OF OPTIMAL APPROXIMATIONS

IN SARD CORNER SPACES

Richard Franke

Technical Report for Period

July - September 1976

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This report investigates computation of optimal approximations in the Sard corner spaces $B_{\lceil 1,1 \rceil}$ and $B_{\lceil 2,2 \rceil}$. Use of the representers of point evaluation functionals is shown to be possible for up to 100 points or so in $B_{\lceil 1,1 \rceil}$. Two schemes for introducing basis functions which are zero in certain regions, including one set which have compact support, are investi-

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gated. Again, these are primarily useful for $B_{[1,1]}$. In the space $B_{[2,2]}$, which contains only continuously differentiable functions, use representers is possible only for small data sets unless one can use a great deal of precision in solving the system of linear equations which arises. The generation of basis functions with compact support is also possible in $B_{[2,2]}$. The general conclusion is that local schemes mus employed, particularly for smooth approximations.

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1.0 Introduction

The subject of linear optimal approximation has received considerable attention in recent years [4], [5], [6], [7], [8]. The subject of multivariate approximation for scattered data, including optimal approximations, is reviewed in [9]. The idea is appealing since the optimal approximation in a certain space of functions minimizes the norm of the error functional for approximations in that space. When the space is a Hilbert space, the computation of optimal approximations becomes rather simple, in theory [2]. A known reproducing kernel function provides the representers of linear functionals defined on the space. The optimal approximation satisfies the system of equations obtained by requiring that the approximation be exact for the representers of the functionals being used for the approximation, usually point evaluation functionals.

In practice, it seems that optimal approximations have not been used very much. This is perhaps partly because of a lack of experience with them, as well as the fact that use of the representers as a basis set for optimal approximation is the analog of the use of truncated power functions as a basis set for univariate spline approximation.

The particular space of functions of two variables to be considered here are the Sard "corner spaces", $B_{\lceil p,q \rceil}$ [8]. A suitable completion of these spaces into a Hilbert space and construction of the reproducing kernel was recently accomplished [1]. Since these spaces are made up of functions whose partial derivatives, up to a certain order in each variable, are absolutely continuous, these spaces contain spline functions in two variables, and the optimal approximations are splines. The Sard corner spaces have the property that the representers reduce to products of functions of one variable, thus

simplifying matters somewhat. Associated with the Sard space $B_{\lceil p,q \rceil}$ is a base point (a,b) in the region of interest. While this point can theoretically be anywhere, practically it is desirable for it to be at one of the corners of the region of interest (assumed rectangular). There are two reasons for this; (i) Computation of the representers is simplified; and (ii) the representers have continuous partial derivatives in x only up through order $p - 1$ at $x = a$ and in y only up through order $q - 1$ at $y = b$, while at other points partial derivatives in x and y are continuous up through order $2p - 2$ and $2q - 2$, respectively. The second reason is the primary reason we assume that (a,b) is the origin and that we are only interested in (x,y) points in the first quadrant.

In connection with the previous paragraph, we note that approximations in $B_{\lceil p,q \rceil}$ are not invariant with respect to translation (unless the base point (a,b) is also translated), nor with respect to stretching or shrinking of the coordinate system. The author had previously commented that they were invariant [3]. The base point (a,b) is the point at which Taylor series (with remainder) for the functions exist, and the approximations are clearly dependent on that point.

In the general case, the reproducing kernel function for $B_{\lceil p,q \rceil}$ is of the form $K(a,b;u,v,x,y) = g_p(a;u,x)g_q(b;v,y)$ where for $a \le u,x$,

$$g_p(a;u,x) = (-1)^p(x - u)_+^{(2p-1)} + \sum_{i<p} \{(u - a)^{(i)}(x - a)^{(i)} + (-1)^i(x - a)^{(i+1)}(u - a)^{(p-1-i)}\} .$$

Here the notation $w^{(i)}$ means $w^i/i!$ while

$$w_+ = \begin{cases} w & , w \ge 0 \\ 0 & , w < 0 \end{cases} .$$

For the case we consider, with $a = b = 0$, $p = q$, and where the functionals are point evaluations, say at the point (x_i,y_i), the representers are $K_i(x,y) = g_q(0;x_i,x)g_q(0;y_i,y)$. As in what follows we have here used simplified notation for the representer associated with (x_i,y_i). We will consider $q = 1$ and $q = 2$ and thus note that $g_1(0;x_i;x) = 1 + x - (x - x_i)_+$ and $g_2(0;x_i;x) = 1 + x_i x + \frac{1}{2} x_i x^2 - \frac{1}{6} x^3 + \frac{1}{6}(x - x_i)_+^3$.

One observation about optimal approximations is now in order, and should lead to increased interest in their use for approximations where the data is irregularly spaced. The data points themselves generate the representers, and hence a set of basis functions for the approximations. One does not need to be concerned about whether or not the basis functions have the interpolation property on the set. Unlike more common basis functions, e.g., polynomials, the representers naturally form a linearly independent set over the data points. This is not meant to imply that the representers of point evaluation functionals are well suited to computation, however. We treat this problem in more detail in later sections.

1.1 The interpolation problem

The underlying problem we shall be considering is that of function approximation by interpolation for functions of two or more variables. The case of more than two variables is a straightforward, if tedious, generalization, and the discussion is limited to two independent variables. Assume that the points (x_k,y_k,z_k), $k = 1,\ldots,N$ are given. No assumptions are generally made as to the spacing of the points, although in some instances we will consider special cases. We assume that if $i \neq k$, $(x_i,y_i) \neq (x_k,y_k)$

Approximation problems other than interpolation are treated in identical fashion. One obtains the same coefficient matrix for approximate integration or differentiation, for example.

2.0 Optimal approximation in $B_{[1,1]}$

In this section we will consider in some detail the problem of computing optimal approximations in the Sard space $B_{[1,1]}$. Again we emphasize that the region of interest is assumed to lie in the first quadrant and that the base point (a,b) is taken to be the origin.

2.1 Representers of point evaluation functionals as a basis

As noted in the introduction, the representers of point evaluation functionals have the form $K_j(x,y) = [1 + x - (x - x_j)_+][1 + y - (y - y_j)_+]$. These functions K_j are continuous, with first partial derivatives in x and y which have jumps at $x = x_j$ and $y = y_j$, respectively. In each of the rectangles $[0,x_j]x[0,y_j]$, $[0,x_j]x[y_j,\infty)$, $[x_j,\infty)x[0,y_j]$, and $[x_j,\infty)x[y_j,\infty)$ the function is bilinear. In the latter rectangle it takes on the constant value $(1 + x_j)(1 + y_j)$.

Use of the $K_j(x,y)$ as basis functions appears to be suspect. The coefficient matrix (the Gram matrix) is symmetric, but casual observation would lead one to suspect it is not particularly well conditioned. We shall see that it is better than the author's inclination toward it. The system of equations has the form

$$(1) \quad \sum_{j=1}^{N} A_j K_j(x_i,y_i) = z_i, \ i = 1,\ldots,N \ .$$

Some numerical experiments were conducted to compute the condition number (with respect to max row sum norm) of some Gram matrices of various sizes. Points were generated by a random number generator in the square $[0,10]x[0,10]$, the Gram matrix was formed, and the condition number computed. In case (i) the points were allowed to be anywhere in the square. In case (ii) the square was subdivided into $[\sqrt{N}]^2$ squares, then one point was generated at random in each

smaller square, and any remaining points were generated at random throughout $[0,10]x[0,10]$. In cases (iii) and (iv) the points in (i) and (ii) were translated to the square $[90,100]x[90,100]$. In cases (v) and (vi) the points in (i) and (ii) we stretched over the interval $[0,100]x[0,100]$. In case (vii) points with integer coordinates in $[0,10]x[0,10]$ were selected by a random number generator. These cases give some indication of condition numbers and the effects of translation and stretching. No claim is made that the cited results are representative nor that an exhaustive set of calculations were made. Other computations, not tabulated here, follow the same trend, however. The results of the calculations are tabulated in Table 1. Not all combinations were computed.

Case N →	10	25	50	100
(i)	170	3400	6700	67000
(ii)	420	1500	5700	36000
(iii)	2400	48000	180000	
(iv)	3600	37000	200000	
(v)	190	3200	8600	
(vi)	600	6100	6200	
(vii)		2900	15000	

Table 1: Condition Numbers of $(K_j(x_i,y_i))$ for $B_{[1,1]}$

One can make several observations from the table. First, for moderate values of N , even up to 50 or 100, satisfactory results can easily be obtain by computing with the representers as basis functions. Depending on the accuracy required in the computed answer and the precision one can (or is willing) to use in the computations, N could be quite a large number, perhaps as large as is feasible to consider for a global approximation. Second, the effect of stretching on the condition number is rather mild. Third, the translation of the points away from the base point increases the condition

number, indicating that one should select the base point (a,b) close to a corner of the region of interest, as we have done.

2.2 A set of basis functions with compact support

It is possible to construct a set of basis functions with compact support which leads to a block tridiagonal system of equations, where each block is tridiagonal, for approximations in $B_{[1,1]}$. Unfortunately the scheme is practical only in the special instance that the data points lie on a grid. Such a grid exists for any set of points, but ordinarily only one point lies on each horizontal and each vertical grid line. For the scheme to be practical it is not necessary that each grid point be a data point, but many points should be on most grid lines. This will be made more explicit after the development.

We must alter our usual notation slightly. Suppose we have grid points (x_i,y_j), $i = 1,\ldots, n$, $j = 1,\ldots,m$. We assume that the x_i and y_j are in increasing order. Denote the set of subscript pairs corresponding to data points by I. Then corresponding to each $(k,\ell)\in I$ there is a known function value $z_{k\ell}$. Nondata points on the grid will be denoted as $(i,j)\notin I$, where we will always assume that $1 \le i \le n$ and $1 \le j \le m$.

It is easy to obtain functions of one variable of the appropriate form which have compact support. They are

$$G_i(x) = \sum_{r=1}^{n} \alpha_{ir}\, g_1(0;x_r,x),\ i = 1,2,\ldots,n,$$

where as before $g_1(0;x_r,x) = 1 + x - (x - x_r)_+$,

with $\alpha_{11} = \dfrac{1 + x_2}{(1 + x_1)(x_2 - x_1)}$, $\alpha_{12} = \dfrac{1}{x_1 - x_2}$

$$\alpha_{ii-1} = \frac{1}{x_{i-1} - x_i},\ \alpha_{ii} = \frac{x_{i+1} - x_{i-1}}{(x_i - x_{i+1})(x_{i-1} - x_i)},\ \alpha_{ii+1} = \frac{1}{x_i - x_{i+1}}$$

$$\alpha_{nn-1} = \frac{1}{x_{n-1} - x_n} \ , \ \alpha_{nn} = \frac{1}{x_n - x_{n-1}}$$

and $\alpha_{ir} = 0$ if $|i - r| > 1$. For convenience, let $x_0 = 0$ and $x_{n+1} = \infty$ then we note that $G_i(x_j) = \delta_{ij}$ as well as $G_i(x) \neq 0$ only over the interva (x_{i-1}, x_{i+1}) . Also construct the dual functions $H_j(y)$, with $H_j(y) = \sum_s \beta_{js}\, g_1(0;y_s,y)$. Note that the G_i and H_j are linear B-splines.

To satisfy the interpolation requirements in terms of the local basis functions $G_i(x)H_j(y)$,we obtain the equations $\sum_{i=1}^{n} \sum_{j=1}^{m} a_{ij} G_i(x_k)H_j(y_\ell) \equiv$ $a_{k,\ell} = z_{k,\ell}$, $(k,\ell) \in I$. For $(k,\ell) \notin I$, the products $g_1(0;x_k,x)g_1(0;y_\ell,y$ cannot appear in the approximation. Substituting for $G_i(x)$ and $H_j(y)$ the approximation becomes $\sum_{i=1}^{n} \sum_{j=1}^{m} a_{ij}\, G_i(x)H_j(y) = \sum_{i,j} a_{ij} \sum_r \alpha_{ir}\, g_1(0;x_r,x) \sum_s$

$\beta_{js}\, g_1(0;y_s,y) = \sum_{i,j} a_{ij} \sum_{r,s} \alpha_{ir}\, \beta_{js}\, g_1(0;x_r,x)g_1(0;y_s,y)$.

We then set the coefficient of $g_1(0;x_k,x)g_1(0;y_\ell,y)$ for $(k,\ell) \notin I$, equal to zero, obtaining as the system of mn equations for the a_{ij} ,

$$(2) \qquad \begin{cases} a_{k,\ell} = z_{k,\ell} \ , \ (k,\ell) \in I \\ \sum_{i,j} a_{ij}\ \alpha_{ik}\ \beta_{j\ell} = 0, \ (k,\ell) \notin I \ . \end{cases}$$

If we order the equations and variables in some logical fashion, say $(1,1),..,\ (1,m),\ (2,1),...,\ (2,m),...,\ (n,m)$, for $m \leq n$, the fact that $\alpha_{ik} = 0$ if $|i - k| > 1$ and $\beta_{j\ell} = 0$ if $|j - \ell| > 1$ shows that the

resulting coefficient matrix of the system of equations is of the form

$$\bar{T} = \begin{pmatrix} T_{11} & T_{12} & 0 & -- & -- & 0 \\ T_{21} & T_{22} & T_{23} & 0 & -- & 0 \\ 0 & \diagdown & & & & \vdots \\ \vdots & & \diagdown & T_{nn-1} & & T_{nn} \end{pmatrix},$$

Where each $T_{ij} = \alpha_{ij}$

$$\begin{pmatrix} \beta_{11} & \beta_{12} & 0 & -- & -- & 0 \\ \beta_{21} & \beta_{22} & \beta_{23} & 0 & -- & 0 \\ 0 & \diagdown & & & & \vdots \\ \vdots & & \diagdown & & & \vdots \\ 0 & & & \beta_{mm-1} & & \beta_{mm} \end{pmatrix},$$

except that a row of $\bar{T}$ corresponding to $(k,\ell) \in I$ is replaced by a new row with zeros everywhere except for the unit diagonal element.

We can note that β_{jj-1} and β_{jj+1} are nonpositive and except for $j = 1$ and $j = m$, $\beta_{jj} = -\beta_{jj-1} - \beta_{jj+1}$, while for $j = 1$, $\beta_{11} > -\beta_{12}$ and for $j = m$, $\beta_{mm} = -\beta_{mm-1}$. Thus each block T_{ij} is diagonally dominant. An interesting aside is that the system (2) looks very much like the system of equations obtained when solving Laplace's equation by finite differenceson a rectangular grid, where a 9 point approximation to the Laplacian is used. Here, of course, the "boundary conditions" may be scattered throughout the region. In this respect the coefficient matrix is well suited to the use of iterative methods for solution of (2).

For the usual case, the above leads to a system of N^2 equations, and the amount of computation will be too large for even moderate N. Use of a block elimination equation solver reduces the problem to one of repeated solution of n systems of equations of m equations (usually full, but not always).

Although the original problem can be solved in about $N^3/6$ operations, the new basis requires about $n\,m^3$, or about N^4 operations if $m = n = N$. One should point out, however, that after obtaining the a_{ij} one has sufficie information to obtain the value of the interpolating function by bilinear inte polation since a_{ij} represents the function value at (x_i,y_j).

We will consider the case in which there are several data points on each grid line. The amount of work required to solve (2) decreases somewhat as the fraction of known grid values increases, however, the major effect is that the relationship between (n,m) and N changes. Assume that the fraction of grid points at which data is known is p, $0 < p \leq 1$. The total number of data points is then $N = pmn$, and the number of operations required to solve (1) is about $(pmn)^3/6$. For $(pmn)^3/6 \approx nm^3$ we see that $p^3 \approx 6/n^2$, or $p \approx \sqrt[3]{6}\, n^{-2/3}$. Representative values of these fractions are given in Table 2, along with the total number of equations in (1) and (2). If p is larger than the listed value in Table 2, (2) can be solved in fewer operations than (1).

n	p	N	mn
4	.721	2.88m	4m
8	.454	3.63m	8m
15	.299	4.48m	15m
50	.134	6.69m	50m
100	.0843	8.43m	100m
1000	.0182	18.2m	1000m

Table 2: Fraction p of grid points to be known for comparable number of operations in solving (1) and (2).

2.3 Basis functions which are zero in some regions

Because the representers of point evaluation functionals are constant for large enough values of the independent variables, a certain linear combination of any two can be made zero for large enough values of the independent variables. In particular, consider $K_i(x,y)$ and $K_j(x,y)$. The function $(1 + x_i)(1 + y_i)K_j(x,y) - (1 + x_j)(1 + y_j)K_i(x,y)$ is zero for $x \geq \max(x_i,x_j)$ and $y \geq \max(y_i,y_j)$. Thus a new set of basis functions with zero values over part of the region of interest can easily be constructed. It is desirable to first order the data points in terms of their "distance" from the origin. It seems reasonable to order the (x_k,y_k) in terms of nondecreasing values of $(1 + x_k)(1 + y_k)$. This is not the only ordering which can be used, but it carries the assurance that the new j^{th} basis function will be nonzero at the j^{th} point, and also has an added benefit we will discuss later.

Assume that the data points are ordered so that $\rho_k = (1 + x_k)(1 + y_k)$, $k = 1,\ldots,N$ is a nondecreasing sequence.

Then define $L_j(x,y) = \dfrac{\rho_j K_{j+1}(x,y) - \rho_{j+1} K_j(x,y)}{\rho_j K_{j+1}(x_j,y_j) - \rho_{j+1} K_j(x_j,y_j)}$, $j = 1,\ldots,N-1$

and $L_N(x,y) = \dfrac{K_N(x,y)}{K_N(x_N,y_N)}$.

Then the $L_j(x,y)$ satisfy $L_j(x_j,y_j) = 1$, $j = 1,\ldots,N$ and $L_j(x,y) = 0$ for $x \geq x_j^* = \max(x_j,x_{j+1})$ and $y \geq y_j^* = \max(y_j,y_{j+1})$, $j = 1,\ldots,N-1$

Using these basis functions, the interpolation problem requires the solution of the system

$$(3) \qquad \sum_{j=1}^{N} A_j L_j(x_i,y_i) = z_i, \quad i = 1,\ldots,N .$$

Because of the above ordering on the data points, there is some possibility of the entry $L_j(x_i,y_i)$ being zero for $j < i$, that is an element below the diagonal of the coefficient matrix.

The basis functions $L_j(x,y)$ have a property which is desirable and which arises out of our ordering of the data points.

Proposition: In the first quadrant, $|L_j(x,y)| \leq 1$.

We note that it is also true that if $\rho_{j+1} < \rho_j$, then $|L_j(x,y)| < 1$ except along one of the rays which start at (x_j,y_j) and extend horizontally to the right and vertically upward. The proof is simple and will not be given. It consists of considering the value of $L_j(x,y)$ at all points where the first partial derivatives are discontinuous, since any extrema must occur at such a point. These points are $(0,0)$, $(x_j,0)$, $(x_{j+1},0)$, $(0,y_j)$, $(0,y_{j+1})$, (x_j,y_j), (x_j,x_{j+1}), (x_{j+1},y_j), and (x_{j+1},y_{j+1}). Typical function values are shown in figure 1. Recall that in each rectangle $L_j(x,y)$ is bilinear and hence determined by its values at the corners.

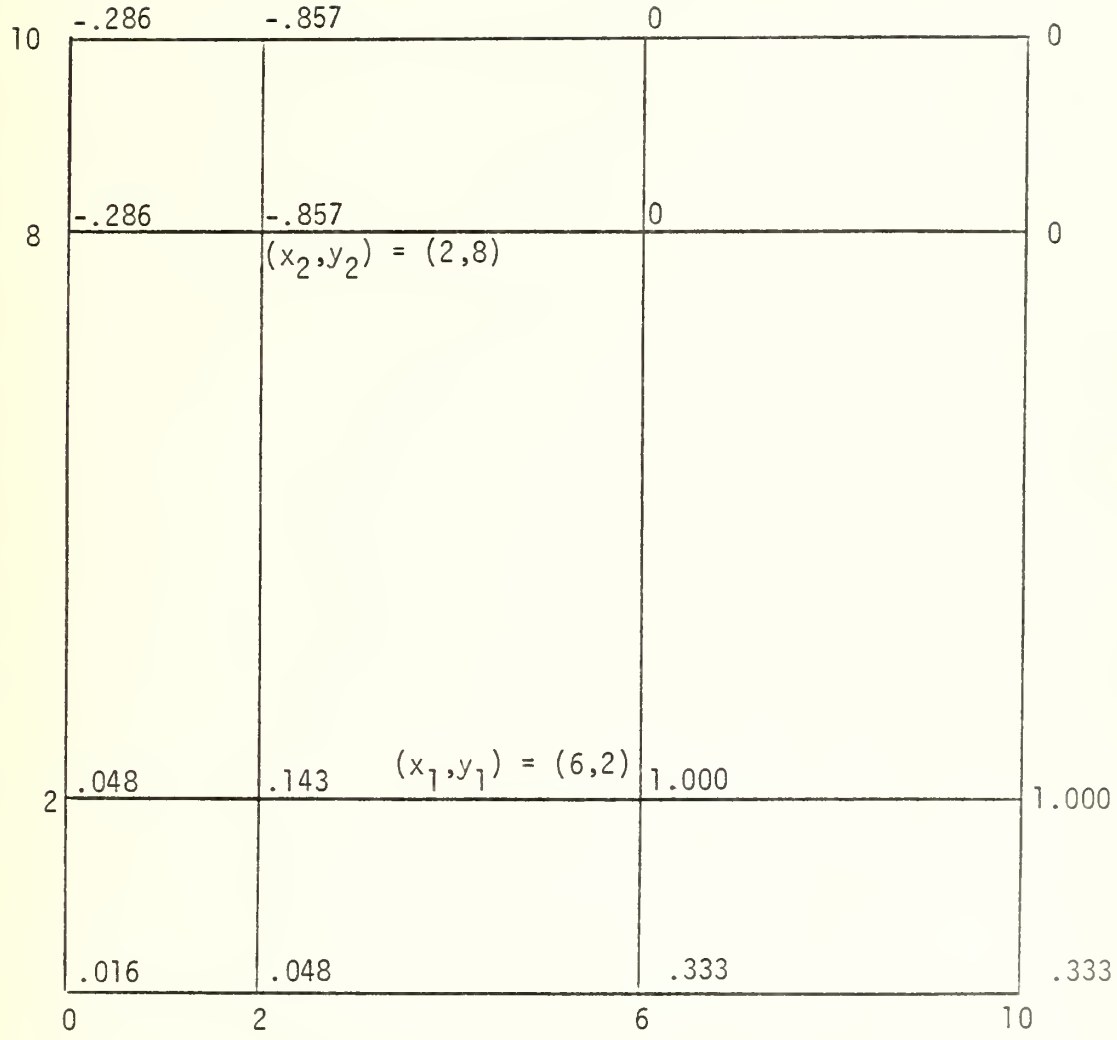

Figure 2: Values of $L_1(x,y)$

Certain behavior can be classified further. For example: (i) if $x_{j+1} > x_j, y_{j+1} > y_j$, then $|L_j(x,y)| < 1$ except at (x_j,y_j); (ii) if $\rho_{j+1} = \rho_j$, then $L_j(x,y) = 0$ for $x \leq \min(x_j,x_{j+1})$ and $y \leq \min(y_j,y_{j+1})$ and $L_j(x_{j+1},y_{j+1}) = -1$.

The coefficient matrix of the system (3) has its largest element (in magnitude) on the diagonal, and some zeros may occur below the diagonal. The data points used to generate Table 1 were used to test the effectiveness of the introduction of zeros and to determine the condition numbers associated with the new basis functions. The results are shown in Table 3, with the number in parenthesis indicating the number of leading zeros in the matrix. By reordering the columns of the matrix it is sometimes possible to introduce many more leading zeros, and while a scheme of this sort has not been implemented, in many cases it would substantially reduce the number of operations required for solution of the system (3).

The system (3) does not have a symmetric coefficient matrix, and unless approximately 30% leading zeros are introduced, it will take fewer operations to solve (1) than to solve (3). However, the condition number of the new coefficient matrix has been smaller in every case examined, sometimes by a factor of 15 or more, but more commonly by a factor of 2-5. Reordering columns for a maximum number of leading zeros will often result in 30% or more leading zeros, based on some hand computations where no effort was made to obtain the maximum number of leading zeros.

Case N →	10	25	50	100
(i)	36(4)	250(20)	1800(261)	18000(535)
(ii)	32(18)	460(32)	2000(162)	12000(582)
(iii)	1800(6)	2600(39)	30000(315)	
(iv)	160(18)	3300(62)	19000(288)	
(v)	23(2)	150(20)	1400(195)	
(vi)	28(18)	390(16)	1500(105)	
(vii)		620(21)	6200(146)	

Table 3: Condition Numbers of $(L_j(x_i,y_i))$ and Number of Leading Zeros (in Parenthesis)

3.0 Optimal Approximation in $B_{\lceil 2,2\rceil}$.

The problem of computing optimal approximations in the Sard space $B_{\lceil 2,2\rceil}$ is somewhat more difficult than for $B_{\lceil 1,1\rceil}$. The reproducing kernel functions are seen to be piecewise bicubic functions, reducing to bilinear functions for sufficiently large values of the independent variables. We shall investigate the feasibility of extending the results of the previous section to $B_{\lceil 2,2\rceil}$ in this section.

3.1 Representers of point evaluation functionals as a basis

The representer of the point evaluation functional at the point (x_j,y_j) is

$$K_j(x,y) = \left[1 + x_j x + \frac{1}{2} x_j^2 x - \frac{1}{6} x^3 + \frac{1}{6}(x - x_i)_+^3\right]\left[1 + y_j y + \frac{1}{2} y_j^2 y - \frac{1}{6} y^3 + \frac{1}{6}(y - y_j)_+^3\right].$$

This function is cubic in x for $0 \le x \le x_j$, and linear in x for $x > x_j$, and the dual holds in y. These functions increase rapidly since the point $x = x_j$ is the inflection point of the cubic in x, and thus when the

function is linear in x, it has slope the same as the maximum slope of the cubic. Because of this, the Gram matrix is not well conditioned. Following a similar path to that taken in the previous section, some condition numbers for the Gram matrix were computed for some sets of randomly generated points. The results are given in Table 4. The point description column refers to the descriptions in section 2.1.

Case N →	10	25	50	100
(i)	$3.94 \cdot 10^4$	$3.51 \cdot 10^7$	$8.27 \cdot 10^7$	$4.58 \cdot 10^9$
(ii)	$3.89 \cdot 10^5$	$1.89 \cdot 10^6$	$4.47 \cdot 10^9$	$9.95 \cdot 10^8$

Table 4: Condition Numbers of $(K_j(x_i,y_i)$ for $B_{\lceil 2,2 \rceil}$

The observations which we wish to make are that: (1) One will quickly be in numerical trouble in Real *4 on the IBM 360, and (2) While the condition number is large, meaningful computations can be done in Real *8.

3.2 A set of basis functions with compact support

A similar construction for $B_{\lceil 2,2 \rceil}$ as was pursued for $B_{\lceil 1,1 \rceil}$ in section 2.2 can be done. There is some question as to what conditions should be imposed on G_1, G_2, G_3, G_4, G_{N-3}, G_{N-2}, G_{N-1}, and G_N, (as well as the corresponding dual functions H_i), but several reasonable options are open. In the general case one wants $G_i(x) = \sum_{r=1}^{n} \alpha_{ir}\, g_2(0;x_r,x)$ so that $G_i(x_i) = 1$, $G_i(x) = 0$, $x \leq x_{i-2}$ or $x \geq x_{i+2}$. Proceeding in similar fashion one will obtain a system of equations which in block form has 5 non-zero blocks, each block being a square matrix with 5 non-zero elements per row. Numerically this is somewhat more complicated than before, but there are instances where it could be useful.

No details of the construction have been carried out here, but the generalization is straightforward. Also note that the G_i and H_j here are cubic B-splines.

3.3 Basis functions which are zero in some regions

It seems natural to be able to extend this idea to $B_{\lceil 2,2 \rceil}$. The basis functions $K_j(x,y)$ are bilinear for $x \geq x_j$ and $y \geq y_j$, thus it seems possible a certain linear combination of them could be made identically zero to the right and above all points (x_j,y_j) associated with those five basis functions. Proceeding in the obvious fashion, ordering the points (x_k,y_k) by some rule, we then wish, for $j \leq N - 4$, to construct functions

$$L_j(x,y) = \sum_{k=j}^{j+4} \gamma_{jk} K_k(x,y) \quad \text{such that}$$

$$L_j(x,y) = 0 \quad \text{for} \quad x \geq x_j^* = \max (x_{j+4}, x_{j+3}, x_{j+2}, x_{j+1}, x_j)$$

$$\text{and} \quad y \geq y_j^* = \max (y_{j+4}, y_{j+3}, y_{j+2}, y_{j+1}, y_j)$$

and $L_j(x_j,y_j) = 1$. Unfortunately, if the points (x_{j+4},y_{j+4}), (x_{j+3},y_{j+3}), (x_{j+2},y_{j+2}), (x_{j+1},y_{j+1}) lie on any bilinear curve, this system of equations generally has no solution. Thus the ordering imposed earlier would at least have to restrict one away from four successive points lying on a bilinear curve. In general, this is not possible.

Because of the rather more restrictive region where the function is zero, less benefit is likely to accrue anyway. In addition, even when one can construct such sets of basis functions, it is not possible to bound the function $L_j(x,y)$ as in the $B_{\lceil 1,1 \rceil}$ case, and in particular it cannot be bounded by one.

4.0 Conclusions

This investigation has determined that use of the representers of point evaluation functionals can be used as a basis for some problems without encountering severe computational problems. For smooth approximations this is probably not generally true, however. In addition, because the computational burden for global approximations is likely to be quite large, it is the author's opinion that local approximations must be investigated for smooth interpolation. The time is perhaps propitious for an investigation into the underlying mathematical basis for some previously suggested schemes for local smooth interpolation.

The use of optimal approximations in Sard corner spaces for the interpolation of irregularly spaced data results in an approximation which has discontinuities along the lines parallel to the axes through each data point. This would seem to the author to be unnecessarily complicated, and the author intends to investigate global approximations in which the discontinuities are less numerous and local in character rather than extending along lines to infinity.

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